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Preprint MPS-2015-16

31 August 2015

On Symmetries of the Feinberg-Zee Random Hopping Matrix

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Dedicated to Roland Duduchava on the occasion of his 70th birthday

Abstract. In this paper we study the spectrum Σ of the infinite Feinberg-Zee random hopping matrix, a tridiagonal matrix with zeros on the main diagonal and random ± 1's on the first sub- and super-diagonals; the study of this non-selfadjoint random matrix was initiated in Feinberg and Zee (*Phys. Rev. E* 59 (1999), 6433–6443). Recently Hagger (arXiv:1412.1937, to appear in *Random Matrices: Theory and Applications*) has shown that the so-called periodic part π_∞ of Σ, conjectured to be the whole of Σ and known to include the unit disk, satisfies $p^{-1}(\pi_\infty) \subset \pi_\infty$ for an infinite class S of monic polynomials p. In this paper we make very explicit the membership of S, in particular showing that it includes $P_m(\lambda) = \lambda U_{m-1}(\lambda/2)$, for $m \geq 2$, where $U_n(x)$ is the Chebychev polynomial of the second kind of degree n. We also explore implications of these inverse polynomial mappings, for example showing that π_∞ is the closure of its interior, and contains the filled Julia sets of infinitely many $p \in S$, including those of P_m, this partially answering a conjecture of the second author.

Mathematics Subject Classification (2000). Primary 47B80; Secondary 37F10, 47A10, 47B36, 65F15.

Keywords. random operator, Jacobi operator, non-selfadjoint operator, spectral theory, fractal, Julia set.

1. Introduction

In this paper we study spectral properties of in nite matrices of the form

$$A_c = \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 0 & 1 & & & \\ & c_0 & \boxed{0} & 1 & & \\ & & c_1 & 0 & \ddots & \\ & & & \ddots & \ddots & \end{pmatrix}; \tag{1.1}$$

where $c \in \; := \{ \; 1\}^{\mathbb{Z}}$ is an in nite sequence of $\;1$'s, and the box marks the entry at $(0;0)$. Let ℓ^2 denote the linear space of those complex-valued sequences $\; : \mathbb{Z} \to \mathbb{C}$ for which $\| \; \|_2 := \{\sum_{n \in \mathbb{Z}} | \;_n|^2\}^{1/2} < \infty$, a Hilbert space equipped with the norm $\| \; \|_2$. Then to each matrix A_c with $c \in \;$ corresponds a bounded linear mapping $\ell^2 \mapsto \ell^2$, which we denote again by A_c, given by the rule

$$(A_c \quad \|_2 :=$$

$(;)$ is the inner product on ℓ^2, is given by

$$W(A_c) = \quad := \quad \{x + iy : x, y \in \mathbb{R},\ |x| + |y| < 2\}. \qquad (1.3)$$

This gives the upper bound that $\quad \overline{\quad}$, the closure of . Other, sharper upper bounds on are discussed in Section 2 below.

This current paper is related to the problem of computing lower bounds for via (1.2). If $b \in$ is constant then A_b is a Laurent matrix and spec$A_b = [\ 2, 2]$ if $b_m \quad 1$, while spec$A_b = i[\ 2, 2]$ if $b_m \quad 1$; thus, by (1.2), $_1 := [\ 2, 2] \cup i[\ 2, 2]$. Generalising this, if $b \in$ is periodic with period n then specA_b is the union of a finite number of analytic arcs which can be computed by calculating eigenvalues of $n \quad n$ matrices (see Lemma 2.2 below). And, by (1.2), $_n$, where $_n$ is the union of specA_b over all b with period n. This implies, since is closed, that

$$:= \overline{\ _1\ } \quad ; \qquad (1.4)$$

where $_1 := \cup_{n \in \mathbb{N}}\ _n$.

We will call the periodic part of , noting that [3] conjectures that equality holds in (1.4), i.e. that $_1$ is dense in and $=$. Whether or not this holds is an open problem, but it has been shown in [5] that $_1$ is dense in the open unit disk $D := \{\ \in \mathbb{C} : |\ | < 1\}$, so that

$$\overline{D} \qquad : \qquad (1.5)$$

For a polynomial p and $S \quad \mathbb{C}$, we define, as usual, $p(S) := \{p(\ ) : \ \in S\}$ and $p^{\ 1}(S) := \{\ \in \mathbb{C} : p(\ ) \in S\}$. (We will use throughout that if S is open then $p^{\ 1}(S)$ is open (p is continuous) and, if p is non-constant, then $p(S)$ is also open, e.g., [22, Theorem 10.32].) The proof of (1.5) in [5] depends on the result, in the case $p(\ ) = \ ^2$, that

$$p^{\ 1}(\ _1) \quad _1 ; \quad \text{so that also } p^{\ 1}(\quad) \qquad : \qquad (1.6)$$

This implies that $S_n \quad _1$, for $n = 0, 1, \ldots$, where $S_0 := [\ 2, 2]$ and $S_n := p^{\ 1}(S_{n\ 1})$, for $n \in \mathbb{N}$. Thus $\cup_{n \in \mathbb{N}} S_n$, which is dense in $\overline{D}$, is also in $_1$, giving (1.5).

Hagger [17] makes a large generalisation of the results of [5], showing the existence of an infinite family, S, containing monic polynomials of arbitrarily high degree, for which (1.6) holds. For each of these polynomials p let

$$U(p) := \bigcup_{n=1}^{\infty} p^{\ n}(D). \qquad (1.7)$$

(Here $p^{\ 2}(S) := p^{\ 1}(p^{\ 1}(S))$, $p^{\ 3}(S) := p^{\ 1}(p^{\ 2}(S))$, etc.) Hagger [17] observes sequence $(p^n(\ ))_{n \in N}$, the orbit of , is bounded. Further, the boundary of $K(p)$, $J(p) := \partial K(p)$ $K(p)$, is the Julia set of p.)

The de nition of the set S in [17], while constructive, is rather indirect. The rst contribution of this paper (Section 3) is to make explicit the membership of S. As a consequence we show, in particular, that $P_m \in S$, for $m = 2; 3; \ldots$, where $P_m(\ ) := U_{m-1}(\ =2)$, and U_n is the Chebychev polynomial of the second kind of degree n [1].

The second contribution of this paper (Section 4) is to say more about the interior points of . Previous calculations of large subsets of $_1$, precisely calculations of $_n$ for n as large as 30 [3, 4], suggest that lls most of the square , but int( ), the interior of , is known only to contain D. Using that the whole family f

T2 5528 1.295 Td [(for)]TJ/F11 9.9626 Tf 15.038 0 Td [(n1)] [(f)]TJ/F11 9]-331(the)-21 1/F8 the)96268 9.9627526 838 Tf 7.07 0

this last equation obtained using (2.3). The following lemma, proved using these representations, makes clear that many different vectors k correspond to the same polynomial p_k.

Lemma 2.6. If $k = (k_1, \dots, k_n) \in \{\pm 1\}^n$, for some $n \in \mathbb{N}$, and $\ell = k'$ or ℓ is a cyclic permutation of k, then $p_k = p_\ell$. If $\ell = -k$ then $p_\ell(\lambda) = i^{-n} p_k(i\lambda)$.

Proof. Using (2.6) and (2.3) we see that

$$\begin{aligned} p_\ell(\lambda) - p_k(\lambda) &= q_{\ell(1:n-1)}(\lambda) - q_{k(1:n-1)}(\lambda) + k_n q_{k(2:n-2)}(\lambda) - \ell_n q_{\ell(2:n-2)}(\lambda) \\ &= \lambda q_{\ell(2:n-1)}(\lambda) - \lambda q_{k(1:n-2)}(\lambda) - \ell_1 q_{\ell(3:n-1)}(\lambda) \\ &\quad + k_{n-1} q_{k(1:n-3)}(\lambda) + k_n q_{k(2:n-2)}(\lambda) - \ell_n q_{\ell(2:n-2)}(\lambda). \end{aligned}$$

If ℓ is a cyclic shift of k, i.e., $\ell_j = k_{j-1}$, $j = 2, \dots, n$, and $\ell_1 = k_n$, then this last line is identically zero. Thus $p_\ell = p_k$ if ℓ is a cyclic permutation of k.

If $\ell = k'$ then that $p_k = p_\ell$ follows from (2.7) and Lemma 2.4. If $\ell = -k$ then that $p_\ell(\lambda) = i^{-n} p_k(i\lambda)$ follows from (2.6) and Lemma 2.4.

Call $k \in \{\pm 1\}^n$ *even* if $\prod_{j=1}^n k_j = 1$, and *odd* if $\prod_{j=1}^n k_j = -1$. Then [17, Corollary 5], it is immediate from Lemmas 2.2 and 2.5 that

$$\operatorname{spec} A_k^{per} = p_k^{-1}([-2,2]), \text{ if } k \text{ is even}, \quad \operatorname{spec} A_k^{per} = p_k^{-1}(i[-2,2]), \text{ if } k \text{ is odd}. \tag{2.8}$$

Complex dynamics. In Section 5 below we show that filled Julia sets, $K(p)$, of particular polynomials p, are contained in the periodic part π_∞ of the almost sure spectrum of the Feinberg-Zee random hopping matrix. To articulate and prove these results we will need terminology and results from complex dynamics.

Throughout this section p denotes a polynomial of degree ≥ 2. We have defined above the compact set that is the filled Julia set $K(p)$, the Julia set $J(p) = \partial K(p) \subset K(p)$, the Fatou set

If w is an attracting periodic point we denote by $A_p(w)$ the basin of attraction of the cycle $C = \{z_0, \ldots, z_{n-1}\}$ of z, by which we mean $A_p(w) := \{z \in \mathbb{C} : d(p^n(z), C) \to 0$ as $n \to \infty\}$. Here, for $S \subset \mathbb{C}$ and $z \in \mathbb{C}$, $d(z, S) := \inf_{w}$

fate under iterations of p of the components of the Fatou set it is helpful to understand the possible behaviours of a periodic component. This is achieved in the classification theorem (e.g. [2]). To state this theorem we introduce further terminology. Let us call a fixed component U of $F(p)$ a parabolic component if there exists a neutral fixed point $w \in \partial U$ with multiplier 1 such that the orbit of every $z \in U$ converges to w. Call a fixed component U of $F(p)$ a Siegel disk if it is conjugate to an irrational rotation on U, which means that there exists a conformal mapping $\varphi : U$

and so $p^n(z) \in S$ for some n. In the case that the orbit of z is eventually in a Siegel disk then also $p^n(z) \in S$ for some n for, if the orbit of every critical point $w \in K(p)$ is eventually in T, it follows that the boundary of every Siegel disk is in T, and (as S is simply connected) that every Siegel disk is in S.

Previous upper bounds on Σ. We have noted above that, if $c \in \Omega$ is pseudo-ergodic, then $\Sigma = \mathrm{spec}\, A_c \subset \overline{W(A_c)} = \overline{\Delta}$, given by (1.3). Similarly, the spectrum of A_c^2 is contained in the closure of its numerical range, so that[3]

$$\Sigma \subset \{\pm\sqrt{z} : z \in \mathrm{spec}(A_c^2)\} \subset N_2 := \{\pm\sqrt{z} : z \in \overline{W(A_c^2)}\}. \tag{2.12}$$

Clearly, f $_n$: n 2 Ng is a convergent family of upper bounds for that is in principle computable; deciding whether 2 $_n$ requires only computation of smallest singular values of n n matrices (see [4, (39)]). Explicitly $_1$ = 2 $\overline{D}$, and $_n$ is plotted for n = 6; 12; 18 in [4]. But for these values $_n$, and computing $_n$ for larger n is challenging, requiring computation of the smallest singular value of $2^{n \quad 1}$ matrices of order n to decide whether a particular 2 $_n$. Substantial numerical calculations in [4] established that 1:5 + 0:5i 62 $_{34}$, providing the rst proof that is a strict subset of $\overline{\ }$, this con rmed now by the simple explicit bound (2.12) and (2.13).

3. Lower Bounds on and Symmetries of and

Complementing the upper bounds on that we have just discussed, lower bounds on have been obtained by two methods of argument. The rst is that (1.2) tells us that specA_b for every b 2 . In particular this holds in the case when b is periodic, when the spectrum of A_b is given explicitly by Lemmas 2.2 and 2.5, so that, as observed in the introduction,

$$_n := \bigcup_{k2f \quad 1g^n} \mathrm{spec}A_k^{per} :$$

Explicitly [4, Lemma 2.6], in particular,

$$_1 = [\quad 2; 2] [\quad i[\quad 2; 2] \quad \text{and} \quad _2 = \quad _1 [f \ x \quad ix : \quad 1 \quad x \quad 1g: \qquad (3.1)$$

In the introduction we have de ned $_1$:= $[^1_{n=1}$ $_n$ and have termed := $\overline{\ _1\ }$, also a subset of since is closed, the periodic part of . We have also recalled the conjecture of [3] that = . Let

$$_n := \bigcup_{m=1}^{n} \quad _m \quad _1 :$$

Then it follows from Lemma 2.1 that

$$_n \% \quad \text{as } n ! 1 \quad : \qquad (3.2)$$

If, as conjectured, = , then (3.2) complements (2.16); together they sandwich by convergent sequences of upper ($_n$) and lower ($_n$) bounds that can both be computed by calculating eigenvalues of n n matrices. Figures 2 and 3 include visualisations of $_{30}$, indistinguishable by eye from $_{30}$, but note that the solid appearance of $_{30}$, which is the union of a large but nite number of analytic arcs, is illusory. See [3, 4] for visualisations of $_n$ for a range of n, suggestive that the convergence (3.2) is approximately achieved by n = 30.

The same method of argument (1.2) to obtain lower bounds was used in [3], where a special sequence b 2 was constructed with the property that spec A_b $\overline{D}$, so that, by (1.2), $\overline{D}$. The stronger result (1.5), that this new lower bound on is in fact also a subset of , was shown in [5], via a second method of argument for constructing lower bounds, based on surprising symmetries of and

. We will spell out in a moment these symmetries (one of these described rst in [5], the whole in nite family in [17]), which will be both a main subject of study and a main tool for argument in this paper. But rst we note more straightforward but important symmetries. In this lemma and throughout $\bar{}$ denotes the complex conjugate of $\in \mathbb{C}$.

Lemma 3.1. [4, Lemma 3.4] (and see [19], [5, Lemma 4]). All of $_n$, $_n$, , and are invariant with respect to the maps $\mapsto i$ and $\mapsto \bar{}$, and so are invariant under the dihedral symmetry group D_2 generated by these two maps.

To expand on the brief discussion in the introduction, [17] proves the existence of an in nite set S of monic polynomials of degree 2, this set de ned constructively in the following theorem, such that the elements $p \in S$ are symmetries of $_1$ and in the sense that (3.3) below holds.

Theorem 3.2. [17] Let S denote the set of those polynomials p_k, de ned by (2.5), with $k = (k_1; \ldots; k_n) \in \{ \quad 1\}^n$ for some n 2, for which it holds that: (i) $k_{n\ 1} =$ 1 and $k_n = 1$; (ii) $p_k = p_{\hat{k}}$, where $\hat{k} \in \{ \quad 1\}^n$ is the vector identical to k but with the last two entries interchanged, so that $\hat{k}_{n\ 1} = 1$ and $\hat{k}_n =$ 1. Then

$$p(\ ) \quad \text{and } p^{\ 1}(\ _1) \quad _1; \tag{3.3}$$

for all $p \in S$.

We will call S Hagger's set of polynomial symmetries for .

We remark that if $p \in S$ then it follows from (3.3), by taking closures and recalling that p is continuous, that also

$$p^{\ 1}(\quad) \qquad \text{and } p^{\ 1}(\text{int}(\quad)) \quad \text{int}(\quad): \tag{3.4}$$

We note also that $p^{\ 1}(\ _1)$ $_1$ implies that $_1$ $p(\ _1)$, but not vice versa, and that $p(\ )$ i

$$p^{\ 1}(\{\ \}) \cap \quad \neq ;; \quad \text{for all} \quad 2 \quad :$$

Further, we note that it was shown earlier in [5] that (3.3) holds for the particular case $p(\ ) = \ ^2$ (this the only element of S of degree 2, see Table 1); in [5] it was also shown, as an immediate consequence of (3.3) and Lemma 3.1, that

$$p^{\ 1}(\ ) \qquad ;$$

for $p(\ ) = \ ^2$. Whether this last inclusion holds in fact for all $p \in S$ is an open problem.

Our rst result is a much more explicit characterisation of S.

Proposition 3.3. The set S is given by $S = \{ p_k : k \in K \}$, where K consists of those vectors $k = (k_1; \ldots; k_n) \in \{ \quad 1\}^n$ with n 2, for which: (i) $k_{n\ 1} =$ 1 and $k_n = 1$; and (ii) $n = 2$, or n 3 and $k_j = k_{n\ j\ 1}$, for 1 j n 2, so that $(k_1; \ldots; k_{n\ 2})$ is a palindrome. Moreover, if $k \in K$, then

$$p_k(\ ) = \quad q_{k(1:n\ 2)}(\ ): \tag{3.5}$$

Proof. It is clear from Theorem 3.2 that what we have to prove is that, if $k \in \{\pm 1\}^n$ with $n \geq 2$ and $k_{n-1} = -1$, $k_n = 1$, then $p_k = p_{\tilde{k}}$ if $n = 2$ or 3; further, if $n \geq 4$, then $p_k = p_{\tilde{k}}$ iff $(k_1, \ldots, k_{n-2})$ is a palindrome.

If $k \in \{\pm 1\}^n$ with $n \geq 2$ and $k_{n-1} = -1$, $k_n = 1$, then, from (2.6) and (2.3),

$$\begin{aligned} p_k(\lambda) &= \lambda q_{k(1:n-1)}(\lambda) - k_n q_{k(2:n-2)}(\lambda) \\ &= \lambda^2 q_{k(1:n-2)}(\lambda) - k_{n-1} q_{k(1:n-3)}(\lambda) - k_n q_{k(2:n-2)}(\lambda) \\ &= \lambda^2 q_{k(1:n-2)}(\lambda); \end{aligned}$$

since $q_{k(1:n-3)}(\lambda) = q_{k(2:n-2)}(\lambda)$, this a consequence of the definitions (2.2) in the cases $n = 2$ and 3, of Lemma 2.4 in the case n

Proof. This follows easily by induction from (3.5) and (2.3).

We note that, using the standard trigonometric representations for the Chebychev polynomials [1], for $m \in \mathbb{N}$,

$$P_m(2\cos\theta) = 2\cos\theta\, U_{m-1}(\cos\theta) = 2\cot\theta\,\sin m\theta =: r_m(\theta). \qquad (3.6)$$

A similar representation in terms of hyperbolic functions can be given for the polynomial p_k when k has length $2m-1$ and $k_j = (-1)^j$; we denote this polynomial by Q_m. Clearly, for $m \geq 2$, $Q_m \in S$ by Proposition 3.3, and Q_m is an odd function by Lemma 2.5. The proof of the following lemma, like that of Lemma 3.5, is a straightforward induction that we leave to the reader.

Lemma 3.6. $Q_1(\lambda) = \lambda$, $Q_2(\lambda) = -\lambda^3 + \lambda$, and $Q_{m+1}(\lambda) = -\lambda^2 Q_m(\lambda) + Q_{m-1}(\lambda)$, for $m \geq 2$. Moreover, for $m \in \mathbb{N}$ and $\theta > 0$,

$$Q_m\left(\sqrt{2\sinh\theta}\right) = \begin{cases} \sqrt{2\sinh\theta}\,\dfrac{\sinh(m\theta) + \cosh((m-1)\theta)}{\cosh\theta}; & \text{if } m \text{ is even},\\[2ex] \sqrt{2\sinh\theta}\,\dfrac{\cosh(m\theta) + \sinh((m-1)\theta)}{\cosh\theta}; & \text{if } m \text{ is odd}. \end{cases}$$

The following lemma leads, using Lemmas 3.5 and 3.6, to explicit formulae for other polynomials in S. For example, if $\tilde{P}_m$ denotes the polynomial p_k when k has length $m \geq 2$, $k_{m-1} = -1$, $k_m = 1$, and all other entries are -1's, then, by Lemmas 3.5 and 3.7,

$$\tilde{P}_m(\lambda) = i^{-m} P_m(i\lambda) = i^{1-m}\lambda\, U_{m-1}(i\lambda/2). \qquad (3.7)$$

Lemma 3.7. If $k \in \{\pm 1\}^n$ and $p_k \in S$, then $p_{-k} \in S$ and $p_{-k}(\lambda) = i^{-n} p_k(i\lambda)$.

Proof. Suppose that $k \in \{\pm 1\}^n$ and $p_k \in S$. If $n = 2$, then $\hat{k} = -k$ and $p_{-k} = p_{\hat{k}} = p_k \in S$. If $n \geq 3$, defining $\ell \in \{\pm 1\}^n$ by $\ell_{n-1} = -1$, $\ell_n = 1$, and $\ell_j = -k_j$, for $j = 1, \ldots, n-2$, $p_\ell \in S$ by Proposition 3.3, so that $p_{-k} = p_{\hat{\ell}} = p_\ell \in S$. That $p_{-k}(\lambda) = i^{-n} p_k(i\lambda)$ comes from Lemma 2.6.

We note that Proposition 3.3 implies that there are precisely $2^{\lceil \frac{n}{2} \rceil - 1}$ vectors of length n in K, so that there are between 1 and $2^{\lceil \frac{n}{2} \rceil - 1}$ polynomials of degree n in S, as conjectured in [17]. Note, however, that there may be more than one $k \in K$ that induce the same polynomial $p_k \in S$. For example, $p_k(\lambda) = \lambda^6 - \lambda^2$ for $k = (-1; 1; 1; -1; -1; 1)$, and, defining $\ell = (1; -1; -1; 1; -1; 1)$ and using Lemma 3.7, also

$$p_\ell(\lambda) = p_{\hat{\ell}}(\lambda) = p_{-k}(\lambda) = -p_k(i\lambda) = \lambda^6 - \lambda^2.$$

In Table 1 (cf. [17]) we tabulate all the polynomials in S of degree ≤ 6.

If $p, q \in S$, so that p and q are polynomial symmetries of Σ in the sense that (3.3) holds, then also the composition $p \circ q$ is a polynomial symmetry of Σ in the same sense. But note from Table 1 that, while $P_3 \circ P_2 \in S$, none of $P_2 \circ P_2$, $P_2 \circ P_2 \circ P_2$, $Q_2 \circ P_2$, $P_2 \circ P_3$, or $P_2 \circ Q_2$ are in S. Thus S does not contain all polynomial symmetries of Σ, but whether there are polynomial symmetries that are not either in S or else compositions of elements of S is an open question.

k	$p_k( )$
(1; 1)	$^2 = P_2( )$
(1; 1; 1)	$^3 = P_3( )$
(1; 1; 1)	$^3 + = Q_2( ) = P_3( )$
(1; 1; 1; 1)	$^4 \; 2 \; ^2 = P_4( )$
(1; 1; 1; 1)	$^4 + 2 \; ^2 = P_4( )$
(1; 1; 1; 1; 1)	$^5 \; 3 \; ^3 + = P_5( )$
(1; 1; 1; 1; 1)	$^5 \; ^3 + = iQ_3(i)$
(1; 1; 1; 1; 1)	$^5 + \; ^3 + = Q_3( )$
(1; 1; 1; 1; 1)	$^5 + 3 \; ^3 + = P_5( )$
(1; 1; 1; 1; 1; 1)	$^6 \; 4 \; ^4 + 3 \; ^2 = P_6( )$
(1; 1; 1; 1; 1; 1)	$^6 \; ^2 = P_3(P_2( ))$
(1; 1; 1; 1; 1; 1)	$^6 + 4 \; ^4 + 3 \; ^2 = P_6( )$

Table 1. The elements p_k 2 S of degree 6.

We nish this section by showing in subsection 3.1 the surprising result that S 1

Proposition 3.9. *Suppose $a, b, c, d \in \{\pm 1\}$ and $k \in \{\pm 1\}^n$, for some $n \geq 2$, and let $\tilde{k} := (k_1, \ldots, k_{n-1})$. Then*

$$\operatorname{spec} A_k^{(n)} \subset \operatorname{spec} A_\ell^{per}, \quad \text{for } \ell = (\tilde{k}', a, b, \tilde{k}, c, d) \in \{\pm 1\}^{2n+2}, \tag{3.9}$$

where $\tilde{k}' = \tilde{k} J_{n-1} = (k_{n-1}, \ldots, k_1)$. Further, $\operatorname{spec} A_k^{(n)} \subset \bigcup_{2n+2}$.

Proof. The proof modifies [4, Theorem 4.1] where the same result is proved for the special case that $a = -c = -1$, $b = -d = 1$. Following that proof, suppose that λ is an eigenvalue of $A_k^{(n)}$

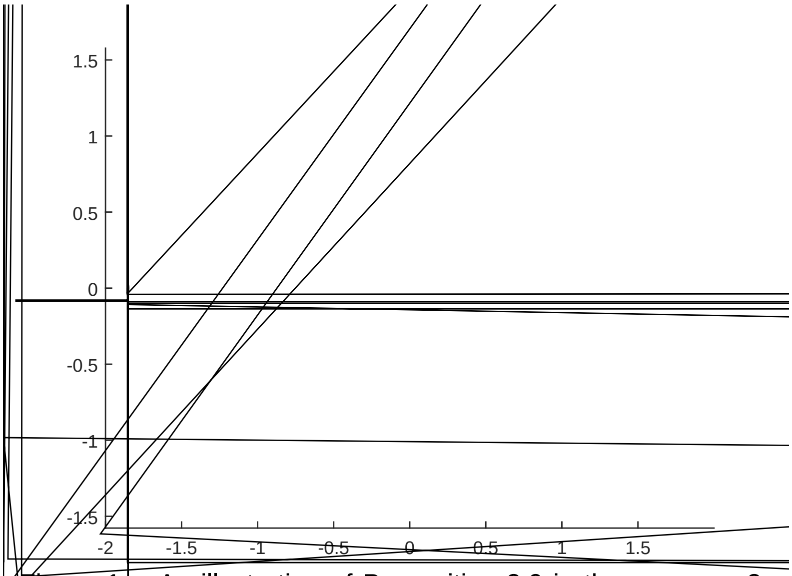

Figure 1. An illustration of Proposition 3.9 in the case $n = 3$, $k_1 = k_2 = 1$. The red circles indicate the eigenvalues, 0 and $\pm\sqrt{2}$, of $A_k^{(3)}$. The black lines are the spectra of A_ℓ^{per}, for the different choices of ℓ defined by (3.9). In this case there are 7 distinct polynomials p_ℓ and 7 associated distinct spectra spec A_ℓ^{per}, each of which contains the eigenvalues of $A_k^{(3)}$. One cannot see all the spectra as separate curves because some of them overlap.

spectrum spec A_ℓ^{per}. In particular, if $a = b = \pm 1$, then the choices $c = d = 1$ and $c = d = -1$ lead to the same polynomial by Proposition 3.3 and the definition of S. But, if $a \neq b$, again by Proposition 3.3 and the definition of S, the choices $c =$ (3)

nite subset of S. Since, by (1.5), D , it follows from (3.3) that all these sets are subsets of .

It is easy to check that

$$\bigcup_{m=0}^{3} i^m (W_1 \cup W_2) = E \cup iE \cup \sqrt{i}E \quad \mathrm{int}(\ ). \tag{3.12}$$

Next note from Table 1 that $p \in S$ where $p(\ ) := \ ^5 \ \ ^3 + \ $ factorises as

$$p(\ ) = \ (\ \ e^{i/6})(\ + e^{i/6})(\ \ e^{i/6})(\ + e^{i/6}).$$

Thus, for $\ = \exp(\ i/6) + w$ with $|w| \ \ $,

$$|p(\ )| \ \ (1+\ )\ (2+\ )(2\sin(\ /6)+\ )(2\cos(\ /6)+\ ) = \ (1+\ )^2(\sqrt{3}+\ )(2+\ ) =: g(\ ).$$

Let $\ \ 0.174744$ be the unique positive solution of $g(\ ) = 1$. Clearly $|p(\ )| < 1$ if $\ = \exp(\ i/6) + w$, with $|w| < \ $, so that

$$\exp(\ i/6) + \ D \quad p^{-1}(D) \quad \mathrm{int}(\ ). \tag{3.13}$$

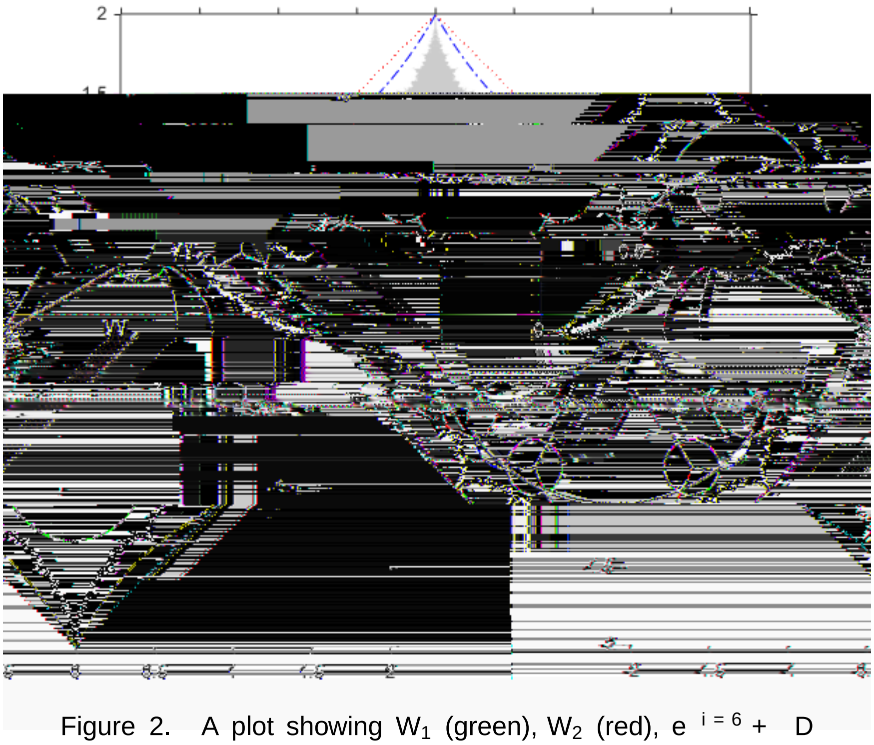

Figure 2. A plot showing W_1 (green), W_2 (red), $e^{i=6}$ + D (blue), and their rotations by multiples of = 2. The union of the green, red, and blue regions is W , defined by (3.14). W contains $1{:}1\overline{D}$, indicated by the black circle, see Proposition 3.13. In the background in grey one can see $_{30}$. The dotted and dashed-dotted curves are the boundaries of and N_2, respectively, defined by (1.3) and (2.12), with N_2 .

last inequality holds since $\cos(=24) = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}} > 0{:}991$ and $g(0{:}174) < 1$ so that $> 0{:}174$.

4. Interior points of

We have just, in Proposition 3.13, extended to a region W $1{:}1\overline{D}$ the part of the complex plane that is known to consist of interior points of . In this section we explore the relationship between and its interior further. We show first of all, using (3.4) and that P_n 2 S for every n 2, that [0; 2) int( ). Next we use this result to show that, for every n 2, all but finitely many points in $\overset{S}{n}$ are interior points of . Finally, we prove, using Theorem 3.8, that is the closure of its interior. If indeed it can be shown, as conjectured in [3], that $\pi_\infty = \Sigma$, then the result will imply the truth of another conjecture in [3], that Σ is the closure of its interior.

Our technique for establishing that $[0,2) \subset \mathrm{int}(\pi_\infty)$ will be to use that $P_m^{-1}((-1,1)) \subset \mathrm{int}(\pi_\infty)$, for every $m \geq 3$, this a particular instance of (3.4). This requires first a study of the real solutions of the equations $P_m(\lambda) = \pm 1$ and their interlacing properties, which we now undertake.

From (3.6),

$$P_m(2) = 2m; \quad P_m(2\cos(\pi/m)) = 0; \quad \text{and} \quad P_m(2\cos(3\pi/(2m))) = -2\cot(3\pi/(2m)):$$

This implies that the equation $P_m(\lambda) = 1$ has a solution in $(2\cos(\pi/m), 2)$. Let λ_m^+ denote the largest solution in this interval. Further, if $m \geq 5$ then $P_m(\lambda) = -1$ has a solution in $(2\cos(3\pi/(2m)), 2)$ since $-2\cot(3\pi/(2m)) < -1$. For $m \geq 4$ let λ_m^- denote the largest solution to $P_m(\lambda) = -1$ in $(0,2)$, which is in the interval $(2\cos(3\pi/(2m)), 2)$ if $m \geq 5$, while an explicit calculation gives that $\lambda_4^- = 1$.

Throughout the following calculations we use the notation $r_n(\theta)$ from (3.6).

Lemma 4.1. For $m \geq 4$ it holds that $P_m'(\lambda) > 0$ for $\lambda_m^- < \lambda < 2$, that $\lambda_m^- < \lambda_m^+$, and that $-1 < P_m(\lambda) < 1$ for $\lambda_m^- < \lambda < \lambda_m^+$.

Proof. Explicitly $P_4(\lambda) = \lambda U_3(\lambda/2) = \lambda^4 - 2\lambda^2$, so that $P_4'(\lambda) = 4(\lambda^3 - 1)$ and these claims are clear for $m = 4$.

Suppose now that $m \geq 5$. We observe first that, by induction, it follows that, for $n \in \mathbb{N}$, $r_n(\theta)$ is strictly decreasing on $(0, \pi/n + \pi/n^2)$. For $r_1(\theta) = 2\cos\theta$, so that this is clearly true for $n = 1$, and if it is true for some $n \in \mathbb{N}$ then

$$r_{n+1}(\theta) = 2\cot\theta - \sin((n+1)\theta)/\cos\theta\,(r_n(\theta) + 2\cos n\theta)$$

is strictly decreasing on $(0, \pi/(n+1) + \pi/(n+1)^2) \subset (0, \pi/n)$. We observe next that

$$\begin{aligned} r_m(\pi/m + \pi/m^2) &= -2\cos(\pi/m + \pi/m^2)\sin(\pi/m)/\sin(\pi/m + \pi/m^2) \\ &< -\frac{2m\cos(\pi/5 + \pi/25)}{m+1} < -10\cos(6\pi/25)/6 < -1; \end{aligned}$$

where we have used that $\sin a/\sin b > a/b$ for $0 < a < b < \pi$. Since $r_m(\theta) = P_m(2\cos\theta)$, these observations imply that, on $(2\cos(\pi/m + \pi/m^2), 2)$, $P_m(\lambda)$ is strictly increasing, and that $\lambda_m^- > 2\cos(\pi/m + \pi/m^2)$. Thus $P_m'(\lambda)$ λ_m^- λ_m^+

Lemma 4.3. For m

As an example of the above lemma, suppose that $k = (\; 1;1) \; 2 \; K_2$. Then (see Table 1) $p_k(\;) = \; ^2$ and, from (2.8), $\mathrm{spec} A_k^{per} = f x \; ix : \; 1 \; x \; 1g$. There are precisely four points, $1 \; i \; 2 \; \mathrm{spec} A_k^{per} \; n \, \mathrm{int}(\;)$. These are not interior points of since they lie on the boundary of $\overline{}$.

Combining the above lemma with Theorem 3.8, we obtain the last result of this section.

Theorem 4.6. is the closure of its interior.

Proof. Suppose 2 . Then, by Theorem 3.8, is the limit of a sequence $(\; _n) \; \overset{S}{_1}$, and, by Lemma 4.5, for each n there exists $\; _n \; 2 \; \mathrm{int}(\;)$ such that $j \; _n \; \; _n j < n \; ^1$, so that $\; _n \; ! \;$ as $n \; ! \; 1$.

5. Filled Julia sets in

It was shown in [17] that, for every polynomial symmetry $p \; 2 \; S$, the corresponding Julia set $J(p)$ satisfies $J(p) \; \overline{U(p)}$, where $U(p)$ is defined by (1.7). (The argument in [17] is that $J(p) \; \overline{U(p)}$ by (2.10), and that $\overline{U(p)}$ by (3.4).) It was conjectured in [17] that also the filled Julia set $K(p) \; \overline{U(p)}$, for every $p \; 2 \; S$. In this section we will first show by a counterexample that this conjecture is false; we will exhibit a $p \; 2 \; S$ of degree 18 for which $K(p) \; 6 \; \overline{U(p)}$. However, we have no reason to doubt a modified conjecture, that $K(p)$, for all $p \; 2 \; S$. And the main result of this section will be to prove that $K(p)$ for a large class of $p \; 2 \; S$, including $p = P_m$, for $m \; 2$.

Our first result is the claimed counterexample.

Lemma 5.1. Let $k = (1; \; 1;1;1;1; \; 1;1; \; 1; \; 1;1; \; 1;1;1;1; \; 1;1; \; 1;1)$, so that (by Proposition 3.3) $p_k \; 2 \; S$, this polynomial given explicitly by

$$p_k(\;) = \; ^{18} \; 4 \; ^{16} + 16$$

for 1.215 1.216. But this implies that $|p'(\ )|$ $|p'(1.2155)| + 0.0005$ 400 0.91, so that is an attracting fixed point.

Numerical results suggest that amongst the polynomials $p \in S$ of degree 20, there is only one other similar counterexample of a polynomial with an attracting fixed point outside the unit disk, the other example of degree 19.

We turn now to positive results. Part of our argument will be to show, for every $p \in S$, that f z, that f z0005

in Section 2, $J(p) = \partial K(p) = \partial A_p(0) = \partial A_p(\infty)$, and, since $K(p)$ has more than one component, $F_B(p)$ has infinitely many components [2, Theorem IV 1.2].

The above example is a particular instance of a more general result. It is straightforward to see that if p is a polynomial with zeros only on the real line, then all the critical points are also on the real line. Since, by Lemma 3.5, $P_m(\lambda) = U_{m-1}(\lambda/2)$, and all the zeros of the polynomial U_{m-1} are real, it follows that all the zeros of P_m are real, so all its critical points are also real, and so the orbits of all the critical points are real. Further, by Corollary 5.3, the orbits of the critical points in $K(p)$ stay in $(-2;2)$. Likewise, as (see (3.7)) $P_m(\lambda) = i^{-m} P_m(i\lambda)$, all the critical points of P_m lie on $i\mathbb{R}$, and so the orbits of these critical points are real if m is even, pure imaginary if m is odd. Further, by Corollary 5.3, the orbits of the critical points in $K(p)$ stay in $(-2;2) \cup i(-2;2)$. Applying Theorem 5.5 we obtain:

Corollary 5.6. $K(P_m)$ and $K(P_m)$, for $m \geq 2$.

Numerical experiments carried out for the polynomials $p \in S$ of degree 6 (exhibited in Table 1) appear to confirm that these polynomials satisfy the conditions of Theorem 5.5, i.e., it appears

; 2) i(Applying [

$_1$ and to and and their boundaries. Further, [17] has shown that contains the Julia sets of all polynomials in S, and Proposition 5.5 and Corollary 5.6 show that contains the lled Julia sets, many of which have fractal boundaries, of the polynomials in an in nite subset of S.

Regarding these polynomial symmetries we make two further conjectures:

5. $K(p)$ for all $p \in S$.
6. $p^{-1}(\ ) \subset$ $p \in S$ all

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Acknowledgments

We thank our friend and scienti c collaborator Marko Lindner for introducing us to the study of, and for many discussions about, this beautiful matrix class.

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