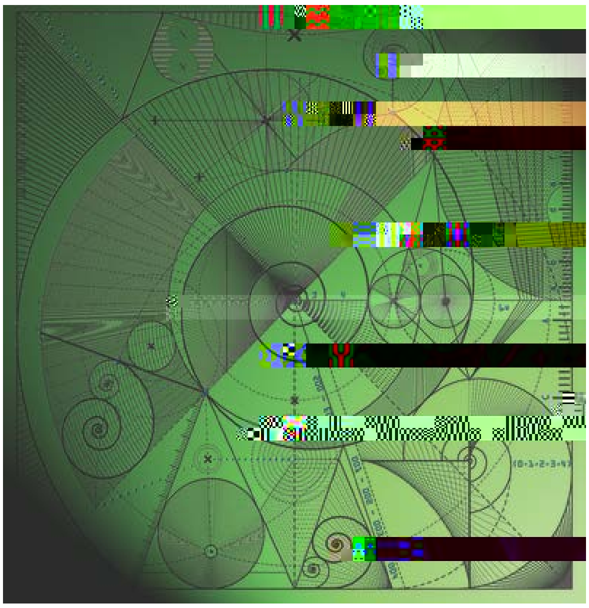
Numerically stable computation of embedding formulae for scattering by polygons

by

A. Gibbs, and A. Moiola



Numerically stable computation of embedding formulae for scattering by polygons

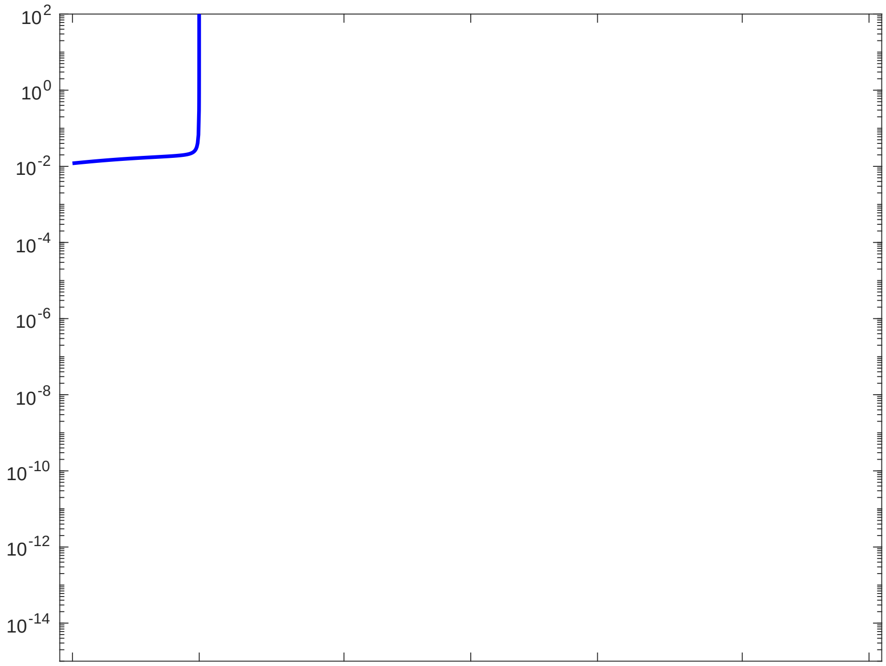
A. Gibbs · S. Langdon · A. Moiola

Abstract For prob e s of t e ar on c scatter ng by po ygona obstac es e bedd ng for u ae prov de a usefu eans of co put ng t e far e d coef c ent nduced by any nc dent p ane wave g ven t e far e d coef c ent of a re at ve y s a set of canon ca prob e s– e nu ber of suc prob e s to be so ved depends on y on t e geo etry of t e scatterer– st t e for u ae t e se ves are exact n t eory any p e entat on w n er t nu er ca error fro t e et od used to so ve t e canon ca prob e s– s error can ead to nu er ca nstab t es–Here we present an effect ve app

nce $|\text{s n}(px/\ )|$ s sy etr c about t e po nts $x \Theta$ t fo ows t at $|\text{s n}(p(\theta+\theta)/\ )| = |\text{s n}(p(\theta-\theta)/\ )|$

ence

$$\text{s n}\ \frac{p}{\ }(\theta+\theta\ )\ \leq$$


10^2
10^0
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14

. e now ave

$$\big|D(\theta,\alpha) - \mathscr{E}_{\mathscr{P}} D(\theta,\alpha\ \theta\ ,N_T)\big| =$$
$$\left|\frac{\theta-\theta}{\Lambda(\theta,\alpha)}\right|\left|\sum_{m=}^{M}\sum_{n=}^{N_T}\frac{(\theta-\theta\ )^{n-}}{n}\left(B_m(\alpha)\frac{\partial^n\hat{D}}{\partial\theta^n}(\theta\ ,\alpha_m) - b_m(\alpha)\frac{\partial^n}{\partial\theta^n}\mathscr{P}\hat{D}(\theta,\alpha_m)\right)+\mathscr{R}_{\mathscr{P}}D(\theta,\alpha\ \theta\ ,N_T)\right|,$$

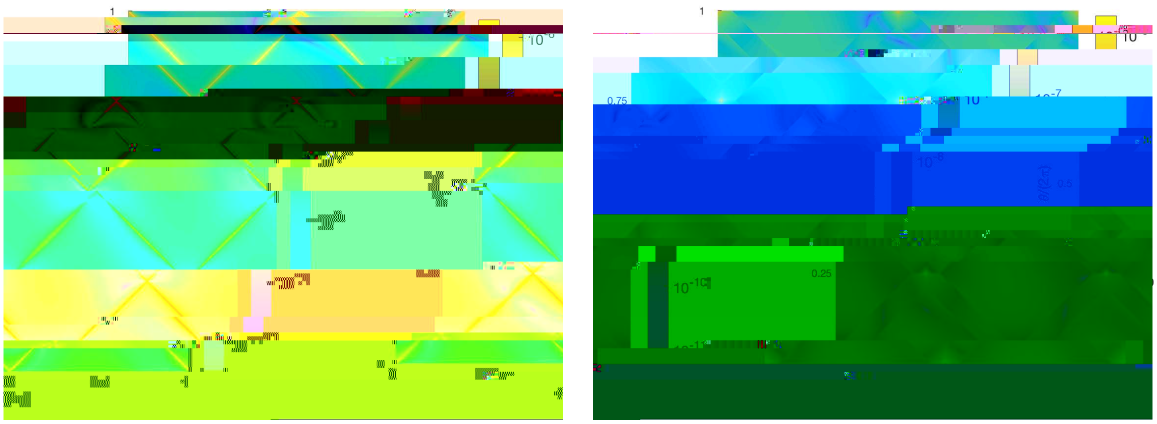
w ere

$$\mathscr{R}_{\mathscr{P}}D(\theta,\alpha\ \theta\ ,N_T) = \sum_{m=}^{M} B_m(\alpha)\sum_{n=N_T+}^{\infty}\frac{(\theta-\theta\ )^{n-}}{n}\frac{\partial^n\hat{D}}{\partial\theta^n}(\theta\ ,\alpha_m).$$

It fo ows ed ate y fro Le a ? t at

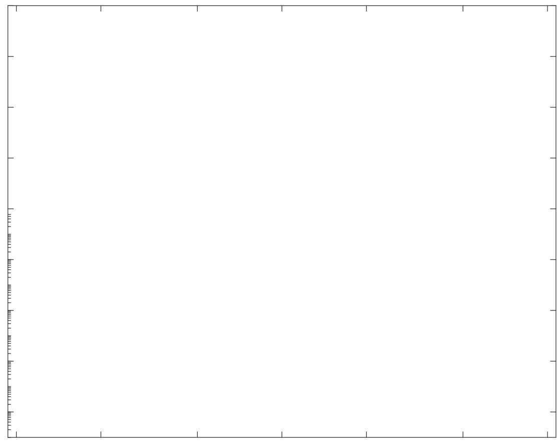
$$\left|\frac{\theta-\theta}{\Lambda(\theta,\alpha)}\right| \leq \frac{}{p\ |\theta_{\,} - \theta\ |},\quad \text{for } \theta_{\,} \in \Theta_{\,},$$

w c co pares favourab y to ? w en θ s c ose to $\theta\ \in \Theta_\alpha$ prov ded t at θ s suf c ent y far fro any $\theta_{\,} \in \Theta_{\,}$ w c s enforced by t e second cond t on n –



a b

F g– – ,· wo p ots of re at ve error for d fferent p e entat on of e bedd ng for u ae for t e prob e of scatter ng by a square of s de engt wavenu ber $k =$ –P ot a dep cts accuracy for a na ve p e entat on



5 More general incident waves

e now de onstrate ow e bedd ng for u ae ay be used to approx ate t e far e d coef c ent of a far broader c ass of nc dent waves t an ust p ane waves by eans of a genera for u a and so e nu er ca ex a p es–In part cu ar we w be nterested n nc dent waves of t e fo ow ng for see e–g–(De n t on –) w c can be t oug t as cont nuous near co b nat ons of p ane waves–

Definition 5.1 (Herglotz wave functions) G ven $g_{\mathrm{Herg}} \in L^2(\ ,\ \pi)$ t e funct on

$$u^i_{\mathrm{Herg}}(\mathbf{x}\ g_{\mathrm{Herg}}) = \int^{\pi} g_{\mathrm{Herg}}(\alpha) \mathrm{e}^{k\mathbf{x}\cdot\mathbf{d}_\alpha}\,\mathrm{d}\alpha, \quad \text{for } \mathbf{x} \in \mathbb{R}^2,$$

w ere $\mathbf{d}_\alpha = -(\cos\alpha, \mathrm{s\,n}\,\alpha)$ s ca ed a *Herglotz wave function* or *Herglotz incident field* w t Herg otz erne $g_{\mathrm{Herg}} \in L^2(\ ,\ \pi)$–

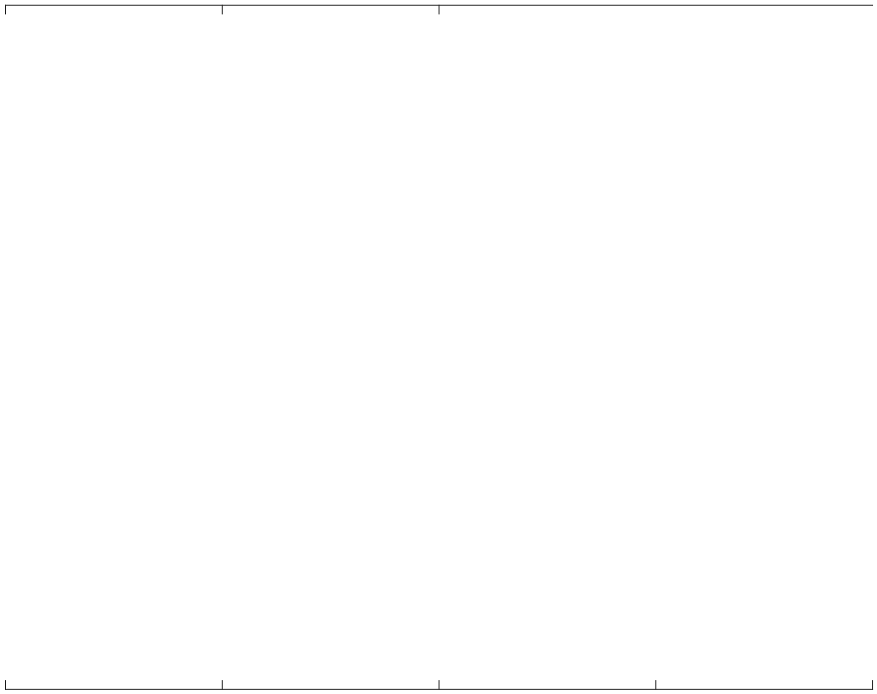
A second concept w c we w nd usefu s t e *far-field map* $\mathscr{F}_\infty$– e rst extend t e scatter ng boundary va ue prob e – – to genera nc dent e ds– f $u^i \in C^\infty(\mathbb{R}^2)$ s an ent re so ut on of t e He o tz equat on we denote by $u = u^i + u^s$ t e so ut on of t e He o tz equat on – n t e co p e ent of t e scatterer w t $u =$ on $\partial\Omega$ and suc t at u^s sat s es t e o erfe d rad at on cond t on– en we wr te $\mathscr{F}_\infty u^i \in C^\infty(\ ,\ \pi)$ for t e far e d coef c ent of t e scattered e d u^s–For exa p e n ter s of fa ar p ane wave notat on – and – we ave $\mathscr{F}_\infty u^i_\alpha = D(\cdot, \alpha)$–

By De n t on – t e far e d coef c ent of a Herg otz wave funct on p ng ng on Ω can be co puted by ntegrat ng t e p ane wave far e d coef c ent aga nst t e Herg otz erne so t can be approx ated us ng t e co b ned e bedd ng approx at on of De n t on – .

$$\left[\mathscr{F}_\infty u^i_{\mathrm{Herg}}(\cdot\ g_{\mathrm{Herg}})\right](\theta) = \int^{\pi} g_{\mathrm{Herg}}(\alpha) D(\theta, \alpha)\,\mathrm{d}\alpha \approx \int^{\pi} g_{\mathrm{Herg}}(\alpha) \mathscr{E}^{\circledast}_{\mathscr{P}} D(\theta, \alpha\ N_T)\,\mathrm{d}\alpha. \qquad –$$

In practg

. st t e so ver s DOFs per s de of t e scatterer are t e sa e as for t e s a est errors observed n F gure – f



w c can be obta ned by so v ng on y M prob e s of p ane wave nc dence–F gure – s ows t e output of t e co b nat on of atro w t our e bedd ng so ver and t e MP pac so ver for t e prob e of p ane wave scatter ng w t t ree regu ar po ygons– s requ red ? and so ves on t e tr ang e square and pentagon respect ve y a tota of so ves for a s ng e scatterer eac a nu ber ndependent of k–For wavenu ber $k =$ us ng MP pac w t out so v ng v a t e e bedd ng for u ae resu ts n a tota of ? so ves on eac scatterer ence a tota of so ves– o even at a re at ve y ow wavenu ber t e e bedd ng for u ae can reduce t e nu ber of so ves requ red by atro as $\hat{N}$ grows w t $O(k)$ t e nu ber of so ves requ red by e bedd ng for u ae s ndependent of k– ere s one dden cost w t e bedd ng for u ae na e y t at t e nu ber of ter s requ red n t e ay or ser es – ? and – n order to a nta n accuracy w need to grow w t k we eave deta ed cons derat on of t s to future wor –

F g– – ;. ea part of tota e d for a con gurat on of u t p e po ygons w t nc dent e d $u^i_{\pi/}$? so ved us ng atro coup ed w t MP pac and t e co b ned e bedd ng approx at on of De n t on – as t e so ver used for t e e bedd ng p e entat on w c n turn s used as t e so ver for atro – e representat on s on y va d outs de of t e un on of ba s conta n ng eac obst

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– B ggs N– – –E bedd ng for u ae for scatter ng