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Rigidity and flatness of the image of certain classes of mappings having tangential Laplacian

by

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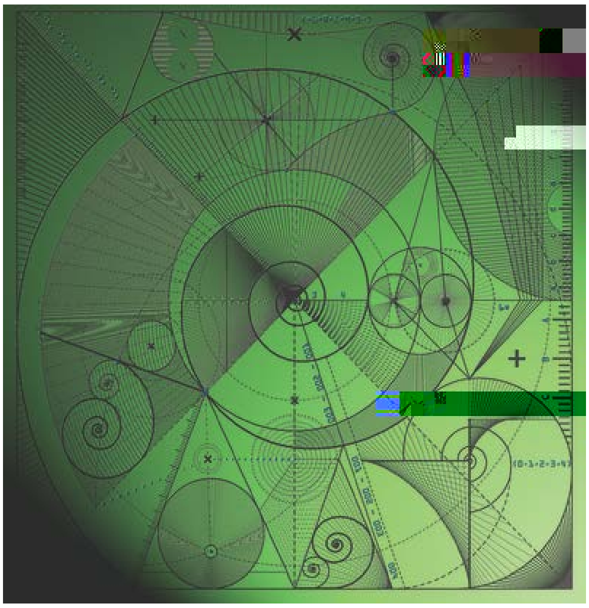

discontinuous on $\mathbb{R}^{N\times n}$; therefore, the PDE system (1.1) has discontinuous coefficients. The geometric meaning of (1.1) is that the Laplacian vector field Δu is tangential to the image $u(\Omega)$ and hence (1.1) is equivalent to the next statement: there exists a vector field

$$A : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^n$$

such that

$$\Delta u = \mathrm{D}u\, A \quad \text{in } \Omega.$$

As we show later, the vector field is generally discontinuous (Lemma 7).

Our interest in (1.1) stems from the fact that it is a constituent component of the p-Laplace PDE system for all $p \in [2,\infty]$. Further, contrary perhaps to appearances, (1.1) is in itself a variational PDE system but in a non-obvious way. Deferring temporarily the specifics of how exactly (1.1) arises and what is the variational principle associated with it, let us recall that, for $p \in [2,\infty)$, the celebrated p-Laplacian is the divergence system

$$\Delta_p u := \mathrm{Div}\big(|\mathrm{D}u|^{p-2}\mathrm{D}u\big) = 0 \quad \text{in } \Omega \tag{1.3}$$

and comprises the Euler-Lagrange equation which describes extrema of the model p-Dirichlet integral functional

$$E_p(u) := \int_\Omega |\mathrm{D}u|^p, \quad u \in W^{1,p}(\Omega;\mathbb{R}^N), \tag{1.4}$$

in conventional vectorial Calculus of Variations. Above and subsequently, for any $X \in \mathbb{R}^{N\times n}$, the notation $|X|$ symbolises its Euclidean (Frobenious) norm:

$$|X| = \left(\sum_{\alpha=1}^{N}\sum_{i=1}^{n}(X_{\alpha i})^2\right)^{1/2}.$$

The pair (1.3)-(1.4) is of paramount important in applications and has been studied exhaustively. The extremal case of $p \to \infty$ in (1.3)-(1.4) is much more modern and intriguing. It turns out that one then obtains the following nondivergence PDE system

$$\Delta_\infty u := \Big(\mathrm{D}u \otimes \mathrm{D}u + | \tag{1.5}$$

(1.4)

The effect of (1.1) to the flatness of the image can be easily seen in the case of $n = 1 \leq N$ as follows: since

$$[\![u']\!]^{\perp} u'' = 0 \quad \text{in } \Omega \subseteq \mathbb{R}$$

and in one dimension we have

$$[\![u']\!]^{\perp} = \begin{cases} I - \dfrac{u' \otimes u'}{|u'|^2}, & \text{on } \{u' \neq 0\}, \\ I, & \text{on } \{u' = 0\}, \end{cases}$$

we therefore infer that $u'' = f u'$ on the open set $\{u' \neq 0\} \subseteq \Omega \subseteq \mathbb{R}$ for some function f, readily yielding after an integration that $u(\Omega)$ is necessarily contained in a piecewise polygonal line of $\mathbb{R}^N$. As a generalisation of this fact, our first main result herein is the following:

Theorem 2 (Rigidity and flatness of rank-one maps with tangential Laplacian)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $n, N \geq 1$. Let $u \in C^2(\Omega; \mathbb{R}^N)$ be a solution to the nonlinear system (1.1) in Ω, satisfying that the rank of its gradient matrix is at most one:*

$$\mathrm{rk}(\mathrm{D}u) \leq 1 \quad \text{in } \Omega.$$

Then, its image $u(\Omega)$ is contained in a polygonal line in, s5o4g5007(the91449pist 5.95 6 0 G /62(ca50 9.962t0 9

As a consequence of Theorem 2, we obtain the next result regarding the rigidity of p-Harmonic maps for $p \in [2, \infty)$ which complements one of the results in the paper [K3] wherein the case $p = \infty$ was considered.

Corollary 4 (Rigidity of p-Harmonic maps, cf. [K3]). Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $n, N \geq 1$. Let $u \in C^2(\Omega; \mathbb{R}^N)$ be a p-Harmonic map in Ω for some $p \in [2, \infty)$, that is u solves (1.3). Suppose that the rank of its gradient matrix is at most one:

$$\mathrm{rk}(\mathrm{D}u) \leq 1 \quad \text{in } \Omega.$$

Then, the same result as in Theorem 2 is true.

In addition, there exists a partition of Ω to at most countably many Borel sets, Note that our result is trivial in the case that $N = n = 2$ since the codimension $N - n$ vanishes. Further, one might also restrict their attention to domains of rectangular shape, since any map with separated form can be automatically extended to the smallest rectangle containing the domain.

Also, herein we consider only the illustrative case of $n = 2 < N$ and do not discuss more general situations, since numerical evidence obtained in [KP] suggests that Theorem 5 does not hold in general for solutions in non-separated form. However, as a consequence of Theorem 5 we have the next particular result:

Corollary 6 (Rigidity and flatness of p-Harmonic maps in separated form). Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $n, N \geq 1$. Suppose that $u \in C^2(\Omega; \mathbb{R}^N)$ is a p-Harmonic map in Ω for some $p \in [2, \infty]$, that is u solves (1.3) if $p < \infty$ and (1.5) if $p = \infty$.

Then, if u has the separated Td69626 Tf 41o(ate)51(d)-426 Tf 326 Tf 9.963 0 Td [(.)]TJ -340.56rd [uIS

The claim being obvious for $\{\mathrm{rk}(\mathrm{D}u) = 0\} = \{\mathrm{D}u = 0\}$, it suffices to consider only the set $\{\mathrm{rk}(\mathrm{D}u) = 1\}$ in order to conclude. Thereon we have that $\mathrm{D}u$ can be written as

$$\mathrm{D}u = \quad a, \quad \text{in } \{\mathrm{rk}(\mathrm{D}u) = 1\},$$

for some non-vanishing vector fields and

We may now prove our first main result.

Proof of Theorem 2. Suppose that $u : \mathbb{R}^n \supseteq \Omega \to \mathbb{R}^N$ is a solution to the nonlinear system (1.1) in $C^2(\Omega;\mathbb{R}^N)$ which in addition satisfies that $\mathrm{rk}(\mathrm{D}u) \leq 1$ in Ω. Since $\{\mathrm{D}u = 0\}$ is closed, necessarily its complement in Ω which is $\{\mathrm{rk}(\mathrm{D}u) = 1\}$ is open.

By invoking Theorem 8, we have that there exists a partition of the open subset $\{\mathrm{rk}(\mathrm{D}u) = 1\}$ to countably many Borel sets $(B_i)_1^\infty$ with respective functions $(f_i)_1^\infty$ and curves $(\gamma_i)_1^\infty$ as in the statement such that (2.1)-(2.2) hold true and in addition

$$\mathrm{D}f_i \neq 0 \quad \text{on } B_i,\ i \in \mathbb{N}.$$

Consequently, on each B_i we have

$$[\![\mathrm{D}u]\!]^\perp = [\![(\gamma_i' \circ f_i) \otimes \mathrm{D}f_i]\!]^\perp = \mathrm{I} - \frac{(\gamma_i' \circ f_i) \otimes (\gamma_i' \circ f_i)}{|\gamma_i' \circ f_i|^2},$$
$$\Delta u = (\gamma_i'' \circ f_i)|\mathrm{D}f_i|^2 + (\gamma_i' \circ f_i)\Delta f_i.$$

Hence, (1.1) becomes

$$\left(\mathrm{I} - \frac{(\gamma_i' \circ f_i) \otimes (\gamma_i' \circ f_i)}{|\gamma_i' \circ f_i|^2}\right)\left((\gamma_i'' \circ f_i)|\mathrm{D}f_i|^2 + (\gamma_i' \circ f_i)\Delta f_i\right) = 0,$$

on B_i. Since $|\gamma_i'|^2 \equiv 1$ on $f_i(B_i)$, we have that γ_i'' is orthogonal to γ_i' thereon and therefore the above equation reduces to

$$(\gamma_i'' \circ f_i)|\mathrm{D}f_i|^2 = 0 \quad \text{on } B_i,\ i \in \mathbb{N}.$$

Therefore, γ_i is affine on the interval $f_i(B_i) \subseteq \mathbb{R}$ and as a result $u(B_i) = \gamma_i(f_i(B_i))$ is contained in an affine line of $\mathbb{R}^N$, for each $i \in \mathbb{N}$. On the other hand, since

$$u(\Omega) = u\big(\{\mathrm{D}u = 0\}\big) \cup \bigcup_{i \in \mathbb{N}} u(B_i)$$

and u is constant on each connected component of the interior of $\{\mathrm{D}u = 0\}$, the conclusion

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on B_i. Since γ_i'' is orthogonal to γ_i' and also γ_i' has unit length, the above reduces to

$$(\gamma_i' \circ f_i)\Big(Df_i \otimes Df_i : D^2 f_i + \frac{|Df_i|^2}{p-2}\Delta f_i\Big) + \frac{1}{p-2}(\gamma_i'' \circ f_i)|Df_i|^4 = 0,$$

on B_i. Again by orthogonality, the above is equivalent to the pair of independent systems

$$(\gamma_i' \circ f_i)\Big(Df_i \otimes Df_i : D^2 f_i + \frac{|Df_i|^2}{p-2}\Delta f_i\Big) = 0\,, \quad (\gamma_i'' \circ f_i)|Df_i|^4 = 0,$$

on B_i. Since $|\gamma_i'| \equiv 1$ of $f_i(B_i)$, it follows that $\Delta_p f_i = 0$ on B_i and since $(B_i)_1^\infty$ is a partition of Ω of the form described in the statement, the result ensues by invoking Theorem 2.

We may now prove our second main result.

Proof of Theorem 5. We begin by temporarily assuming that Ω is a rectangle of the form

$$Q = (a,b) \times (c,d) \subseteq \mathbb{R}^2$$

and we fix $(x_0,y_0) \in Q = \Omega$. Let us also assume that the rank of the gradient is full throughout:

$$\mathrm{rk}(\mathrm{D}u) \equiv 2 \quad \text{in } Q.$$

Later we will remove both these extra assumptions. Let $f : (a,b) \longrightarrow \mathbb{R}^N$ and $g : (c,d) \longrightarrow \mathbb{R}^N$ be such that $u(x,y) = f(x) + g(y)$. Then, the gradient matrix then has the form

$$\mathrm{D}u(x,y) = \big(f'(x), g'(y)\big) \in \mathbb{R}^{N \times 2}.$$

By assumption, we have that $[\![\mathrm{D}u]\!]^\perp \Delta u = 0$ in Ω. By Lemma 7, there exists a vector field $\mathrm{A} : \mathbb{R}^2 \supseteq \Omega \longrightarrow \mathbb{R}^2$ such that $\Delta u = \mathrm{D}u\,\mathrm{A}$. If A has components (a,b), this means that the functions f and g satisfy

$$f''(x) + g''(y) = a(x,y)f'(x) + b(x,y)g'(y). \tag{2.3}$$

Although we will not utilise this in the sequel, it is instructive and quite possible to express the coefficients a,b in terms of f,g,f',g' along the lines of Lemma 7 but more concretely, as follows. By applying the $\mathbb{R}^{N \times N}$ matrix

$$|g'(y)|^2 \mathrm{I} - g'(y) \otimes g'(y)$$

to (2.3), the summand $b(x,y)g'(y)$ on the right hand side is annihilated and we obtain

$$\Big[|g'(y)|^2 \mathrm{I} - g'(y) \otimes g'(y)\Big]\big(f''(x) + g''(y)\big) = a(x,y)\Big[|g'(y)|^2 \mathrm{I} - g'(y) \otimes g'(y)\Big] f'(x).$$

Hence, on $\{\mathrm{rk}(\mathrm{D}u) = 2\}$ we have

$$a(x,y) = \left(\frac{|g'(y)|^2 \mathrm{I} - g'(y) \otimes g'(y)}{\big|\big[|g'(y)|^2 \mathrm{I} - g'(y) \otimes g'(y)\big] f'(x)\big|}\right) \cdot \big(f''(x) + g''(y)\big).$$

Arguing symmetrically, we may obtain

$$b(x,y) = \left(\frac{|f'(x)|^2 \mathrm{I} - f'(x) \otimes f'(x)}{\big|\big[|f'(x)|^2 \mathrm{I} - f'(x) \otimes f'(x)\big] g'(y)\big|}\right) \cdot \big(f''(x) + g''(y)\big).$$

Similar expression can be obtained on $\{\mathrm{rk}(\mathrm{D}u) \le 1\}$ as well.

Let us consider (2.3) as a first order ODE with unknown function f' in the variable x. By integrating the equation in x, we view its first integral as an first order ODE with solution the function g' in the variable y, which can be integrated again. Therefore, by sparing the reader the tedious computations, we arrive at the following integral identity

$$
\left\{
\begin{aligned}
&g'(y)\, e^{\int_{y_0}^{y} \frac{B(x;s)}{A(x;s)}\,\mathrm{d}s} + f'(x) \int_{y_0}^{y} \frac{e^{\int_{x_0}^{x} a(s;t)\,\mathrm{d}s}}{A(x;t)}\, e^{\int_{y_0}^{t} \frac{B(x;\tau)}{A(x;\tau)}\,\mathrm{d}\tau}\,\mathrm{d}t \\
&\qquad = g'(y_0) + f'(x_0) \int_{y_0}^{y} \frac{1}{A(x;t)}\, e^{\int_{y_0}^{t} \frac{B(x;\tau)}{A(x;\tau)}\,\mathrm{d}\tau}\,\mathrm{d}t;
\end{aligned}
\right.
\tag{2.4}
$$

where

$$
\left\{
\begin{aligned}
A(x;y) &:= \int_{x_0}^{x} e^{\int_{x_0}^{s} a(\tau;y)\,\mathrm{d}\tau}\,\mathrm{d}s\,;\\
B(x;y) &:= \int_{x_0}^{x} b(s;y)\, e^{\int_{x_0}^{s} a(\tau;y)\,\mathrm{d}\tau}\,\mathrm{d}s.
\end{aligned}
\right.
$$

Note that

$$A(x;y) \gtrless 0 \quad \text{if and only if} \quad x \gtrless x_0 \quad \forall\, x;y$$

For brevity, we set

$$\begin{cases} H(x,y) := e^{\int_{x_0}^{x} \frac{G(s,y)}{F(s,y)}\,ds}, \\ I(x,y) := \displaystyle\int_{x_0}^{x} \frac{e^{\int_{y_0}^{y} b(t,s)\,ds}}{F(t,y)}\, e^{-\int_{x_0}^{t} \frac{G(\sigma,y)}{F(\sigma,y)}\,d\sigma}\, dt, \\ J(x,y) := \displaystyle\int_{x_0}^{x} \frac{1}{F(t,y)}\, e^{-\int_{x_0}^{t} \frac{G(\sigma,y)}{F(\sigma,y)}\,d\sigma}\, dt. \end{cases}$$

Then (2.6) can be re-written in the simpler form

$$f'(x) = H^{-1}\Big(g'(y_0)J + f'(x_0) - g'(y)I\Big).$$

Substituting the above into (2.5), after some elementary calculations we obtain the equation

$$\big(C - IH^{-1}\big)g'(y) = \big(E - DH^{-1}\big)f'(x_0) + \big(1 + JH^{-1}\big)g'(y_0), \tag{2.7}$$

for all $(x,y) \in Q$. Note that

$$I < 0 \quad \text{if} \quad (x - x_0)(y - y_0) < 0, \quad (x,y) \in Q.$$

Moreover, we have that $C > 0$ and also $H > 0$, for all $(x,y) \in Q$. Hence, if $y < y_0$, we may choose $x > x_0$ and if $y > y_0$, we may choose $x < x_0$. In either case, we can arrange

$$C - IH^{-1} > 0$$

for all $y \neq y_0$ such that $(x - x_0)(y - y_0) < 0$. Therefore, from (2.7) we deduce that

$$g'(y) = \left(\frac{E - DH^{-1}}{C - IH^{-1}}\right) f'(x_0) + \left(\frac{1 + JH^{-1}}{C - IH^{-1}}\right) g'(y_0),$$

which yields

$$g'(y) \in \mathrm{span}[\{f'(x_0), g'(y_0)\}].$$

By an integration, the above inclusion implies that the curve g is valued in an affine plane of the form

$$g(y) \in g(y_0) + \mathrm{span}[\{f'(x_0), g'(y_0)\}].$$

By arguing similarly for f, we also infer that

$$f'(x) \in \mathrm{span}[\{f'(x_0), g'(y_0)\}]$$

and hence

$$f(x) \in f(x_0) + \mathrm{span}[\{f'(x_0), g'(y_0)\}].$$

Conclusively, by putting the above together we have obtained

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g((x0) + span[g is)R2vxM

open subset of Ω given by $\{\mathrm{rk}(Du) = 2\}$. We cover each component with countably many (overlapping) rectangles $\{Q_i : i \in \mathbb{N}\}$ where

$$Q_i = (a_i, b_i) \times (c_i, d_i), \quad i \in \mathbb{N};$$

and respective points inside the rectangles

$$\{(x_{0i}, y_{0i}) \in Q_i \, : \, i \in \mathbb{N}\}.$$

On each rectangle, by the previous reasoning the solution will be contained in an affine plane $\Pi_i \subseteq \mathbb{R}^N$. By choosing (x_{0i}, y_{0i}) on the overlaps of each Q_i with all its neighbouring rectangles, connectedness of the component allows us to conclude that all the planes coincide.

It remains to consider the complement $\Omega \setminus \{\mathrm{rk}(Du) = 2\}$, which we decompose to the sets

$$\mathrm{int}\,\{\mathrm{rk}(Du) \leq 1\} \bigcup \partial\{\mathrm{rk}(Du) \leq 1\};$$

where "int" denotes topological interior. If the interior is non-empty, on each connected component of it we apply Theorem 2 to infer that $u(\mathrm{int}\{\mathrm{rk}(Du) \leq 1\})$ is contained in a polygonal curve of $\mathbb{R}^N$, given by an at most countable union of affine straight line segments. Finally, by continuity of u, we have that $u(\partial\{\mathrm{rk}(Du) \leq 1\})$ is also contained in the union of planes. The result ensues.

We conclude by establishing the remaining corollary.

Proof of Corollary 6. Suppose that u is a C^2 p-Harmonic mapping as in the hypotheses of the corollary for some $p \in [2, \infty]$, namely that it has additively separated form and solves either (1.3) if $p < \infty$ or (1.5) if $p = \infty$. By (1.9)-(1.10), we deduce that u solves the PDE system

$$|Du|^2 [\![Du]\!]^{\perp} \Delta u = 0 \quad \text{in } \Omega.$$

On the open set $\{Du \neq 0\}$, we readily have that u solve the system (1.1). On the other hand, we decompose its complement to

$$\mathrm{int}\,\{Du = 0\} \bigcup \partial\{Du = 0\};$$

If $\mathrm{int}\{Du = 0\}$ is non-empty, then u is constant on each connected component of it. Finally, again by the regularity of u, we have that $u(\partial\{Du = 0\})$ is also contained in the previous union of planes. The claim has been established.

Acknowledgement.