

Department of Mathematics and Statistics

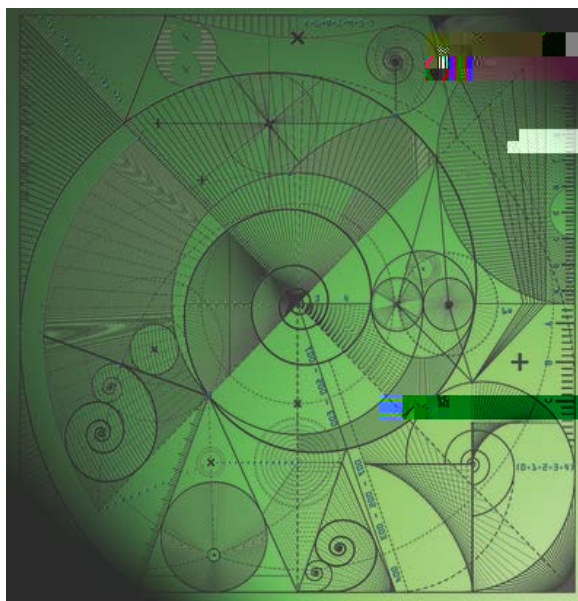
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Accumulation of complex eigenvalues of an indefinite Sturm-Liouville operator with a shifted Coulomb potential

by

Michael Levitin and **Marcello Seri**



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Michael Levitin

Marcello Seri

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Abstract

For a particular family of long-range potentials V , we prove that the eigenvalues of the indefinite Sturm-Liouville operator $A = \text{sign}(x)(-\frac{d^2}{dx^2} + V(x))$ accumulate to zero asymptotically along specific curves in the complex plane. Additionally, we relate the asymptotics of complex eigenvalues to the two-term asymptotics of the eigenvalues of associated self-adjoint operators.

1 Introduction

Given a real-valued potential V such that

$$V \in L^1(\mathbb{R}); \quad \lim_{x \rightarrow -\infty} V(x) = \infty; \quad \limsup_{x \rightarrow -\infty} x^2 V(x) < \frac{1}{e}; \quad (1)$$

consider a one-dimensional Schrodinger operator in $L^2(\mathbb{R})$

$$\begin{aligned} T &:= T_V := -\frac{d^2}{dx^2} + V(x); \\ \text{Dom}(T) &:= \{f \in L^2(\mathbb{R}) \mid f, f' \in AC(\mathbb{R}); T f \in L^2(\mathbb{R})\}. \end{aligned} \quad (2)$$

It is well known that in this case the spectrum $\text{Spec}(T)$ is bounded from below, the essential spectrum $\text{Spec}_{\text{ess}}(T) = [\infty; 1)$, and the negative spectrum $\text{Spec}(T) \setminus (\infty; 1)$ consists of eigenvalues accumulating to zero from below.

Let $J := \text{sign}(x)$ be the multiplication operator by J on $\mathbb{R}$. In what follows we consider the point spectrum of the operator

$$A := A_V := J T_V; \quad \text{Dom}(A) = \text{Dom}(T). \quad (3)$$

This operator is not self-adjoint (and not even symmetric) on $L^2(\mathbb{R})$, and its spectrum need not therefore be real. However, as $J = J^{-1} = J$, A can be treated as a self-adjoint operator in the Krein space $(L^2(\mathbb{R}); [\cdot, \cdot])$ with indefinite inner product

$$[f, g] := \int_{\mathbb{R}} J f \overline{g} \, dx = \int_{\mathbb{R}} f(x) \overline{g(x)} \text{sign}(x) \, dx$$

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ML: Department of Mathematics and Statistics, University of Reading, Whiteknights, PO Box 601, Reading RG6 6AX, UK; m.levitin@reading.ac.uk; <http://www.personal.reading.ac.uk/~ny901965/>

MS: Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK; m.seri@ucl.ac.uk; <http://academic.mseri.me/>

or equivalently as a $\mathcal{J}$ -self-adjoint operator [Azlo]. Operators of type (3) have been studied both in the framework of operator pencils, cf. [DaLe, Ma], and of indefinite Sturm-Liouville problems [BeKaTr, BeTr, KaTr, La].

In both settings the literature is extensive, starting mostly with Soviet contributions in the 1960s, including those by Krein, Langer, Gohberg, Pontryagin and Shkalikov. We refer to [Ma, La] for reviews and bibliographies. In particular due to its many applications, for example in control theory, mathematical physics and mechanics, the field is still very active, with recent works on the theoretical, as well as numerical, aspects, (see e.g. [DaLe, ElLePo, HiTrVD, Ve] and references therein).

In the special case of indefinite Sturm-Liouville operators, it is well known that for positive potentials, $V \geq 0$, the spectrum of A_V is real and the operator A_V is similar to a self-adjoint operator [CuLa, CuNa, Py, Ko]. At a very basic level, this can be seen from the following abstract construction: if R and S are self-adjoint operators with $R > 0$, then, under mild restrictions, the spectrum of $R^{-1}S$ is the same as the spectrum of the self-adjoint operator $R^{1/2}SR^{-1/2}$, and is therefore real.

The case $V \in L^1(\mathbb{R}; (1 + |x|)dx)$

Figure 1: A numerical example showing accumulation to 0 of complex eigenvalues (red diamonds) of the operator A , $\lambda = c\lambda$. The magenta and white circles on the negative real axis are the eigenvalues of T corresponding to the eigenfunctions which are even or odd with respect to zero, cf. [BeKaTr].

Proposition 1(e)(i) holds. Moreover we also prove (Theorem 6) that the complex eigenvalues of

$$T = T := T_V$$

accumulate to zero asymptotically along specific curves in the complex plane, and that the explicit asymptotics of complex eigenvalues of T can be obtained from the asymptotics of eigenvalues of the self-adjoint operator

$$A = A := A_V \quad (1)$$

(or, more precisely, from the eigenvalues of its restriction on either even or odd (with respect to zero) subspace). We also extend these results to the more general non-symmetric potentials

$$V_{\pm}(x) = \begin{cases} \frac{+}{1 + |x|} & \text{if } x > 0 \\ \frac{-}{1 + |x|} & \text{if } x < 0 \end{cases}; \quad +; \quad 2 \in \mathbb{R}_+ : \quad (6)$$

with account of multiplicities. Let $\lambda_n^\#(\cdot)$ denote the eigenvalues of $T^\#$, $\# = D$ or N , ordered increasingly. In what follows we often drop the explicit dependence on $\cdot$.

It is well-known that $\lambda_n^\# < \lambda_{n+1}^\#$ and $\lambda_n^D < \lambda_n^N < \lambda_{n+1}^D$ for all $n \geq 1$, and also that $\lambda_n^\# \rightarrow 0$ as $n \rightarrow \infty$.

Before stating our main results, we need some additional notation.

Definition c. Let $\mathcal{F}$ denote the class of piecewise smooth functions $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ which have a discrete set of singularities (with no finite accumulation points). At each singularity both one-sided limits of F are finite and differ by sign. Assume for simplicity that 0 is not a singularity of F , and that $F(0) = 0$. For $F \in \mathcal{F}$ we denote by $\arctan_F(x)$ the continuous branch of the multi-valued $\text{Arctan}(F(x))$ such that $\arctan_F(0) = 0$.

Remark 1. Away from the singularities of F , the function $\arctan_F(x)$ can be written in terms of the ordinary $\arctan(F(x))$ (which takes the values in $(-\frac{\pi}{2}; \frac{\pi}{2})$) and the total signed index of F on $[0; x]$, which we denote by $Z_F(x)$, and which is defined as the total number of jumps from $+1$ to -1 on $[0; x]$ minus the total number of jumps in the opposite direction:

$$Z_F(x) := \left[\sum_{0 < t_1 < \dots < t_j \leq x} \text{sgn}(F(t_i^+) - F(t_i^-)) \right]: \quad (7)$$

Then

$$\arctan_F(x) = \arctan(F(x)) + Z_F(x): \quad (8)$$

Obviously, $\arctan_F(x) = \arctan_F(x)$.

Our first result gives sharp two-term asymptotics of eigenvalues (accumulating to zero) of the self-adjoint operator T_f :

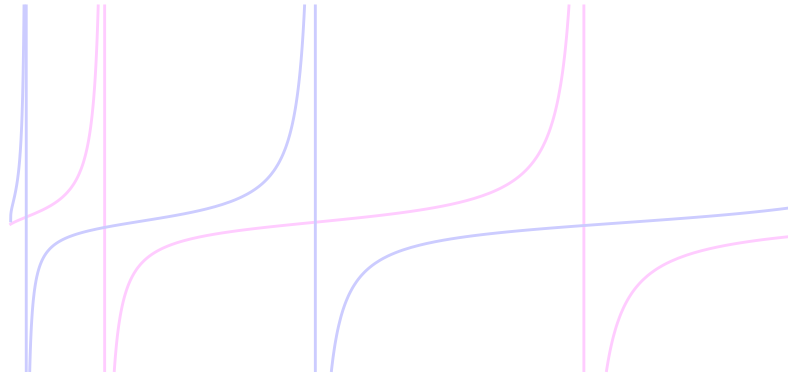


Figure c: $R_1()$ and $R_0()$

Before proceeding to the proofs, we want to discuss the statements of Theorem 6 in more detail.

Remark 7. (a) It is immediately seen from (J₁) and (J₂) that

$$(\varphi) = \operatorname{Im} \quad (\varphi) = \operatorname{Im} + (\varphi):$$

(b) If we introduce two functions $\varphi : \mathbb{R}_+ \rightarrow \mathbb{C}$ by

$$(\varphi)(t) = t + t^{3/2},$$

then

$$\operatorname{Im} \quad (\varphi) = \operatorname{Im} + (\varphi) = t^{3/2};$$

e. The Jost solutions of (16)

It is well known, see e.g. [OlLoBoCl, §7.] and [7.c], that the first order asymptotic behaviour of the Kummer Hypergeometric Functions is given, as $|w| \rightarrow 1$, by

$$U(a; b; w) \sim w^{-a}; \quad \frac{\pi}{c} < \arg(w) < \frac{\pi}{c};$$

$$M(a; b; w) \sim \frac{e^{w/b} w^{b-a}}{\Gamma(a)} + \frac{e^{-ia} w^{-a}}{\Gamma(b-a)}; \quad \frac{\pi}{c} < \arg(w) < \frac{\pi}{c}; \quad a, b \in \mathbb{N} \setminus \{0\};$$

where $\Gamma(\cdot)$ stands for the usual Gamma function.

For $z \in \mathbb{C} \setminus \mathbb{R}_+$, we have $\operatorname{Re} z < 0$, and therefore

$$U\left(\frac{1}{2}\right) \sim \frac{1}{c^{1/2}}; \quad c; \quad cy^{1/2} \sim \frac{1}{cy^{1/2}} e^{2y^{1/2}};$$

and

$$M\left(\frac{1}{2}\right) \sim \frac{1}{c^{1/2}}; \quad c; \quad cy^{1/2} \sim \frac{(cy^{1/2})^{2^{1/2}+1}}{1} e^{2y^{1/2}}$$

as $y \rightarrow 1$.

This in turn implies that the f and g

control their oscillations. A quick look at (c5) and (c6) shows that we require asymptotic formulas, as $n \rightarrow +\infty$, for

$$U_{\frac{p}{c}} = c^{\frac{p}{c}} \left(\frac{1}{c} \right)^{\frac{p}{c}} ; \quad c^2 f \left(\frac{p}{c} \right) ; \quad]g; \quad (c7)$$

Unfortunately, it is a difficult task to find the corresponding formulas, are not, in fact, in the standard references. We rely, instead, on the results from the forthcoming book [Te] which we summarise and adapt in the Appendix.

6.c Asymptotic solutions of a transcendental equation

A crucial element of our analysis is the investigation of the large n -roots of the equation

$$\tan \left(\frac{p}{c} \right) = G \left(\frac{p}{c} \right) \quad (c8)$$

where $\frac{p}{c}$ is treated as a parameter, and where G depends analytically on $\frac{p}{c}$ in the vicinity of $\frac{p}{c} = 1$ and, to leading order, is of class F as a function of $\frac{p}{c}$. The required results are summarised in the following

Lemma 8. Let $G \left(\frac{p}{c} \right)$ be an analytic function of $\frac{p}{c}$ around $\frac{p}{c} = 1$ such that

$$G \left(\frac{p}{c} \right) = G_0 \left(\frac{p}{c} \right) + O \left(\frac{1}{n} \right); \quad \text{as } n \rightarrow +\infty;$$

$G_0 \in F$, and the O terms are regular in $\frac{p}{c}$. Then the solutions $n \left(\frac{p}{c} \right)$, ordered increasingly, of the equation (c8), are given, as $n \rightarrow +\infty$, by

$$n \left(\frac{p}{c} \right) = \frac{n}{2} + \frac{1}{2} G_0 \left(\frac{p}{c} \right) + O(n^{-1}); \quad (c9)$$

The proof of Lemma 8 is in fact immediate as soon as we recall Definition c of $\tan$ and the fact that $\tan$ is π -periodic.

Considering additional terms in the expansion of G one can get additional terms in the expansion of n . This is in fact what we do in more detail in Section 6.c.

6.d Approximation of Dirichlet eigenvalues

We can use the asymptotic approximation obtained in (A.8) to reduce (c5) to the simpler form

$$\cos \frac{p}{c} = \left(J_1 \left(c^{\frac{p}{c}} \right) + O \left(\frac{1}{n} \right) \right) + \sin \frac{p}{c} = \left(Y_1 \left(c^{\frac{p}{c}} \right) + O \left(\frac{1}{n} \right) \right) = 0; \quad (d0)$$

This in turn can be rewritten as

$$\tan \frac{p}{c} = \frac{J_1 \left(c^{\frac{p}{c}} \right)}{Y_1 \left(c^{\frac{p}{c}} \right)} + O \left(\frac{1}{n} \right); \quad (d1)$$

Applying Lemma 8 with

$$\frac{p}{c}; \quad G_0 \left(\frac{p}{c} \right) = \frac{J_1 \left(c^{\frac{p}{c}} \right)}{Y_1 \left(c^{\frac{p}{c}} \right)} = R_1 \left(\frac{p}{c} \right);$$

we obtain, after a minor effort,

$$\begin{aligned} n &= \frac{2}{\pi} \left(n^2 + G_0 \left(\frac{p}{c} \right)^2 + O(n^{-4}) \right) \\ &= \frac{2}{\pi n^2} \left[\right] + \frac{c}{n} G_0 \left(\frac{p}{c} \right) + O(n^{-2}) \end{aligned} \quad (d2)$$

$$= \frac{2}{\pi n^2} \left[\right] + \frac{c}{n} R_1 \left(\frac{p}{c} \right) + O(n^{-2}) \quad (d3)$$

as $n \rightarrow +\infty$, thus proving the first part of Theorem e.

1.e Approximation of Neumann eigenvalues

The analysis for Neumann eigenvalues is slightly more complicated. Again we can use (A.8) to reduce (c6) to

$$\tan \frac{\pi}{c} p = \frac{P(\cdot; \cdot)}{Q(\cdot; \cdot)} \quad (1e)$$

where

$$\begin{aligned} P(\cdot; \cdot) &:= p^{p-8} (c^{p-8} + 1) J_1(c^{p-8}) + (1) p^{p-8} (c^{p-8} - 1) J_2(c^{p-8}) \\ &\quad - 8 p^{p-8} (c^{p-8}) J_3(c^{p-8}) + O(c^{3-2}); \\ Q(\cdot; \cdot) &:= p^{p-8} (c^{p-8} + 1) Y_1(c^{p-8}) + (1) p^{p-8} (c^{p-8} - 1) Y_2(c^{p-8}) \\ &\quad - 8 p^{p-8} (c^{p-8}) Y_3(c^{p-8}) + O(c^{3-2}); \end{aligned}$$

Applying once again Lemma 8 with

$$= \frac{1}{c} p; \quad G_0(\cdot) = \frac{P(\cdot; \cdot)}{Q(\cdot; \cdot)} = \frac{J_0(c^{p-8})}{Y_0(c^{p-8})} = R_0(\cdot);$$

we quickly arrive at

$$n = \frac{2}{\pi n^2} \left[1 + \frac{c}{n} G_0(\cdot) + O(n^{-2}) \right] = \frac{2}{\pi n^2} \left[1 + \frac{c}{n} R_0(\cdot) + O(n^{-2}) \right] \quad (11)$$

as $n \rightarrow \infty$, thus proving the second part of Theorem e.

6 Proof of the asymptotic results of the non-self-adjoint operator

6.1 Eigenvalues and the Jost solutions

Lemma 9. The eigenvalues of (1) are the zeroes of the determinant

$$M(\cdot) = M(\cdot) = \begin{pmatrix} \psi^0(\cdot; \cdot) \\ \psi^1(\cdot; \cdot) \end{pmatrix} + \begin{pmatrix} \psi^0(\cdot; \cdot) \\ \psi^1(\cdot; \cdot) \end{pmatrix} \quad (16)$$

Proof. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of A , and that $g(x) \in L^2(\mathbb{R})$ is a corresponding eigenfunction. Then g solves the differential equation

$$\frac{d^2}{dx^2} g(x) - \frac{1}{1+|x|} g(x) = \text{sign}(x) g(x);$$

If $g_{\pm}$ denote the restrictions of g on $\mathbb{R}_+$ and $\mathbb{R}_-$, then by integrability we must have

$$g_{+}(x) = C_{+} \psi^{-}(\cdot; x); \quad g_{-}(x) = C_{-} \psi^{-}(\cdot; x); \quad x \in \mathbb{R}_{+};$$

where $\psi^{-}(\cdot; x)$ is the Jost solution (ce).

As an eigenfunction should be continuously differentiable at zero, we obtain

$$\begin{aligned} C_{+} \psi^{-}(\cdot; 0) - C_{-} \psi^{-}(\cdot; 0) &= 0; \\ C_{+} \psi^{0}(\cdot; 0) + C_{-} \psi^{0}(\cdot; 0) &= 0; \end{aligned}$$

which has a non-trivial solution if and only if $M(\cdot) = 0$. □

Remark 10. (a)

- (b) By [BeTr, Proposition e.6] one can instead look for the eigenvalues of (16) as the zeroes of the m-function

$$m(\lambda) = \frac{h^{(0)}(\lambda; \alpha)}{h^{(1)}(\lambda; \alpha)} + \frac{h^{(0)}(\lambda; \beta)}{h^{(1)}(\lambda; \beta)}. \quad (17)$$

The use of half-line m-functions is natural and has been already suggested elsewhere, and described in great generality for indefinite Sturm-Liouville problems with turning point at α in [KaTr] (see also references therein).

- (c) In what follows we assume that α is in the upper half plane $\mathbb{C}_+$ and look for the eigenvalues on the first quadrant. The final result will follow by symmetry (see Proposition 3(a) and Proposition 3(f)).

6.c The determinant

We can use (ce) and the known relations [OILoBoCl, x]1.1 between Kummer hypergeometric functions to rewrite (16) as

$$M(\lambda) = \frac{2^{p-1} e^{p-\alpha} \Gamma(\alpha) \Gamma(p-\alpha)}{\Gamma(5-2\alpha)} {}_2F_1\left(\begin{matrix} p \\ p-\alpha \end{matrix}; \frac{p-\alpha}{c}\right) + c^\alpha U\left(\begin{matrix} p \\ p-\alpha \end{matrix}; \frac{p-\alpha}{c}\right); c^{p-\alpha}$$

where we have dropped the lower order terms.

Simplifying, writing $S = T - K$, and collecting terms in K , we get

$$\begin{aligned}
& K + ([+ i)j_1^{2p-} (p- ([+ i)) 6e i^2 e([+ i) p- + c \\
& + T y_1 8i^2 + ([+]6i) p-] (j_2(8]]^{p-}) + 8j_3^{p-p-}) \\
& j_1]6ij_3^{p-p-} e([+ i)^2 p- + ([i) \\
& + cj_2 6e^3]c8([i) 2p-]i + ([([+ i) 3=2 \\
& + T (8 p- ([+ i) p- (p- ([+ i)) (8 + i p-) y_1 \\
& + (+ ci p-) (8^{p-p} \bar{y}_3 + y_2 (]]^{p-} + 8i)) \\
& K y_1 8j_3^{p-p-} 8i^2 + ([6 + i) p-] \\
& + j_2 6e^3 (e c]6i) 2p- ([76 +]i) +]] 3=2 \\
& + ([+ i)T p- (p- ([+ i)) 6e^2 e([i) p- ci y_1 \\
& + 8([i) p-p \bar{y}_3 e([+ i)^2 p- + ([i) \\
& + y_2 6e([+ i)^3 c6 2p- +]i ([i) +]]i 3=2 \\
& + j_1 (8 + i p-) (k c p-) (y_2 (8]]^{p-}) + 8^{p-p} \bar{y}_3) \\
& + ([i) p- (p- ([+ i)) (8 p-) y_1 = 0:
\end{aligned} \tag{e}$$

In what follows, we essentially replicate the reasoning in Lemma 8, but working to a higher

We want to derive a similar expansion for $\tanh \frac{p}{2 + \sqrt{3+2} + 2}$. We use

$$\tanh(t_1 - t_2) = \frac{\sinh(t_2) + \cosh(t_2) \tanh(t_1)}{\cosh(t_2) - \sinh(t_2) \tanh(t_1)}$$

with $t_1 := \frac{p}{2}$ and $t_2 := \frac{1}{4} + \frac{3}{4} p^{-1} + O(p^{-2})$.

As $\tanh(\frac{p}{2}) = 1$ for $p \rightarrow \infty$ modulo exponentially small terms, we get (again up to exponentially small errors)

$$\tanh \frac{p}{2 + \sqrt{3+2} + 2} = \sinh(t_2) \tanh$$

7 Generalizations and other remarks

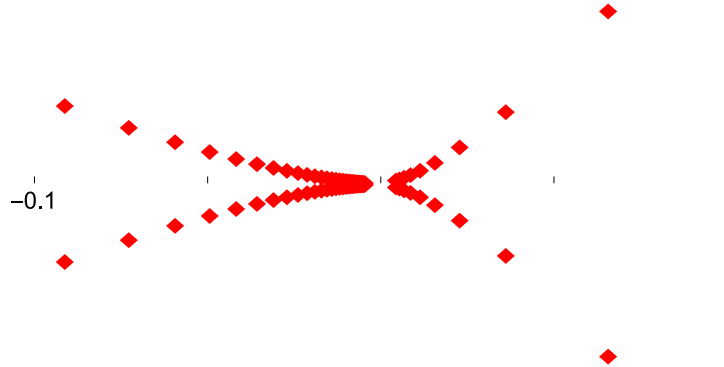


Figure 1: Approximated eigenvalues of $A(\lambda; \mu)$ for $\lambda = 1$, $\mu = 0$.

The procedure used to prove Theorem 6 can be repeated in a completely similar way to obtain a result for the operator

$$A(\lambda; \mu) = JT_V; \quad V(x) = \begin{cases} \frac{1}{1+jxj} & \text{if } x > 0 \\ \frac{1}{1-jxj} & \text{if } x < 0 \end{cases}; \quad \lambda, \mu \in \mathbb{R}_+.$$

In this case the m-function is of the form

$$M(\lambda) = \frac{J_0(\lambda) + iJ_1(\lambda)}{J_0(\lambda) - iJ_1(\lambda)} + \frac{Y_0(\lambda) + iY_1(\lambda)}{Y_0(\lambda) - iY_1(\lambda)}.$$

The curves in the upper (resp. lower) half plane are no more symmetric w.r.t. $i\mathbb{R}$, however for the left quadrants and right quadrants we can extend Theorem 6. The only difference is that now the J_0 and J_1 are now functions of both λ and μ .

Let $\lambda, \mu \in \mathbb{R}_+$. Set

$$f(\lambda; \mu) := \frac{J_1^2(c^{\lambda, \mu}) + J_0^2(c^{\lambda, \mu})}{J_1^2(c^{\lambda, \mu}) + J_0^2(c^{\lambda, \mu})} i + \frac{J_0(c^{\lambda, \mu})J_1(c^{\lambda, \mu}) + Y_0(c^{\lambda, \mu})Y_1(c^{\lambda, \mu})}{J_0(c^{\lambda, \mu})J_1(c^{\lambda, \mu}) + Y_0(c^{\lambda, \mu})Y_1(c^{\lambda, \mu})};$$

$$f_+(\lambda; \mu) := \frac{J_1^2(c^{\lambda, \mu}) + J_0^2(c^{\lambda, \mu})}{J_1^2(c^{\lambda, \mu}) + J_0^2(c^{\lambda, \mu})} i + \frac{J_0(c^{\lambda, \mu})J_1(c^{\lambda, \mu}) + Y_0(c^{\lambda, \mu})Y_1(c^{\lambda, \mu})}{J_0(c^{\lambda, \mu})J_1(c^{\lambda, \mu}) + Y_0(c^{\lambda, \mu})Y_1(c^{\lambda, \mu})};$$

Then the two factor multiplying the term $\text{Re}^{-3/2}$ are given by

$$(\lambda; \mu) := \frac{e^{-\frac{1}{2} \arctan(f(\lambda; \mu))}}{1 + \frac{1}{2} \arctan(f(\lambda; \mu))} \text{ if } \text{Re} > 0;$$

$$+(\lambda; \mu) := \frac{e^{-\frac{1}{2} \arctan(f_+(\lambda; \mu))}}{1 + \frac{1}{2} \arctan(f_+(\lambda; \mu))} \text{ if } \text{Re} < 0;$$

One can immediately see that the asymmetry appearing w.r.t. $i\mathbb{R}$ is reflected in the asymmetric dependence on λ and μ .

It is interesting to observe that for $\operatorname{Re} \lambda > 0$ the effect of λ is much stronger than the one of μ (the latter appears only in the cotangent term, its contribution is bounded, while the former additionally appears as an inverse prefactor). The situation is opposite when $\operatorname{Re} \lambda < 0$.

The expressions for $\mathcal{J}_\pm(\lambda; \mu)$ are more involved than the ones for $\mathcal{J}_\pm(\lambda)$ but, as expected, they simplify to (e^λ) and (e^λ) for $\mu = 0$. As that case, it is possible to use the standard results on Bessel functions to show that the two constants have non-zero real and imaginary part for any $\lambda > 0$.

To answer the general question posed in [Be] for a wider class of potentials one would need good estimates of the Jost functions in a complex half ball containing the origin and the positive and negative real axis. To our knowledge, the best result of this kind is contained in a paper by Yafaev [Ya]. In that work, however, the author needed to exclude two cones containing the real axis for his estimates to hold. Additionally he could get only the first term in the asymptotic expansion, whereas for our result we would need at least the first two.



Figure 6: Plot of real part (left) and imaginary part (right) of $U^{p-1/2}(a; c; c^{p-1/2})$ (black) and its approximation given by (A.8) (dashed red) for small values of a and c .

and $\text{Re}(az) = \text{Re}(a)$.

Observe that for $a \in \mathbb{R}_+$, $\text{Re}(az) > 0$ if $\text{Re}(a) > 0$.

The coefficients A_0 and B_0 also have explicit expressions that can be derived using some symmetry properties and L'Hôpital rule, see [Te, (c7.e.7e)]:

$$A_0 = \frac{1}{c \sin(c)} \frac{c}{\tan(c)} \cos(c)$$

$$B_0 = \frac{1}{c \sin(c)} \frac{c}{\tan(c)} \sin(c)$$

where $\omega = \frac{1}{2} i \omega_0$.

The computation of A_n and B_n for $n \geq 1$ is quite involved, however we will need only A_1 . One can exploit the procedure to compute A_0 and B_0 , and the recursive definition of the coefficients to get a Taylor approximation in negative powers of a for $c \neq 0$. We get

$$\text{if } c = 0; \quad A_0^0 = 1 + O(a^{-2}); \quad A_1^0 = \frac{1}{6} + O(a^{-2}); \quad B_0^0 = 0; \quad (A.6)$$

$$\text{if } c \neq 0; \quad A_0^1 = 1 + O(a^{-4}); \quad A_1^1 = \frac{1}{6} + O(a^{-2}); \quad B_0^1 = \frac{p}{ca} + O(a^{-3}); \quad (A.7)$$

With these, (A.c) can be re-written

$$U^{p-1/2}(a; c; \frac{c^{p-1/2}}{a}) = (a + \frac{1}{2}c) e^{\frac{1}{2}a} C_{c-1}(a; c) (A_0^c + A_1^c) + C_{c-2}(a; c) B_0^c + O(a^{-2}) \quad (A.8)$$

where

$$C(a; c) := \cos(a) J(c^{p-1/2}) + \sin(a) Y(c^{p-1/2}) \quad (A.9)$$

and A_0^c , A_1^c and B_0^c are obtained dropping the error term in the appropriate coefficient in (A.6) and (A.7).

Remark A.1. Here the error is in fact $O(a^{-2})$, which is the same as the error in the approximation of the function by the Taylor series.

In our case

$$t_1 = c_1 i + c_2 + O(\epsilon^2):$$

For $j \in \mathbb{N}$, $\arg t_1$ is in the upper complex half plane. In particular this allows a and z to be in the closure of the first and fourth quadrant.

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