A moving-mesh nite-di erence method for segregated two-phase competition-di usion

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Abstract

A moving-mesh nite-di erence solution of a Lotka-Volterra competition-di usion model of theoretical ecology is described in which the competition is su ciently strong to spatially segregate the two populations, leading to a two-phase problem with coupling conditions at the moving interface. The moving mesh approach preserves the identities of the two species in space and time, so that the parameters always refer to the correct population. The model is implemented numerically with a variety of parameter combinations illustrating how the populations evolve through time.

1 Introduction

We apply a moving mesh nite di erence method based on conservation [4, 6] to a problem in population dynamics. A Lotka-Volterra competition model is considered that describes a two-phase segregated reaction-di usion system with a high competition limit such that the species are completely spatially segregated and only interact using an interface condition [3t7([3t7([3(an)-36trV1(di]Ti559-14.44

secondly, there is a parameter in the Lotka-Volterra model of the interface condition (the equivalent of the latent heat coe cient of the Stefan problem) which is set equal to zero. Unlike the Stefan problem, one species does not transform into another, which means that the competition system has an interface condition that speci es the interface velocity only implicitly.

A moving-mesh approach is an e ective way to model this system because unlike xed mesh descriptions it provides a framework for keeping particular mesh nodes attached to particular species rather than particular parts of space, and the dynamics for any given location are automatically those of the correct species.

We use the Lotka-Volterra model described by Hilhorst et al. in [3] approximated by the moving-mesh nite-di erence method (MMFDM) of [4].

2 The Lotka-Volterra system

The Lotka-Volterra system is the two-component reaction-di usion system

$$\frac{@u_1}{@t} = {}_1\frac{@^2u_1}{@x^2} + f(u_1;u_2)u_1 \qquad x \in R_1(t); \quad t > 0 \qquad (1)$$

$$\frac{@u_2}{@t} = {}_2\frac{@^2u_2}{@x^2} + g(u_1;u_2)u_2 \qquad x \in R_2(t) \quad t > 0 \qquad (2)$$

where $u_1(x;t)$ and $u_2(x;t)$ are the population densities of two competing species in abutting regions $R_1(t)$ and $R_2(t)$, the parameters $_1$, $_2$ are constant di usion coe cients, and

$$f(u_1;u_2) = r_1\left(1 - \frac{u_1 + K_1u_2}{k_1}\right)$$

$$g(u_1;u_2) = r_2\left(1 - \frac{u_2 + K_2u_1}{k_2}\right):$$

are reaction terms in which $K_1;K_2$ are species-speci c competition rates, $k_1;k_2$ are the carrying capacities of the species, and $r_1;r_2$ are reproductive rate parameters.

In [3] it is demonstrated that for two species completely segregated the reaction terms can be reduced to

$$f(u_1;u_2) = r_1(1 - u_1=k_1)$$

$$g(u_1; u_2) = r_2(1 \quad u_2=k_2):$$

so that equations (1) and (2) become

$$\frac{@u_1}{@t} = {}_1\frac{@^2u_1}{@x^2} + \quad r_1 \quad 1 \quad \frac{u_1}{k_1} \qquad u_1 \qquad x \; 2 \; R_1(t); \quad t > 0 \qquad (3)$$

$$\frac{@u_2}{@t} = {}_2\frac{@^2u_2}{@x^2} + \quad r_2 \quad 1 \quad \frac{u_2}{k_2} \qquad u_2 \qquad x \; 2 \; R_2(t) \quad t > 0 \qquad (4)$$

The resulting system represents the limit in which the carrying capacities $K_1; K_2$ values are very large, i.e. the competition rate is high enough that the two species cannot coexist in space and interact only through the interface boundary.

Initial conditions on u_1 and u_2 are selected such that one species is in growth and the other in decline. These are shown in gure (1).

Zero Neumann boundary conditions $@u_1=@x = 0$ and $@u_2=@x = 0$ are applied at xed external boundaries away from the interface.

2.1 The interface conditions

At the interface between the two species there is a condition in [3] that gives the relationship between their uxes. In essence, the species both ow into the interface and annihilate each other in a ratio determined by the competition coe cient . This condition is given as

$$_1\frac{@u_1}{@x} = \qquad {}_2\frac{@u_2}{@x} \qquad (5)$$

where $= K_2=K_1$

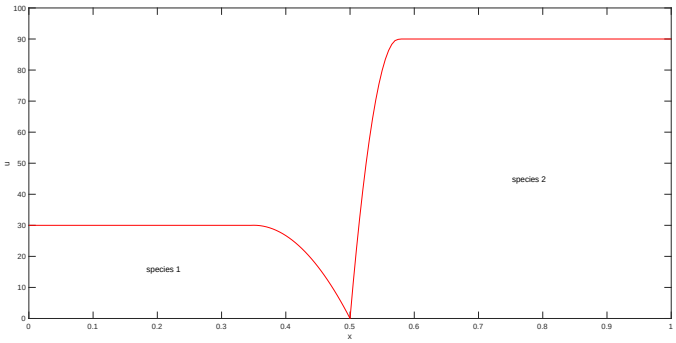

Figure 1: Initial conditions for the competition system, with population density u_1 of species 1 (on the left) and u_2 of species 2 (on the right). The interface node has zero population and must always satisfy the interface condition.

3 The MMFDM conservation method

3.1 A relative conservation principle

De ne the total population of species p as

$$_p(t) = \int_{R_p(t)} u_p(x;t)\,dx$$

$(p = 1;2)$. Then by Leibnitz' Integral Rule,

$$-_p = \frac{d\ _p}{dt} = \frac{d}{dt}\int_{R_p(t)} u_p(x;t)\,dx = \int_{R_p(t)} \frac{@u_p}{@t}\,dx + [u_p v_p]_{R_p(t)}$$

interval $(x(t); b)$ where b is a xed boundary and $x(t)$ is any point in the region $R_2(t)$.

The boundary conditions at the external boundaries2 Tf 648e6ere

From [3] the interface condition is given by (5). Whilst the interface velocity is not given explicitly by (5) this equation does determine the location of the interface implicitly. Thus, if we know $\partial u/\partial x$ adjacent to the interface in each region we may use the condition that $u = 0$ at the interface to infer an interface position such that the values of $\rho_\alpha \partial u/\partial x$ either side of the interface are in the ratio .

We now describe a nite di erence numerical method for the solution of the problem.

4 Numerical solution

Let the domain (a, b) be $(0, 1)$. At time level $t = t^n$ de ne time-dependent mesh points

$$0 = x_0 < x_1^n < \cdots < x_{m-1}^n < x_m^n < x_{m+1}^n < \cdots < x_N^n < x_{N+1}^n = 1$$

where x_m^n is the node at the moving interface, and let u_i^n, $(0 \le i \le N+1)$, approximate $u(x, t)$ by u_i^n at these points.

The total mass approximations $_1^n \quad _1(t)$ and $_2^n \quad _2(t)$ of (12) and (13) are estimated by the composite trapezium rule

$$_1^n = \sum_{i=1}^{m} \frac{1}{2}(u_{i-1} + u_i)(x_i - x_{i-1}); \qquad _2^n = \sum_{i=m}^{N} \frac{1}{2}(u_i + u_{i+1})(x_{i+1} - x_i); \quad (14)$$

and the constant-in-time relative masses $c_{1,i}$ and $c_{2,i}$ in the interval (x_{i-1}^n, x_i^n) by

$$c_{1,i} = \frac{1}{1}\frac{1}{2}(u_{i-1}^0 + u_i^0)(x_i^0 - x_{i-1}^0); \qquad (0 \quad _{,i}; x_{i-1}^n$$

$$
\begin{aligned}
u(x;0) &= 30; && (0 \quad x \quad 0{:}34)\\
u(x;0) &= (x \quad 0{:}2)(0{:}5 \quad x) \quad 170 \quad 7{:}85; && (0{:}35 \quad x \quad 0{:}5)\\
u(x;0) &= 0; && (x = 0{:}51)\\
u(x;0) &= (x \quad 0{:}65)(0{:}5 \quad x) \quad 170 \quad 94; && (0{:}52 \quad x \quad 0{:}58)\\
u(x;0) &= 90; && (0{:}59 \quad x \quad 1)
\end{aligned}
$$

chosen to resemble the one in [2] (see gure 1).

4.1 Rates of change of the total populations

The rates of change of the total populations $_1$; $_2$ of (6) are approximated by composite trapezium rules, in region 1,

$$
{}_1 = \quad {}_1 \quad \frac{u_m^n \quad u_{m \quad 1}^n}{x_m^n \quad x_{m \quad 1}^n}
$$

$$
+ r_1 \sum_{i=1}^{m} \frac{1}{2} \quad u_{i \quad 1}^n \quad 1 \quad \frac{u_{i \quad 1}^n}{k_1} \quad + u_i^n \quad 1 \quad \frac{u_i^n}{k_1} \quad (x_i \quad x_{i \quad 1}) \qquad (17)
$$

from (12), and in region 2,

$$
{}_2 = \quad {}_2 \quad \frac{u_{m+1}^n \quad u_m^n}{x_{m+1}^n \quad x_m^n}
$$

$$
+ r_2 \sum_{i=m}^{N} \frac{1}{2} \quad u_i^n \quad 1 \quad \frac{u_i^n}{k_2} \quad + u_{i+1}^n \quad 1 \quad \frac{u_{i+1}^n}{k_2} \quad (x_{i+1} \quad x_i) \qquad (18)
$$

from (13).

4.2 Approximating the velocities

From (10), using the composite trapezium rule, the velocity v_i^n in region 1 satis es,

$$
c_{1;i} \; {}_1^n + u_i^n v_i^n = \quad {}_1 \frac{@u^i}{@x_m}
$$

$$+ r_1 \sum_{j=2}^{i} \frac{1}{2} \quad u_j^n \quad {}_1 \quad 1 \quad \frac{u_j^n \quad {}_1}{k_1} \quad + u_j^n \quad 1 \quad \frac{u_j^n}{k_1} \qquad (x_j \quad x_j \quad {}_1); \quad (1 < i < m \quad 1);$$

where we have taken the subdomain $_1^n$ to be the interval $(x^n; x_m^n)$. Similarly, from (11),the velocity v_2^n in region 2 satis es

$$c_{2;i} \; {}_2^n + u_i^n v_i^n = \quad {}_2 \frac{@u^i}{@x_m}$$

$$+ r_2 \sum_{j=i}^{N} \frac{1}{2} \quad u_j^n \quad 1 \quad \frac{u_j^n}{k_2} \quad + u_{j+1}^n \quad 1 \quad \frac{u_{j+1}^n}{k_2} \qquad (x_{j+1} \quad x_j); \quad (m+1 < i < N \quad):$$

where we have taken the subdomain $_2^n$ to be the interval $(x_m^n; x^n)$. x_{jn}^n $_i)$. $+1$

where the subscript m denotes the interface node and the x_{m-1}, u_{m-1} are adjacent node positions and solution values. Since $u_m = 0$, from (26) an approximation to the position of the interface node x_m^{n+1} in terms of adjacent nodal values at $m-1$ is

$$x_m^{n+1} = \frac{{}_1u_{m-1}^{n+1} x_{m+1}^{n+1} + {}_2u_{m+1}^{n+1} x_{m-1}^{n+1}}{{}_1u_{m-1}^{n+1} + {}_2u_{m+1}^{n+1}} : \qquad (27)$$

Thus, once the others x_i^{n+1}, u_i^{n+1} have been updated, x_m^{n+1} can be found from (27).

4.5 Algorithm

In summary, the moving mesh finite difference solution of the competition-diffusion problem given by equations (3) and (4) with the interface condition (5) on the moving mesh in 1-D is given by the following algorithm.

From the initial mesh and the initial condition compute the initial values $_p(0)$, $(p = 1, 2)$ of the total populations of the species from (14) and the values of the relative masses $s_{p,i}$ and $\bar{c}_{p,i}$ from (15), (16) and (23).

Then for each time step:

1. Find the rates of change $-_1, -_2$ of the total masses from (17) and (18),
2. Calculate the nodal velocities v_i from (10) and (11),
3. Update $_1$ and $_2$ from $-_1$ and $-_2$ using the explicit Euler scheme (19),
4. Generate the nodal values x_i at the next time-step from the v_i using the explicit Euler scheme (20),
5. Update the population densities u_i at the next time level in each region from (24) and (25),
6. Update the new position of the interface node x_m at the next time level from (27).

5 Results

We nd that the model is stable and robust. Even using the explicit Euler integration scheme we observe minimal oscillations a ecting the smoothness of the results. in [1].

5.1 A parameter choice

In the body of work concerning Lotka-Volterra equations there is a vast range of parameter values in use because there are so many varied but suitable examples of the type of competition that are described here. We select a conservatively representative set of parameters, chosen to demonstrate some of the behaviour that this model is able to describe.

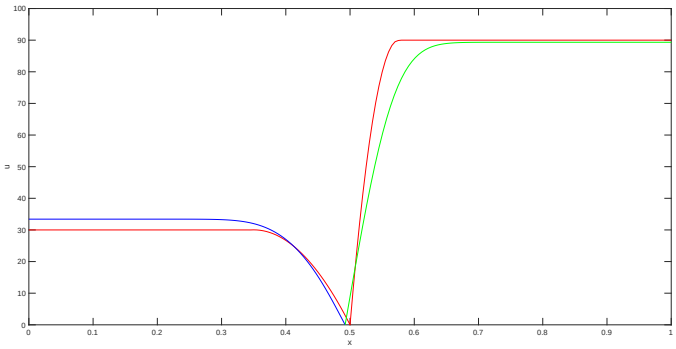

Figure 2: Result of competition model at $t = 1.5$. Here we use $_1 = \ _2 = 0.01$, $k_1 = k_2 = 100$, $r_1 = r_2 = 1$ and $\ = 3$. We run the model with a time step of 000001 for 150000 iterations and plot the results every 01. We see the internal dynamics of the species driving the population densities and interface uxes, and the position of the interface responding to those uxes. The domain is $0 < x < 1$. The initial conditions are shown in red, with species 1 in blue and species 2 in green.

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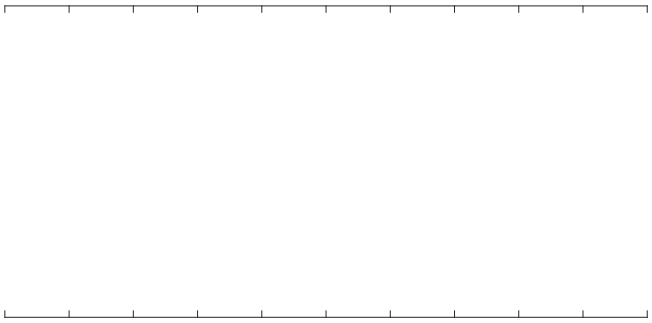

Figure 3: Result of competition model at $t = 4.5$. Here we use $_1 = \ _2 = 0.01$, $k_1 = k_2 = 100$, $r_1 = r_2 = 1$ and $\ = 3$. We run the model with a time step of 000001 for 450000 iterations and plot the results every 01. The interface continues to evolve and the masses of the species are now limited by the respective carrying capacities. The domain is $0 \ x < 1$. The initial conditions are shown in red, with species 1 in blue and species 2 in green.

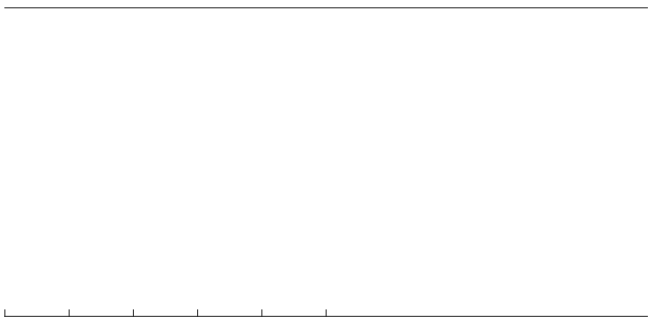

Figure 4: Result of competition model at t

5.2.2 Di usion characteristics

Alternatively we may adjust the di usion characteristics of the system. By allowing species 2 to di use at a higher rate, we observe that species 2 is able to make territorial gains due to this property alone (gure 7). Here we use $_1 = 0:01$; $_2 = 0:05$, $k_1 = k_2 = 100$, $r_1 = r_2 = 1$ and $= 3$. Due to the growth characteristics we can see interesting temporal e ects. Here the interface velocity has actually reversed directions as the system changes from di usion dominated to growth dominated. We observe that species 2 is able to make territory gains initially due to its high di usion rate, even though the competition rate is unaltered. However, as time goes on, the growth and competition characteristics become increasingly important. We see species 1 becoming more dominant over time, so that the interface velocity actually reverses direction.

Figure 8 shows the evolution of the system at $t = 11$, and gure 9 shows the movement of the interface with the direction reversal.

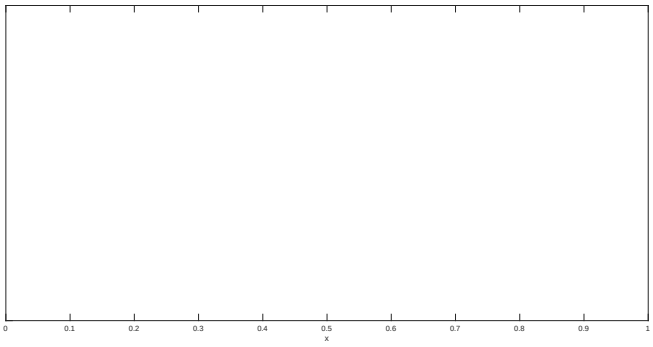

Figure 7: Result of competition model at $t = 1:5$, considering the e ect of an increased di usion rate for species 2. Here we use $_1 = 0:01$; $_2 = 0:05$, $k_1 = k_2 = 100$, $r_1 = r_2 = 1$ and $= 3$. We run the model with a time step of $0:00001$ for 150000 iterations, and plot the results every $0:1$. The gure shows the rapid territorial gains of species 2 over species 1 due to its high di usion rate. The initial conditions are shown in red, with species 1 in blue and species 2 in green. The domain is $0 x < 1$.

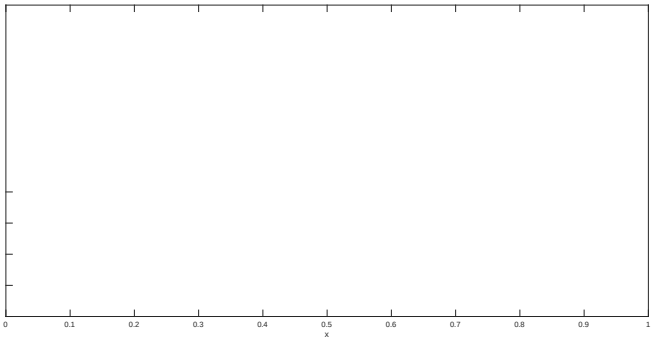

Figure 8: Result of competition model at $t = 11$, considering the effect of an increased diffusion rate for species 2. Here we use $D_1 = 0.01$, $D_2 = 0.05$, $k_1 = k_2 = 100$, $r_1 = r_2 = 1$ and $\alpha = 3$. We run the model with a time step of 0.000001 for 1.100000 iterations, and plot the results every 0.01. We see that the initial diffusion-driven gains by species 2 are reversed, and that the overall growth characteristics are dominating so that species 1 is gaining territory. The initial conditions are shown in red, with species 1 in blue and species 2 in green. The domain is $0 < x < 1$.

These illustrations give confidence that the model is likely to be able to satisfy the requirements of modelling a wide variety of competition systems. It is stable to a large choice of set-up parameters and is able to produce complex behaviours without problems.

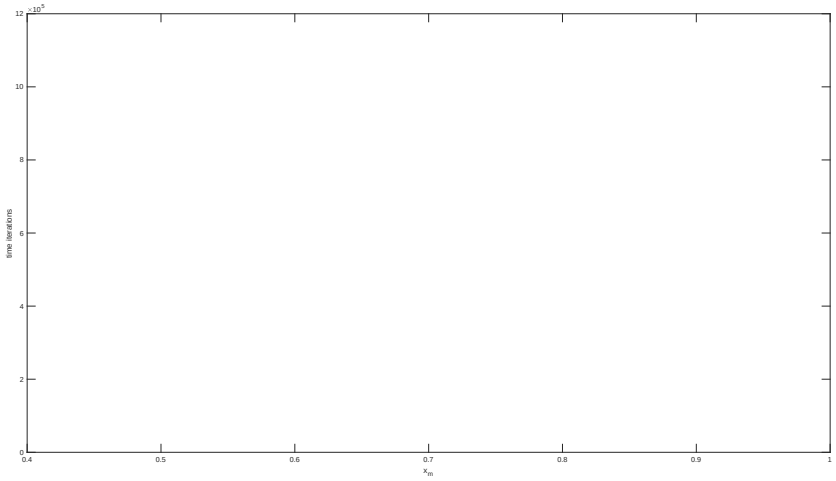

Figure 9: Position of interface x_m against time, showing interface movement for the competition model at up to $t = 11$, considering the effect of an increased diffusion rate for species 2. Here we use $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.05$, $k_1 = k_2 = 100$, $r_1 = r_2 = 1$ and $\gamma = 3$. We run the model with a time step of 000001 for 1100000 iterations. Due to the growth characteristics we can see interesting temporal effects. Here the interface velocity has actually reversed direction as the system changes from diffusion-dominated to growth-dominated.

6 Summary

In this paper we have applied the moving mesh finite difference method based on a conservation principle (MMFDM) of [4] to a two-phase Lotka-Volterra competition system with a high competition limit [3], such that the species are completely spatially segregated and interact solely through an interface condition based on this limit.

In section 2 the model is described in detail. In section 3 the MMFDM implemented. In section 4 numerics. In section 5 illustrations are given for a variety of parameter combinations, observing the various behaviours that dominate as the species evolve through time.

For a set of parameters that favour species 1 we see an increasing interface velocity in the initial stages followed by a long steady phase where the interface velocity is approximately constant. Although the population of species

2 initially grows it is eventually wiped out by the competition with species 1. As the annihilation of species 2 is approached, the interface velocity increases again. The interface continues to evolve and the populations of the species are then limited by the respective carrying capacities. This is due to the low

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