

Department of Mathematics and Statistics

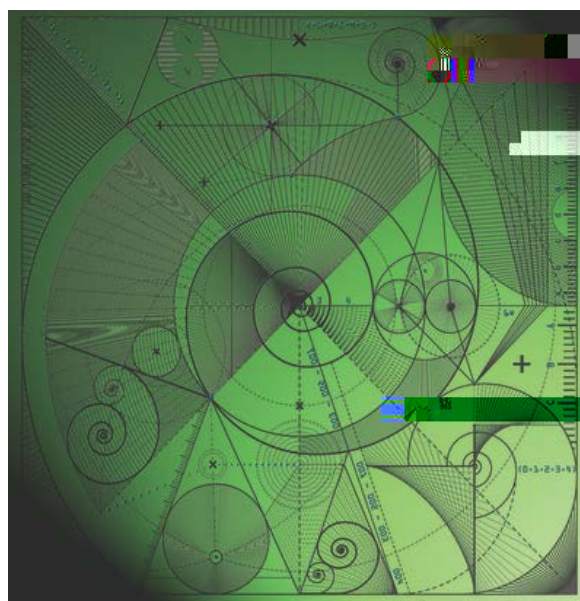
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Solutions of vectorial Hamilton-Jacobi equations are rankone Absolute Minimisers in L^∞

by

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SOLUTIONS OF VECTORIAL HAMILTON-JACOBI EQUATIONS
ARE RANK-ONE ABSOLUTE MINIMISERS IN L^1

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Abstract. Given the supremal functional $E_1(u;$

$\mathbb{R}^n$, the respective PDE is the 1-Laplace equation

$$(1.3) \quad \Delta_1 u := \operatorname{div} u - Du : D^2 u = \sum_{i,j=1}^n D_i u D_j u D_{ij}^2 u = 0:$$

Despite the importance for applications and the deep analytical interest of the area, the vectorial case of $N \geq 2$ remained largely unexplored until the early 2010s. In particular, not even the correct form of the respective PDE systems associated to L^1 variational problem was known. A notable exception is the early vectorial contributions [BJW1, BJW2] wherein (among other deep results) L^1 versions of lower semi-continuity and quasiconvexity were introduced and studied and the existence of Absolute Minimisers was established in some generality with H depending on u itself but for $\min f; N \geq 2$.

different sets of variations. In [K2] we proved the following variational characterisation in the class of classical solutions. $\mathcal{AC}^2$ map $u : \mathbb{R}^n \rightarrow \mathbb{R}^N$ is a solution to

$$(1.5) \quad Du \otimes Du : D^2u = 0$$

if and only if it is a Rank-One Absolute Minimiser on Ω , namely when for all $D \Subset \Omega$, all scalar functions $g \in C_0^1(D)$ vanishing on ∂D and all directions $\xi \in \mathbb{R}^N$, u is a minimiser on D with respect to variations of the form $u + g\xi$ (Figure 1):

$$(1.6) \quad \|Du\|_{L^1(D)} \leq \|Du + Dg\xi\|_{L^1(D)};$$

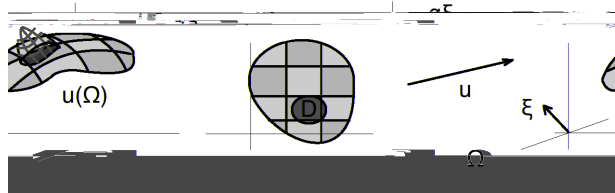


Figure 1.

Further, if $\text{rk}(Du) = \text{const.}$, u is a solution to

$$(1.7) \quad |Du|^2 [Du]^\perp u = 0$$

if and only if $u(\cdot)$ has 1-Minimal Area, namely when for all $D \Subset \Omega$, all scalar functions $h \in C^1(\bar{D})$ (not vanishing on ∂D) and all vector fields $\nu \in C^1(D; \mathbb{R}^N)$ which are normal to $u(\cdot)$, u is a minimiser on D with respect to normal free variations of the form $u + h\nu$ (Figure 2):

$$(1.8) \quad \|Du\|_{L^1(D)} \leq \|Du + D(h\nu)\|_{L^1(D)};$$

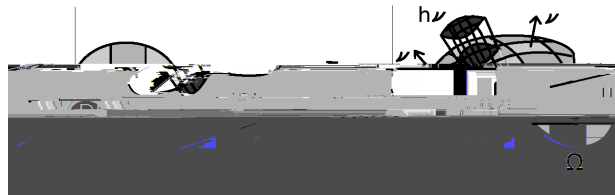


Figure 2.

We called a map 1-Minimal with respect to functional $\|D(u)\|_{L^1(\cdot)}$ when it is a Rank-One Absolute Minimiser on Ω and $u(\cdot)$ has 1-Minimal Area.

Perhaps the greatest difficulty associated to (1.1) and (1.4

In this paper we consider the obvious generalisation of the rank-one minimality notion of (1.6) adapted to the functional (1.1). To this end, we identify a large class of rank-one Absolute Minimisers: for any $c \geq 0$, every solution $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ to the vectorial Hamilton-Jacobi equation

$$(1.9) \quad H(x; Du(x)) = c; \quad x \in \mathbb{R}^n;$$

actually is a rank-one absolute minimiser. Namely, for any $b \geq 0$, any $W_0^{1,1}(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$, we have

$$\operatorname{ess\,sup}_{x \in \mathbb{R}^n} H(x; Du(x)) = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} H(x; Du(x)) + D^-(x) :$$

For the above implication to be true we need the solutions to be in $C^1(\mathbb{R}^n)$ and not just in $W_{loc}^{1,1}(\mathbb{R}^n)$. This is not a technical difficulty: it is well known even in the scalar case that if we allow only for 1 non-differentiability point, strong solutions of the Eikonal equation $|Du| = 1$ are not absolutely minimising for the L^1 norm of the gradient (e.g. the cone function $x \mapsto |x|$). However, due to regularity results which are available in the scalar case, it suffices to assume everywhere differentiability (see [CEG, CC]).

Our only hypothesis imposed on H is that for any $x \in \mathbb{R}^n$ the partial function $H(x; \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ is rank-one level-convex. This means that for any $t \geq 0$, the sublevel sets $\{x \in \mathbb{R}^n : H(x; \cdot) \leq t\}$ are rank-one convex sets in $\mathbb{R}^n$. A set $C \subset \mathbb{R}^n$ is called rank-one convex when for any matrices $A, B \in \mathbb{R}^{n \times n}$ with $\operatorname{rk}(A - B) = 1$, the convex combination $A + (1 - \lambda)B$ is in C for any $0 \leq \lambda \leq 1$. An equivalent way to phrase the rank-one level-convexity of $H(x; \cdot)$ is via the inequality

$$H(x; A + (1 - \lambda)B) \leq \max\{H(x; A), H(x; B)\} \quad x \in \mathbb{R}^n;$$

Theorem 1. Let $\Omega \subset \mathbb{R}^n$ be an open set; $n \geq 2$ and $H : \mathbb{R}^n \times \mathbb{R} \rightarrow [0; 1]$ a continuous function, such that for all $x \in \Omega$, $P \in \mathbb{R}^n$ $H(x; P)$ is rank-one level-convex, that is

$$H(x; \cdot) \text{ is a rank-one convex in } \mathbb{R}^n, \text{ for all } x \in \Omega;$$

Let $u \in C^1(\Omega; \mathbb{R}^N)$ be a solution to the vectorial Hamilton-Jacobi PDE

$$H(x; Du) = c \text{ on } \Omega;$$

for some $c \geq 0$. Then, u is a rank-one Absolute Minimiser of the functional

$$E_1(u; \Omega) = \operatorname{ess\,sup}_{x \in \Omega} H(x; Du(x)); \quad u \in W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^N):$$

In addition, the following marginally stronger result holds true: for any $\Omega \subset \mathbb{R}^n$, any $u \in W_0^{1,1}(\Omega)$ and any $v \in \mathbb{R}^N$, we have

$$E_1(u; \Omega) \leq \inf_{B \subset \Omega} E_1(u + v; B)$$

where $B \subset \Omega$

We illustrate the idea by assuming first in addition that $u \in W^{1,1}_0(\Omega; \mathbb{R}^N) \setminus C^1(\Omega; \mathbb{R}^N)$. In this case, the point x is a critical point of $\Phi(u)$ and we have $D\Phi(u)(x) = 0$. Hence,

$$\begin{aligned} D\Phi(u)(x) &= [\lambda]^p D\Phi(u)(x) + [\lambda]^{2p} D\Phi(u)(x) \\ &= D\Phi(u)(x) + D[\lambda]^{2p}(u)(x) \\ &= 0 \end{aligned}$$

because $[\lambda]^{2p} u \in \mathcal{H}$. Thus,

$$\begin{aligned} E_1(u; \Omega) &= c = H(x; Du(x)) \\ &= H(x; D\Phi(u)(x) + [\lambda]^{2p} Du(x)) \end{aligned}$$

and hence

$$\begin{aligned} E_1(u; \Omega) &= H(x; D\Phi(u)(x) + [\lambda]^{2p} Du(x)) \\ &= H(x; Du(x)) \\ (2.2) \quad &= \operatorname{ess\,sup}_{y \in B(x)} H(y; Du(y)) \\ &= E_1(u; B(x)) \end{aligned}$$

for any $B(x) \subset \Omega$, whence the conclusion ensues.

Now we return to the general case of $u \in W^{1,1}_0(\Omega; \mathbb{R}^N)$. We extend u by 0 on Ω^c and consider the sets

$$(2.3) \quad \begin{aligned} \mathcal{A}_k &:= \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{d_0}{k}; \quad k \geq N; \\ &\quad \emptyset; \quad k = 0; \end{aligned}$$

where $d_0 > 0$ is a constant small enough so that $\mathcal{A}_1 \neq \emptyset$. We set

$$(2.4) \quad V_k := \overline{\mathcal{A}_k \cap \Omega}; \quad k \geq N$$

and consider a partition of unity $(\varphi_k)_{k=1}^{\infty} \subset C_c^1(\Omega)$ over Ω so that

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for $j = 1$, we have

$$\begin{aligned}
 (2.7) \quad k_{C(\bar{V}_1)} &= \sup_{V_1} \left(\sum_{k=1}^{\infty} \frac{1}{k} \int_{V_1} \chi_k \, dx \right) \\
 &= \sup_{V_1} \left(\sum_{k=1}^{\infty} \frac{1}{k} \int_{V_1} \chi_k \, dx \right) \\
 &= \sup_{V_1} \left(\sum_{k=1}^{\infty} \frac{1}{k} \int_{V_1} \chi_k \, dx \right) \\
 &= \sup_{V_1} \left(\sum_{k=1}^{\infty} \frac{1}{k} \int_{V_1} \chi_k \, dx \right)
 \end{aligned}$$

whilst, for $l = 1$ we similarly have

$$(2.8) \quad k_{C(\bar{V}_1)} = 2 \max_{k=1,2} \int_{V_1} \chi_k \, dx$$

By the standard properties of mollifiers, we have that the function

$$(2.9) \quad \eta(t) := \sup_{0 < t < d_0} \int_{V_1} \chi_k \, dx$$

is an increasing continuous modulus of continuity with $\eta(0^+) = 0$. By (2.7)-(2.9), we have that

$$(2.10) \quad k_{C(\bar{V}_1)} \leq \frac{3}{2} \eta(t) \quad \text{for } t \geq \frac{1}{2}$$

Since the C^1 regularity of u is obvious (because by assumption u is such and $[u] \in [u]$), the claim has been established.

Note now that since $u \in W_0^{1,1}([0, R^N])$, the set B is

for any $\alpha \in [0, 1]$, as desired.

Now, by using (2.4)-(2.5), (2.14)-(2.16) and the level-convexity of the function of (2.18), for any $x \in B_{\alpha}(x_0)$ we have the estimate

$$\begin{aligned}
 (2.19) \quad A(x) &= H(x); \\
 &= \sum_{k=1}^K \int_{B_{\alpha}(x)} D_k(x) \, d\mu_k(x) + \int_{B_{\alpha}(x)} D(x) \, d\mu(x) \\
 &= \sum_{k=1}^K \int_{B_{\alpha}(x)} D_k(x) \, d\mu_k(x) + \int_{B_{\alpha}(x)} D(x) \, d\mu(x) \\
 &= \sum_{k=1}^K \int_{B_{\alpha}(x)} D_k(x) \, d\mu_k(x) + \int_{B_{\alpha}(x)} D(x) \, d\mu(x)
 \end{aligned}$$

Since for any x and α , the map

$$\mu_k := \int_{B_{\alpha}(x)} D_k(x) \, d\mu_k(x)$$

is a probability measure on the ball $B_{\alpha}(x)$ which is absolutely continuous with respect to the Lebesgue measure $\mathbb{L}^n$, in view of (2.17), (2.19) gives

$$\begin{aligned}
 (2.20) \quad A(x) &= \sum_{k=1}^K \int_{B_{\alpha}(x)} D_k(x) \, d\mu_k(x) + \int_{B_{\alpha}(x)} D(x) \, d\mu(x) \\
 &= \sum_{k=1}^K \int_{B_{\alpha}(x)} D_k(x) \, d\mu_k(x) + \int_{B_{\alpha}(x)} D(x) \, d\mu(x) \\
 &= \sum_{k=1}^K \int_{B_{\alpha}(x)} D_k(x) \, d\mu_k(x) + \int_{B_{\alpha}(x)} D(x) \, d\mu(x)
 \end{aligned}$$

By the continuity of H and Du , there is a positive increasing modulus of continuity ω with $\omega(0^+) = 0$ such that

$$\begin{aligned}
 &|H(x; P) - H(y; Q)| \leq \omega(|x - y| + |P - Q|); \\
 &|Du(x) - Du(y)| \leq \omega(|x - y|);
 \end{aligned}$$

for all $x, y \in B_{\alpha}(x_0)$ and $|P|, |Q| \leq K D_{L^1}(\alpha) + 1$. By using that $\int_{B_{\alpha}(x)} D(x) \, d\mu(x) \leq \int_{B_{\alpha}(x)} D(x) \, d\mu(x)$ on $B_{\alpha}(x)$, (2.20) and the above give

$$\begin{aligned}
 A(x) &= \int_{B_{\alpha}(x)} H(x; P) \, d\mu(x) + \int_{B_{\alpha}(x)} D(x) \, d\mu(x) \\
 &= \int_{B_{\alpha}(x)} H(x; P) \, d\mu(x) + \int_{B_{\alpha}(x)} D(x) \, d\mu(x)
 \end{aligned}$$

By (2.14), (2.21) gives

$$A(x) = \operatorname{ess\,sup}_{y \in B^-(x)} H(y; D^-(y)) + \sup_{y \in B^-(x)} \|_2 \|x - y\| + Du(y) - Du(x) \\ = \operatorname{ess\,sup}_{y \in B^-(x_0)} H(y; D^-(y)) + \|_2 \|x - x_0\| + \|_2 \|x_0 - y\|; j$$

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