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# Existence of 1D vectorial Absolute

# EXISTENCE OF 1D VECTORIAL ABSOLUTE MINIMISERS IN $L^1$ UNDER MINIMAL ASSUMPTIONS

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**Abstract.** We prove the existence of vectorial Absolute Minimisers in the sense of Aronsson to the supremal functional $E_1(u; \quad ^0) = kL( ; u; Du$

case of maps $u : \quad \mathbb{R}^n \to \quad \mathbb{R}^N$) together with its associated system of equations begun in the early 2010s by the second author in a series of papers, see [K1]-[K6], [K8]-[K12

Theorem 1 generalises two respective results in the both the papers [BJW1] and [K9]. On the one hand, in [BJW1] Theorem 1 was established under the extra assumption $C_2 = C_3 = 0$ which forces $L\ (x;\ ;\ 0) = 0$, for all $(x;\ )\ 2\ R\quad R^N$. Unfortunately this requirement is incompatible with important applications of (1.1) to problems of $L^1$ -modelling of variational Data Assimilation ( 4DVar ) arising in the Earth Sciences and especially in Meteorology (see [B, BS, K9]). An explicit model of $L$ is given by

$$L\ (x;\ ;P\ ) := \quad k(x) \quad K(\ )^{\ 2} + \quad P \quad V(x;\ )^{\ 2}; \tag{1.7}$$

and describes the \error" in the following sense: consider the problem of nding spaces $W_b^{1;qm}(\ ;R^N)$ such that, for any $s$ $1$, $u^m$ $*$ $u^{\ 1}$ weakly as $m\ !\ 1$ in $W^{1;s}(\ ;R^N)$ along a subsequence. Moreover,

$$E_1\ (u^1\ ;\ ) = \ C_1\ = \lim_{m\,!1} C_m : \tag{2.2}$$

By approximate minimiser we mean that $u^m$ satis es

$$E_m(u^m;\ ) \quad C_m \ < \ 2^{\ m^2} : \tag{2.3}$$

Finally, for any $A$ measurable of positive measure the following lower semi-continuity inequality holds

$$E_1\ (u^1\ ;A) \quad \liminf_{m\,!1} \tag{2.4}$$

as $m \to \infty$ along a subsequence. Now, recalling that $^{m;} = u^m$ at the endpoints $\{0;1\}$, and since $u^m$ is an approximate minimiser of (2.1) over $W_b^{1;m}(\ ;\mathbb{R}^N)$ for each $m \in \mathbb{N}$, by utilising minimality, the additivity of the integral and Hölder inequality, we get

$$E_m\ u^m;(0;1) \quad E_m \quad {}^{m;}\ ;(0;1) \ + 2 \ {}^{m^2}$$

and hence

$$E_m\ u^m;(0;1)^{\frac{1}{m}} \quad E_m \quad {}^{m;}\ ;(0;1)^{\frac{1}{m}} + 2\ {}^{m}$$
$$E_\infty \quad {}^{m;}\ ;(0;1) \ + 2\ {}^{m}: \tag{2.6}$$

On the other hand, we have

$$E_\infty \quad {}^{m;}\ ;(0;1) \ = \max\Big\{ E_\infty \quad {}^{m;}\ ;(0;\ )\ ;\ E_\infty \quad {}^{m;}\ ;(\ ;1 \quad )\ ;\ E_\infty \quad {}^{m;}\ ;(1 \quad ;1) \Big\}$$

and since $^{m;} = {}^{\infty}$ on $(\ ;1 \quad )$, we have

$$E_\infty \quad {}^{m;}\ ;(0;1) \quad \max\ E_\infty \quad {}^{m;}\ ;(0;\ )\ ;E_\infty \quad {}^{\infty}\ ;(0;1)\ ;\ E_\infty \quad {}^{m;}\ ;(1 \quad ;1) \ : \tag{2.7}$$

Combining (2.5)-(2.7) and (2.4), we get

$$E_\infty\ u^\infty\ ;(0;1) \quad \liminf_{m\to\infty} \quad \max\Big\{ E_\infty \quad {}^{m;}\ ;(0;\ )\ ;E_\infty \quad {}^{\infty}\ ;(0;1)\ ;\ E_\infty \quad {}^{m;}\ ;(1 \quad ;1)\Big\}$$
$$\max\Big\{ E_\infty \quad {}^{\infty}\ ;(0;1)\ ;E_\infty \quad {}^{\infty;}\ ;(0;\ )\ ;\ E_\infty \quad {}^{\infty;}\ ;(1 \quad ;1)\Big\} \ : \tag{2.8}$$

Let us now denote the difference quotient of a function $v:\mathbb{R}\to\mathbb{R}^N$ as $D^{1;t}v(x) := \frac{1}{t}[v(x+t) \quad v(x)]$. Then, we may write

$$D\ {}^{\infty;}\ (x) = D^{1;} \quad {}^{\infty}(0), \qquad x \in (0;\ );$$
$$D\ {}^{\infty;}\ (x) = D^{1;} \quad {}^{\infty}(1), \qquad x \in (1 \quad ;1),$$

Note now that

$$\begin{cases} E_\infty \quad {}^{\infty;}\ ;(0;\ ) \ = \max\limits_{0\ x} \quad {}^{\infty}(1), \quad x \in (1 \quad ; \quad {}^{x}_{1} \\ \end{cases} \tag{2.9}$$

The rest of the proof is devoted to establishing (2.10). Let us begin by recording for later use that

$$
(2.11)\qquad
\begin{cases}
\max\limits_{0\leq x\leq \delta} \left|\psi^{1,\delta}(x) - \psi^{1}(0)\right| \longrightarrow 0, & \text{as } \delta \to 0,\\[1ex]
\max\limits_{1-\delta\leq x\leq 1} \left|\psi^{1,\delta}(x) - \psi^{1}(1)\right| \longrightarrow 0, & \text{as } \delta \to 0.
\end{cases}
$$

Fix a generic $u \in W^{1,1}(\Omega;\mathbb{R}^N)$, $x \in [0,1]$ and $0<\varepsilon<1/3$ and define

$$A_\varepsilon(x) := [x-\varepsilon, x+\varepsilon] \cap [0,1].$$

We claim that there exist an increasing modulus of continuity $\omega \in C(0,\infty)$ with $\omega(0^+) = 0$ such that

$$
(2.12)\qquad \left| E_1\big(u; A_\varepsilon(x)\big) - \operatorname*{ess\,sup}_{y\in A_\varepsilon(x)} L\big(x, u(x), Du(y)\big)\right| \leq \omega(\varepsilon).
$$

Indeed for a.e. $y \in A_\varepsilon(x)$ we have $|x$

## References


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