Moving nite element approximation of an aggregating population in two dimensions

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Abstract

The evolution of a species is modelled in two dimensions by a Lotka-Volterra equation system in which the random motion of individuals is biased so as to increase their expected rate of reproduction. The system is solved numerically in a xed nite region using a moving mesh nite element method in which the mesh movement is driven by local conservation. With random seeding the population is seen to form clusters which depend on parameters representing di usion and the size of a local survival region.

1 Introduction

In [3, 4] Grindrod noted that the derivation of many dispersion models rests

Results obtained in [4] from this model for a single population in one dimension demonstrated that from an initially random seeding of individuals local clusters are formed.

In this paper we demonstrate the clustering phemonenon numerically in two dimensions using a moving mesh finite element method based on conservation. The numerical approximation uses linear finite elements moving

2 Model equations for a single species

In order to emphasise clustering e ects we assume that births or deaths occur on a much longer time scale than clustering, so are neglected here.

The single species population balance equation for the population density $u(x;t)$ in a region is then

$$\frac{\partial u}{\partial t} = \nabla \quad (uv) \tag{2}$$

where the velocity v is the sum of the optimal velocity $= \nabla q$ and a di usive velocity $(\nabla u)=u$, leading to

$$\frac{\partial u}{\partial t} = \nabla^2 u \quad \nabla \quad (u \nabla q); \tag{3}$$

$x \in \quad ; \quad t \quad 0$ where is a di usion coe cient. Boundary conditions on u and q are the re ective conditions

$$\partial u = \partial n = 0; \qquad \partial q = \partial n = 0; \qquad x \in \partial \quad ; \quad t \quad 0. \tag{4}$$

Note that a consequence of (3) and (4) is that the total population

$$\int u(x;t)\,dx \tag{5}$$

is constant in time.

Given $E(u)$ and the population density $u(x;t)$ at any given time, we can obtain $q(x;t)$ from (1) and use (3) to determine the evolution of u. Following [3, 4] we take $E(u)$ to be of the form

$$E(u) = (1 \quad u)(u \quad a) \tag{6}$$

where $a = 0.2$.

3 Solution procedure on a moving domain

We follow a procedure in which the interior of the domain is allowed to deform in time so as to preserve and track a distribution of the local population.

We write (2) as

$$\frac{\partial u}{\partial t} + \nabla \cdot (uv) = 0\,; \tag{7}$$

an Eulerian conservation law equivalent to constancy in time of the (Lagrangian) local mass

$$\int_{\Omega(t)} u\,dx \tag{8}$$

when the points of the domain $\Omega(t)$ move with velocity $v(x;t)$.

We solve (3) numerically using the moving-mesh finite element procedure based on conservation described in [1, 2, 5], as follows. Comparing (7) with (3), the velocity v satisfies

$$\nabla^2 u + \nabla \cdot (u\ \ ) = \nabla \cdot (uv) \tag{9}$$

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Having found $v(x;t)$ from (9), ...

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$$= \int_{\Omega(t)} \frac{\partial}{\partial t}(wu)\, d\Omega + \oint_{\partial\Omega} w(x;t)u(x;t)v(x;t) \cdot \hat{n}\, dS;$$

yielding

$$\int_{\Omega(t)} \left(w(x;t)\frac{\partial u}{\partial t} + u(x;t)\frac{\partial w}{\partial t} + w(x;t)\nabla \cdot (uv) + u(x;t)v(x;t) \cdot \nabla w \right) d\Omega = 0 \; . \tag{12}$$

Assuming that the weight functions $w(x;t)$ move with the velocity $v(x;t)$ of the points of the domain,

$$\frac{\partial w}{\partial t} + v(x;t) \cdot \nabla w = 0;$$

so that equation (12) reduces to

$$\int_{\Omega(t)} \left(w(x;t)\frac{\partial u}{\partial t} - u(x;t)v(x;t) \cdot \nabla w \right) d\Omega = 0$$

After integration by parts using the boundary condition (4) we obtain the weak form

$$\int_{\Omega(t)} w(x;t)\nabla \cdot (uv)\, d\Omega = -\int_{\Omega(t)} w(x;t)\frac{\partial u}{\partial t} d\Omega \; ; \tag{13}$$

Then, substituting the weak form of the driving PDE (3) into (13),

$$\int_{\Omega(t)} w(x;t)\nabla \cdot (uv)\, d\Omega = \int_{\Omega(t)} w(x;t)(-\nabla^2 u + \nabla \cdot (u\nabla q))\, d\Omega$$

After integration by parts using the boundary conditions (4), we obtain the weak form

$$\int_{\Omega(t)} \nabla w \cdot (uv)\, d\Omega = \int_{\Omega(t)} \nabla w \cdot \nabla u\, d\Omega - \int_{\Omega(t)} u(x;t)\nabla w \cdot \nabla q\, d\Omega$$

for the velocity $v(x;t)$.

For a unique $v(x;t)$ we introduce a velocity potential $\phi(x;t)$ such that $v(x;t) = -\nabla\phi$, leading to the weak form

$$\int_{\Omega(t)} u(x;t)\nabla w \cdot \nabla\phi\, d\Omega = -\int_{\Omega(t)} \nabla w \cdot \nabla u\, d\Omega + \int_{\Omega(t)} u(x;t)\nabla q \cdot \nabla w\, d\Omega \; ; \tag{14}$$

for $(x;t)$, given $u(x;t)$ and $q(x;t)$. Equation (14) has a unique solution for $(x;t)$ given the boundary conditions (4), apart from a constant which differentiates out.

Before (14) is solved for $(x;t)$ we obtain the function $q(x;t)$ from a weak form of equation (1),

$$\int_{(t)} w(x;t)E(u)\,d = \int w(x;t)\nabla^2 q\,d + \int_{(t)} w(x;t)q(x;t)\,d \;:$$

Integrating the right hand side by parts using the boundary condition (4), we obtain the weak form

$$\int_{(t)} \nabla w \cdot \nabla q\,d + \int_{(t)} w(x;t)q(x;t)\,d = \int_{(t)} w(x;t)E(u)\,d \;; \qquad (15)$$

for $q(x;t)$, given $u(x;t)$.

The solution procedure for the velocity $v(x;t)$ is therefore to obtain $q(x;t)$ from (15) and (1), deduce $(x;t)$ from (14) and hence the velocity $v(x;t)$ from a weak form of the relation $v \cdot \nabla = 0$.

3.2 Finite elements

Finite elements are applied on a mesh of triangles within a 2-D polygonal region. Let w be a standard piecewise-linear finite element basis function $W_i(x)$, $(1 \le i \le N)$, on the mesh (the full set forming a partition of unity). The total distributed mass is constant in time through the imposition of zero Neumann natural boundary conditions (4).

A piecewise-linear population density $U(x;t)$ is given by the expansion

$$U(x;t) = \sum_j U_j(t)W_j(x) \qquad (16)$$

The distributed conservation principle (11) then becomes

$$\int_{(t)} W_i(x;t)\,U(x;t)\,d = C(W_i); \qquad (17)$$

constant in time, where

$$C(W_i) = \int_{(0)} W_i(x;0)U(x;0)\,d$$

is obtained from the initial conditions (at $t = 0$)

We take q and $E(u)$ to be the piecewise-linear finite element functions Q and E having expansions

$$Q = \sum_{j} Q_j(t) W_j(x;t); \qquad E = \sum_{j} E_j(t) W_j(x;t);$$

respectively. Note that although $E(u)$ is a nonlinear function of u we calculate discrete values of E at the nodes and accept a linear approximation between nodes. The weak form (15) then becomes

$$\sum_{j=1}^{N} \int_{(\ t)} W_i W_j \, d \quad E_j(t)$$

$$= \sum_{j=1}^{N} \int_{(\ t)} r \ W_i \quad \int$$

in matrix form

$$K(\underline{U})\_ = K\underline{U} + K(\underline{U})\underline{Q} \qquad (20)$$

where _ is a vector with entries $_i(t)$ and $K(\underline{U})$ is the weighted sti ness matrix with entries

$$Z$$

$$_{(\ t)} U(x; t)r$$

where $\dot{\hat{X}}_i(\hat{x};t) = V_i(\hat{X}(x;t);t)$ by the explicit Euler scheme

$$\hat{X}_i^{n+1} = \hat{X}_i^n + \Delta t\,\dot{\hat{X}}_i^n \qquad (22)$$

where Δt is the time step. The time step is chosen sufficiently small to avoid instability.

3.6 The moved solution

Having found the mesh points $\hat{X}_i$ at time t^{n+1} we recover the population density $U(\hat{x};t)$ at time t^{n+1} at the new time step from (16) expanded in terms of $W_j(\hat{x})$ as

$$U(\hat{x};t^{n+1}) = \sum_{j=1}^{N} U_j^{n+1} W_j(\hat{x});$$

using the weak form of the conservation principle (17), obtaining

$$\sum_{j=1}^{N} \left(\int_{\Omega(t)} W_i^{n+1} W_j^{n+1}\, d\Omega \right) U_j^{n+1} = C(W_i) \qquad (23)$$

where $C(W_i)$ is given from the initial conditions by

$$C(W_i) = \int_{\Omega(0)} W_i(x;0)\,U(x;0)\,d\Omega \qquad (24)$$

Equation (23) is equivalent to the matrix system

$$M(\hat{x})\underline{U}^{n+1} = \underline{C} \qquad (25)$$

where $\underline{U}^{n+1}$ is the vector having entries U_i^{n+1}, $\underline{C}$ is the vector having entries $C(W_i)$, and $M(\hat{x})$ is the mass matrix evaluated at $\hat{x}$.

4 Algorithm

Summarising, the algorithm for the moving mesh finite element solution of the single species aggregation model defined by equations (3), (1) and (6) on a mesh in 2-D in a region with fixed boundaries and with internal nodes moved by conservation is as follows.

From the initial mesh $X_i(x;0)$ and initial conditions $U(x;0)$ obtain the constant-in-time values of $C(W_i)$ from (24). Then, at each time step,

1. Calculate the nodal values of the piecewise-linear function $E(x;t)$ from equation (18),
2. Obtain $Q(x;t)$ from equation (19),
3. Find the velocity potential $(x;t)$ from equation (20),
4. Deduce the node velocities $V(x;t)$ from equation (21),
5. Determine the moving coordinates $\hat{x}_i(x;t)$ at the next time-step from (22),
6. Recover the solution $U(\hat{x};t)$ on the moved mesh at the next time step from equation (25).

5 Results

We use a random seeding to provide the initial conditions for the model, selected from a normal distribution with a mean of 0:3 and a standard deviation of 0:01. We are able to run the model sometimes to a blow up and sometimes to a solution where population growth and decline become approximately balanced, depending on the initial values of u_i, and also on the parameters and . The parameter controlling the rate of diffusion has a smoothing effect while from the definition contained within (1) it is apparent that defines the scale of the clusters that are expected to form. We can see this scaling effect in the results, with the number and size of clusters reduced as increases.

An example solution is given in figure 1, for parameters $= 0:005$ and $=$ 0:01. This choice produces four clusters from the initially random seeding.

the two e ects become balanced and the approximately balanced solution is observed.

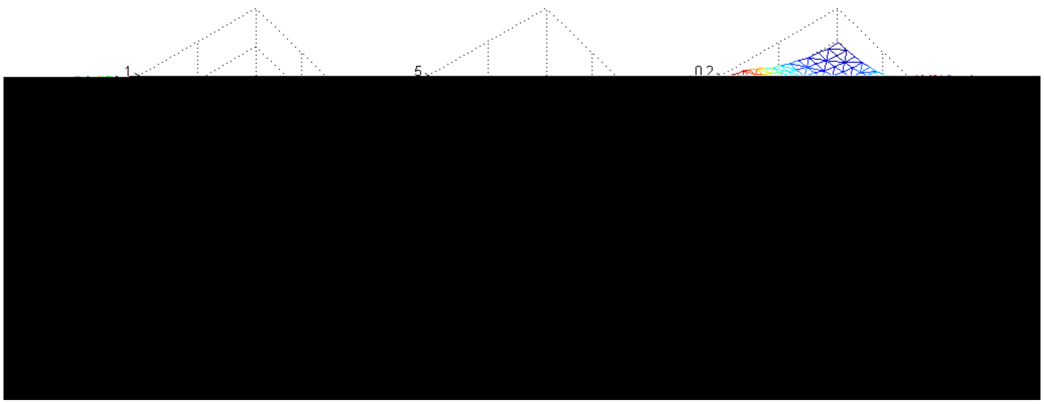

Figure 1: A solution of the 2D population equations after 350 time steps at t = 0:35, with = 0:005 and = 0:01.

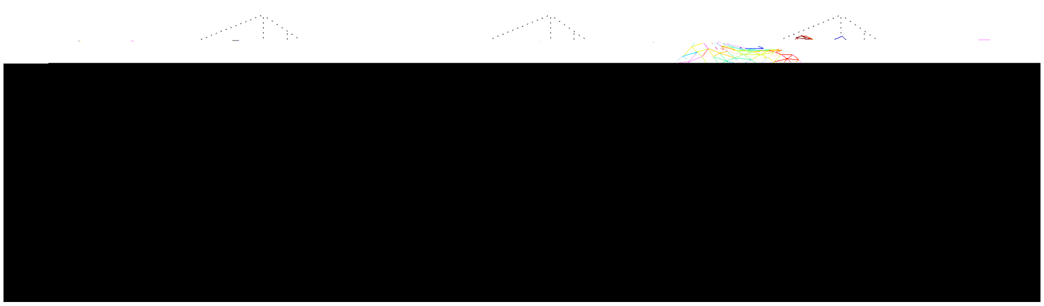

Figure 2: A solution of the 2D population equations after 10 time steps at t = 0:01, with = 0:001 and = 0:01.

world systems which can be described in a similar manner to this model. The aim should be to understand the requirements from both a mathematical and value perspective. Subsequent development will be in the direction of the research requirements of those ecological systems which would most bene t from a study which has access to this modelling capability.

References

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