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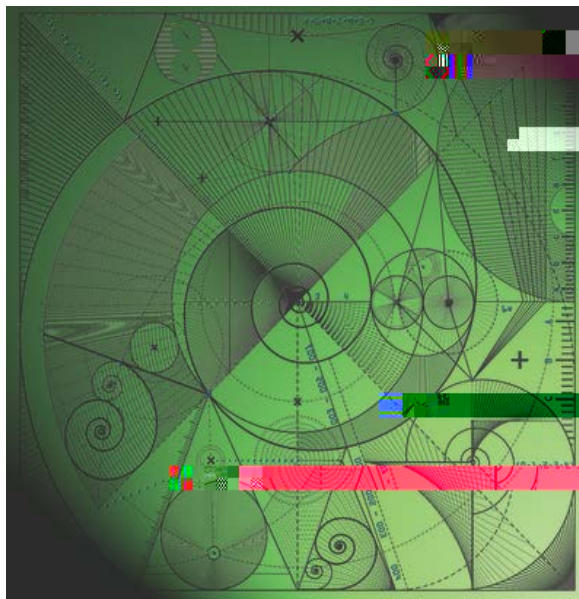
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Generalised golden ratios

by

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Abstract

Given a finite set of real numbers A , the generalised golden ratio is the unique number $G(A) > 1$ for which we only have trivial unique expansions in smaller bases, and non-trivial unique expansions in larger bases. We show that $G(A)$ varies continuously with the alphabet A (of fixed size), and we calculate $G(A)$ for certain alphabets. As we vary a single parameter within A , the generalised golden ratio function may behave like a constant function, a linear function, and even a square root function.

We also build upon the work of Komornik, Lai, and Pedicini (2011) and study generalised golden ratios over ternary alphabets. We give a new proof of their main result, that is we explicitly calculate the function $G(f0; 1; mg)$. (For a ternary alphabet, it may be assumed without loss of generality that $A = f0; 1; mg$.) We also study the set of $m \in (1; 2]$ for which $G(f0; 1; mg) = 1 + \sqrt[m]{m}$ and prove that it is an uncountable set of Hausdorff dimension 0. Last of all we show that the function mapping m to $G(f0; 1; mg)$ is of bounded variation yet has unbounded derivative.

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1 Introduction and statement of results

Let $A := f a_0; a_1 0; \dots; a$

that the golden ratio acts as a natural boundary between the possible cardinalities the set of expansions can take. It is natural to ask whether such a boundary exists for more general alphabets.

Before we state the definition of a generalised golden ratio it is necessary to define the univoque set. Given an alphabet A and $\beta > 1$ we set

$$U(A) := \left\{ (u_k)_{k=1}^{\infty} \in A^{\mathbb{N}} : \sum_{k=1}^{\infty} \frac{u_k}{\beta^k} \text{ has a unique expansion} \right\}$$

We call $U(A)$ the univoque set. Note that for any alphabet A and $\beta > 1$ the points

$$\sum_{k=1}^{\infty} \frac{a_0}{\beta^k} \text{ and } \sum_{k=1}^{\infty} \frac{a_d}{\beta^k}$$

both have a unique expansion, so $\overline{a_0}$ and $\overline{a_d}$ are always contained in the univoque set. Here and throughout $\overline{w}$ denotes the infinite periodic word with period w . We are now in a position to define a generalised golden ratio for an arbitrary alphabet. Given an alphabet A , we call $G(A) \in (1; \beta)$ the generalised golden ratio for A if whenever $\beta \in (1; G(A))$ we have $U(A) = \{ \overline{a_0}, \overline{a_d} \}$, and if $\beta > G(A)$ then $U(A)$

The set M is uncountable, but its Hausdorff dimension is 0.

Theorem 3. We have $\dim_H(M) = 0$.

On certain intervals, the function G has the following simple form.

Theorem 4. Let h be a positive integer and $2^h \leq m < 2^{h+1}$. Then we have

$$G\left(\frac{m}{2^h}\right) = G\left(\frac{1}{2^h}; 1; m\right) = m^{1-h}.$$

In [1] the first author studied $G(m)$ for integer m . In this case the following results hold.

Theorem B. Let $m \in \mathbb{Z}$ with $m \geq 2$. The following statements hold:

$$G(m) = \begin{cases} \frac{\frac{m}{2} + 1}{m+1} & \text{if } m \text{ is even,} \\ \frac{\frac{m^2 + 10m + 9}{4}}{m+1} & \text{if } m \text{ is odd.} \end{cases}$$

If m is odd, then there exists $\delta(m) > 0$ such that for all $x \in (G(m); G(m) + \delta(m))$, the set $U(x)$ consists of $\frac{m-1}{2}$ and a subset of the sequences that end with $\frac{m-1}{2}$.

If m is even, then there exists $\delta(m) > 0$ such that for all $x \in (G(m); G(m) + \delta(m))$, the set $U(x)$ consists of $\frac{m}{2}$ and a subset of the sequences that end with $\frac{m}{2}$.

For each positive integer k , we also calculate $G(m)$ on a small interval to the right of k . These calculations demonstrate that the function G can vary in different ways as we change a single parameter. For now we postpone the statement of these results.

2 Continuity of $G(A)$

Proof of Theorem 1. As $G(A) = G\left(\frac{A - a_0}{a_d - a_0}\right)$, we have $G(f(a_0; a_1; \dots; a_d)g) = G(r(a_0; a_1; \dots; a_d))$, with

$$r : \mathbb{R}_d^+ \rightarrow \mathbb{R}_d^+ : (a_0; a_1; \dots; a_d) \mapsto \frac{a_1 - a_0}{a_d - a_0}, \frac{a_2 - a_0}{a_d - a_0}, \dots, \frac{a_{d-1} - a_0}{a_d - a_0};$$

$$: \mathbb{R}_d^+ \rightarrow \mathbb{R}_d^+ : (a_1; a_2; \dots; a_{d-1}) \mapsto 0; a_1; a_2; \dots; a_{d-1}; 1g;$$

and $\mathbb{R}_d^+ = f(a_1; a_2; \dots; a_{d-1}) \in \mathbb{R}^{d-1} : 0 < a_1 < a_2 < \dots < a_{d-1} < 1g$. As r is continuous on $\mathbb{R}_d^+$, it is sufficient to prove that G is continuous on $\mathbb{R}_d^+$.

Let $a = (a_1; a_2; \dots; a_{d-1}) \in \mathbb{R}_d^+$ and $\epsilon > 0$ arbitrary but fixed. We will show that $|G(b) - G(a)| \leq \epsilon$ for all b in a neighbourhood of a . Let first $X \subset \mathbb{R}_d^+$ be a closed neighbourhood of a such that $|q(b) - q(a)| \leq \epsilon$ for all $b \in X$. (Note that q is continuous on $\mathbb{R}_d^+$.) Set

$$Y = \min_{b \in X} q(b) - \epsilon; \quad Y = \{b \in X : G(b) < q(a) - \epsilon\}.$$

If $Y = \emptyset$, then X is a neighbourhood of a with $|G(b) - G(a)| \leq \epsilon$ for all $b \in X$. Otherwise, let $k \in \mathbb{N}$ be such that $\frac{1}{k} < \epsilon$. Then

$$b_{j+1} - b_j \leq \frac{1}{q(b) - 1} - \frac{1}{q(a) - 1} \leq \sum_{k=1}^X \frac{1}{k} \quad (2.3)$$

for all $(b_1; \dots; b_{d-1}) \in Y$, $0 \leq j < d$, with $b_0 = 0$, $b_d = 1$. Set

$$(a; b) = \min_{0 \leq j < d} (a_{j+1} - a_j) - (b_{j+1} - b_j)$$

(with $b_0 = a_0 = 0$, $b_d = a_d = 1$), and let $Z \subset X$ be a neighbourhood of a such that

$$\frac{a_j}{(q(a) - \epsilon)^k} - \frac{b_j}{(q(b) - \epsilon)^k} \leq (a; b); \quad \frac{b_j}{(q(b) - \epsilon)^k} \leq \frac{a_j}{(q(a) - \epsilon)^k}$$

Similarly, we obtain from (2.2), (2.5) and (2.6) that

$$\sum_{k=1}^{\infty} \frac{1}{(a+b)^k} u_{i+k} = \sum_{k=1}^{\infty} \frac{1}{k} u_{i+k} + (a;b) < b_j \quad b_{j-1} + (a;b) = a_j \quad a_{j-1} \quad \text{when } u_i = b_j \neq 0:$$

Therefore, we have $\sum_{k=1}^{\infty} \frac{1}{k} u_{i+k} = (a;b)$, thus $G(a) = G(b) + 1$ for all $b \in Y \setminus Z$.

For $b \in X \cap Y$, recall that $G(a) = q(a) + 2$ $G(b) + 2$. Similarly, we obtain for all $b \in Z$ that $G(b) = G(a) + 1$ when $a \in Y$, $G(b) = G(a) + 3$ when $a \notin Y$. This gives that $G(b) = G(a) + j - 3$ for all $b \in Z$, thus G is continuous at a . $\square$

3 Generalised golden ratios over ternary alphabets

3.1 Statements

Komornik, Lai and Pedicini [4] described the function $m_{\mathbb{G}}(m)$ on the interval $(1; 2]$. We provide more

Note that all the numbers and sequences do not change if we replace by 0 since $0(0\bar{1}) = 0^-$

$_{[0;n]}(0)$ for all $n \geq 0$, i.e., $u_i u_{i+1} = u$

for all $i \geq 0$ such that $u_i = 1$. By Lemma 3.6 and since u is aperiodic, $u_i = 1$ implies that $u < u_{i+1}u_{i+2} \leq u_1u_2$. These bounds cannot be improved because, for all $n \geq 0$, $1_{[0;n]}(0)$ and $1_{[0;n-1]}(1)$ (which is a suffix of $1_{[0;n]}(0)$) are factors of u . Therefore, we have $2 \leq 1 + p_m(m)$ if and only if

$$p_m = 1 + \sum_{k=1}^{\infty} \frac{u_k}{(1 + p_m)^{k+1}} \quad \text{and} \quad 1 + \sum_{k=1}^{\infty} \frac{u_k}{(1 + p_m)^k} = m:$$

This means that $1 + \sum_{k=1}^{\infty} u_k (1 + p_m)^{-k} = m$, i.e., $m = m_u$. □

Lemma 3.10. Let $2 \leq S$ and $m > 1$. There is a unique number $f(m) > 1$ such that

$$m = 1 + \sum_{k=1}^{\infty} \frac{u_k^{(f(m))}}{f(m)^k}. \quad (3.4)$$

We have $f'(m) < 0$, $f(m_{(0\bar{1})}) = 1 + p_{\overline{m_{(0\bar{1})}}}$, f

Proof. The number α is well defined since $f^0(m) < 0$, $g^0(m) > 0$ on I ,

$$f(m_{(0I)}) = 1 + \frac{p}{m_{(0I)}} > g(m_{(0I)}) \text{ and } f(m_{(\bar{1})}) < 1 + \frac{p}{m_{(\bar{1})}} = g(m_{(\bar{1})}):$$

If $(\bar{1}) =$

Proof. Let $\alpha \in \mathbb{Z}$, $u \in \mathbb{Z} \setminus (m)$ if $0 \leq 1g^N$ and α as in Lemma 3.15. Then $\alpha \in \mathbb{Z} \setminus (m)$ for all $\gamma > 0$. If $\alpha \in \mathbb{Z} \setminus 1$, then Lemma 3.9 gives that $1 + \frac{p}{m}$. If $\alpha = (\bar{1})$,

4 Behaviour at the generalised golden ratio

In this section we discuss the behaviour of the univoque set at the generalised golden ratio. It was observed in [1] that when $\alpha = G(L)$ for some $L \geq 2$, then every $x \in (0; \frac{L}{L+1})$ either has a countable or finite number of expansions or a continuum of expansions. In other words $U_{G(L)}(L)$ is still trivial. However, Lemma 3.9 demonstrates that this is not always the case. Indeed the following result is an immediate consequence of this lemma.

Proposition 4.1. There exists A for which $U_{G(A)}(A)$ is non-trivial.

In [9] it was shown that the smallest $\alpha \in (1; 2)$ for which an x has precisely two expansions over the alphabet $\{0; 1g\}$ was $\alpha \approx 1.71064$. In other words, there is a small gap between the golden ratio for the alphabet $\{0; 1g\}$ and the smallest α for which an x has precisely two expansions. As we show below, for certain alphabets it is possible that an x has precisely two expansions at the golden ratio.

Proposition 4.2. For every $m \geq M$, the number $m = G(m)$ has precisely two expansions in base $\alpha(m)$ over the alphabet $\{0; 1; mg\}$.

Proof. Let $u \in \mathbb{S}$ be such that $m = m_u$, let $\alpha = G(m) = 1 + \frac{1}{m}$ and let $m = \sum_{k=1}^{\infty} \frac{v_k}{\alpha^k}$ be an expansion of m over the alphabet $\{0; 1; mg\}$. Since $m > \frac{m}{1}$, we have $v_1 \neq 1; mg$, thus $\sum_{k=1}^{\infty} \frac{v_k}{\alpha^k} = m$ and 0 has a unique expansion, and 1 has the expansion $u_1 u_2$ by (3.1), which is also unique. $\square$

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