School of Mathematical
and Physical Sciences

University of Reading

Department of Mathematics and Statistics

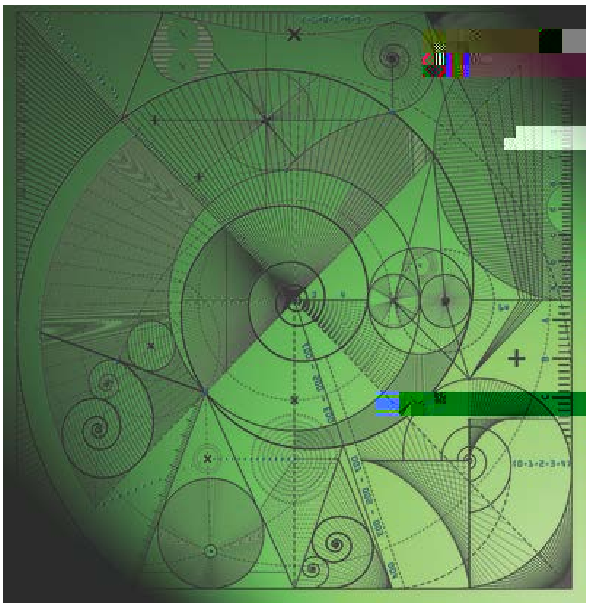
Preprint MPS-2015-07

24 May 2015

Regularization of Descriptor Systems

by

Nancy K. Nichols and Delin Chu



Regularization of Descriptor Systems

Nancy K. Nichols and Delin Chu

Abstract Implicit dynamic-algebraic equations, known in control theory as descriptor systems, arise naturally in many applications. Such systems may not be regular (often referred to as singular). In that case the equations may not have unique solutions for consistent initial conditions and arbitrary inputs and the system may not be controllable or observable. Many control systems can be regularized by proportional and/or derivative feedback. We present an overview of mathematical theory and numerical techniques for regularizing descriptor systems using feedback controls. The aim is to provide stable numerical techniques for analyzing and constructing regular control and state estimation systems and for ensuring that these systems are robust. State and output feedback designs for regularizing linear time-invariant systems are described, including methods for disturbance decoupling and mixed output problems. Extensions of these techniques to time-varying linear and nonlinear systems are discussed in the final section.

1 Introduction

Singular systems of differential equations, known in control theory as *descriptor systems* or *generalized state-space systems*, have fascinated Volker Mehrmann throughout his career. His early research, starting with his habilitation [33, 35],

Nancy K. Nichols
Department of Mathematics, University of Reading, Box 220, Reading, RG6 2AX, UK e-mail: n.nichols@rdg.ac.uk

Delin Chu
Department of Mathematics, National University of Singapore, Singapore e-mail: matchudl@nus.edu.sg

[1] **Published in:** *Numerical Algebra, Matrix Theory, Differential-Algebraic Equations and Control Theory*, (eds P. Benner, M. Bollhoefer, D. Kressner, C. Mehl and T. Stykel), Springer International Publishing, Switzerland, pp. 415–433, 2015. doi:10.1007/978-3-319-15260-8_15

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. Here $x(\cdot)$ is the state, $y(\cdot)$ is the output, and $u(\cdot)$ is the input or control of the system. It is assumed that $m, p \le n$ and that the matrices B, C are of full rank. The matrix E may be *singular*. Such systems are known as *descriptor* or *generalized state-space* systems. In the case $E = I$, the identity matrix, we refer to (1) or (2) as a *standard* system.

We assume initially that the system is time-invariant; that is, the system matrices E, A, B, C are constant, independent of time. In this context, we are interested in proportional and derivative feedback control of the form $u(t) = Fy(t) - G\dot{y}(t) + v(t)$ or $u(k) = Fy(k$

$$\text{rank}\left(\begin{bmatrix} E+BG \\ T_\infty^T (E+BG)(A+BF) \end{bmatrix}\right) = n. \tag{13}$$

$$(E+GC, A+FC), \tag{15}$$

where the matrices F and G must be selected to ensure that the response $\hat{x}$ of the observer converges to the system state x for any arbitrary starting condition; that is, the system must be asymptotically stable. By duality with the state feedback problem, it follows that if the condition O2 holds, then the matrices F and G can be chosen such that the corresponding closed-loop pencil (15) is regular and of index at most one. If condition O1 also holds, then the closed-loop system is S-observable. Furthermore, the remaining freedom in the system can be used to ensure the stability and robustness of the system and the finite eigenvalues of the system pencil can be assigned explicitly by the techniques described for the state feedback control problem.

3.2 Disturbance Decoupling by State Feedback

In practice control systems are subject to disturbances tha

$$UEV = \begin{array}{c c} & \begin{array}{ccccc} t_1 & t_2 & t_3 & s_4 & s_5 \end{array} \\ \begin{array}{c} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} & \begin{bmatrix} E_{11} & 0 & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 \\ E_{31} & E_{32} & E_{33} & E_{34} & 0 \\ E_{41} & E_{42} & 0 & E_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array};$$

$$UBP = \begin{array}{c c} & \begin{array}{cc} t_3 & t_4 \end{array} \\ \begin{array}{c} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ B_{31} & B_{32} \\ 0 & B_{42} \\ 0 & 0 \end{bmatrix} \end{array}; \tag{19}$$

$$\begin{array}{ccc} t_1 & t_2 & t_3 \end{array}$$

$$WCV =$$

$$E + BG\Gamma = \begin{bmatrix} M + B_1 G & 0 \\ K + B_2 G & I \end{bmatrix}, \quad A + BFC = \begin{bmatrix} 0 & I + B_1 F \\ P & B_2 F \end{bmatrix}. \tag{26}$$

Different effects can, therefore, be achieved by feeding back either the derivatives $\dot{x}$ or the states x. In particular, in the case where M is singular, but rank $[M, B_1] = n$, the feedback G can be chosen such that $M + B_1 G$ is invertible and well-conditioned [7], giving a *robust* closed-loop system that is regular and of index zero. The feedback matrix F can be chosen separately to assign the eigenvalues of the system [30], for example, or to achieve other objectives.

The complete solution to the mixed output feedback regularization problem is given in [22]. The theorem and its proof are very technical. Solvability is established using condensed forms derived in the paper. The solution to the output feedback problem given in Theorem 4 is a special case of the complete result for the mixed output case given in [22]. The required feedback matrices are constructed directly from the condensed forms using numerically stable transformations.

Usually the design of the feedback matrices still contains freedom, however, which can be resolved in many different ways. One choice is to select the feedbacks such that the closed-loop system is robust, or insensitive to perturbations, and, in particular, such that it remains regular and of index at most one under perturbations (due, for example, to disturbances or parameter variations). This choice can also be shown to maximize a lower bound on the stability radius of the closed-loop system [13]. Another natural choice would be to use minimum norm feedbacks, which would be a least squares approach based on the theory in [24]. This approach is also

$$
\begin{aligned}
E(t)x(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \\
y(t) &= C(t)x(t),
\end{aligned}
\tag{27}
$$

where $E(t), A(t) \in \mathbb{R}^{n\times n}, B(t) \in \mathbb{R}^{n\times m}, C(t) \in \mathbb{R}^{p\times n}$ are all *continuous* functions of time and $x(t)$ is the state, $y(t)$ is the output, and $u(t)$ is the input or control of the system. (Corresponding discrete-time systems with time-varying coefficients can also be defined, but these are not considered here.)

In this general form, complex dynamical systems including constraints can be modelled. Such systems arise, in particular, as linearizations of a general nonlinear control system of the form

$$
\begin{aligned}
\mathcal{F}(t,x,x,u) &= 0, \qquad x(t_0) = x_0, \\
y &= \mathcal{G}(t,x),
\end{aligned}
\tag{28}
$$

where the linearized system is such that $E(t), A(t), B(t)$ are given by the Jacobians of $\mathcal{F}$ with respect to x, x, u, respectively, and $C(t)$ is given by the Jacobian of $\mathcal{G}$ with respect to x (see [31]).

For the time-varying system (27) and the nonlinear system (28), the system properties can be modified by time-varying state and output feedbacks as in the time-invariant case, but the characterization of the system, in particular the solvability and regularity of the system, is considerably more complicated to define than in the time-invariant case and it is correspondingly more difficult to analyse the feedback

$E; A; B; C;$ are assumed to be analytic functions of t, but these conditions can be relaxed provided the ASVD decompositions remain suf ciently smooth.

In the papers [12, 31], a much deeper analysis of the regularization problem is developed. Detailed solvability conditions for the time-varying system (27) are established and different condensed forms are derived, again using the ASVD. Constant rank assumptions do not need to be applied, although existence of smooth ASVDs are required. The analysis covers a plethora of different possible behaviours of the system. One of the tasks of the analysis is to determine redundancies and inconsistencies in the system in order that these may be excluded from the design process. The reduction to the condensed forms displays all invariants that determine the existence and uniqueness of the solution. The descriptor system is then de ned to be regularizable if there exist proportional or derivative feedback matrices such that the closed-loop system is uniquely solvable for every consistent initial state vector and any given (ilos08(i).46tnnsmmooth.sy eerensyho

5 Conclusions

We have given here a broad-brush survey of the work of Volker Mehrmann on the problems of regularizing descriptor systems. The extent of this work alone is formidable and forms only part of his research during his career. We have concentrated speci cally on results from Volker's own approaches to the regularity problem. The primary aim of his work has been to provide stable numerical techniques for analyzing and constructing control and state estimation systems and for ensuring that these systems s1ns1nwsons. The s tg e o

37. Mehrmann, V., Nichols, N.K.: Mixed output feedback for descriptor systems. Technical Report SPC 95-37, DFG Forschergruppe SPC, Fakultät für Mathematik, TU Chemnitz-Zwickau, D-09107 Chemnitz, FRG, 1995.
38. Nichols, N.K.: Robust control system design for generalised state-space systems. 25th IEEE Conference on Decision and Control, Institute of Electrical and Electronic Engineers, Vol. 1, 538-540, (1986).
39. Nichols, N.K.: Dynamic-algebraic equations and control system design. In: Numerical Analysis–1993, (eds D.F. Griffith and G.A. Watson), Longman Scientific and Technical, pp. 208–224, (1994).
40. Pearson, D.W., Chapman, M.J., Shields, D.N.: Partial singular value assignment in the design of robust observers for discrete time descriptor systems. IMA J. Math. Control Inform., *5*, 203–213 (1988)
41. Rosenbrock, H.H.: Structural properties of linear dynamic systems. Internat. J. Control, *20*, 191–202 (1974)
42. Verghese, G.C., Van Dooren, P., Kailath, T.: Properties of the system matrix of a generalized state space system. Internat. J. Control, *30*, 235–243 (1979)
43. Yip, E.L., R.F. Sincovec, R.F.: Solvability, controllability and observability of continuous descriptor systems. IEEE Trans. Automat. Control, *AC–26*, 702–707 (1981)