

While the literature on Treitz-type elements for time-harmonic wave propagation problems is nowadays quite developed (see e.g. [4, 6, 11, 14, 23, 29, 32] for different approaches using Treitz-type basis functions and e.g. [2, 15, 19, 20] for theoretical analysis

in Remark 2.1 (in one space dimension) and again in Remark 3.4 (in any space dimension).
 Given a space domain $\Omega = (x_L; x_R)$ and a time domain $I = (0; T)$, we set $Q := \Omega \times I$.
 We denote by $n_Q = (n_Q^x; n_Q^t)$ the outward pointing unit normal vector on ∂Q .
 We assume the electric permittivity $\epsilon = \epsilon(x)$

3.1 Mesh and DG notation

We introduce a mesh $\mathcal{T}_h$ on Q , such that its elements are rectangles with sides parallel to the space and time axes, and all the discontinuities of the parameters α and β lie on interelement boundaries (note that the method described in this paper can be generalised to allow discontinuities lying inside the elements as in [25]). The mesh may have hanging nodes.

We denote with $F_h = \bigcup_{K \in \mathcal{T}_h} \partial K$ the mesh skeleton and its subsets:

$F_h^{\text{hor}} :=$ the union of the internal horizontal element sides ($t = \text{constant}$);

$F_h^{\text{ver}} :=$ the union of the internal vertical element sides ($x = \text{constant}$);

$F_h^0 := [x_L; x_R] \times \{0\}$;

$F_h^T := [x_L; x_R] \times \{T\}$;

$F_h^L := \{x_L\} \times [0; T]$;

$F_h^R := \{x_R\} \times [0; T]$;

We define the following broken Sobolev space:

$$H^1(\mathcal{T}_h) := \dots \quad .12:$$

where

a

$$a_{\text{T DG}}^{\text{wave}}(v_{\text{hp}}; \quad_{\text{hp}}; w; \quad) = \quad_{\text{T DG}}^{\text{wave}}(w; \quad) - 8(w; \quad)^2 V_p(T_h); \quad (11)$$

with

$$\begin{aligned} a_{\text{T DG}}^{\text{wave}}(v_{\text{hp}}; \quad_{\text{hp}}; w; \quad) := & \int_{F_h^{\text{hor}}} c^2 v_{\text{hp}} [\![w]\!]_t + \quad_{\text{hp}} [\![\quad]\!]_t \, dx + \int_{F_h^{\text{T}}} (c^2 v_{\text{hp}} w + \quad_{\text{hp}} \quad) \, dx \\ & + \int_{F_h^{\text{ver}}} f v_{\text{hp}} \vartheta [\![\quad]\!]_N + f \quad_{\text{hp}} \vartheta [\![w]\!]_N + [\![v_{\text{hp}}]\!]_N [\![w]\!]_N + [\![\quad_{\text{hp}}]\!]_N [\![\quad]\!]_N \, dS \\ & + \int_{F_h^{\text{D}}} n \quad + v_{\text{hp}} w \, dS; \\ \quad_{\text{T DG}}^{\text{wave}}(w; \quad) := & \int_{F_h^0} (c^2 v_0 w + \quad_0 \quad) \, dx + \int_{F_h^{\text{D}}} g(w \quad n) \, dS; \end{aligned}$$

where $\quad_0 = r \quad U(\quad; 0)$ and $v_0 = \frac{\partial U}{\partial t}(\quad; 0)$ are (given) initial data. Here, the jumps are defined as follows: $[\![w]\!]_t := (w \quad w^+)$ and $[\![\quad]\!]_t := (\quad^+)$ on horizontal faces, $[\![w]\!]_N := w_{j_{K_1}} n_{K_1}^x + w_{j_{K_2}} n_{K_2}^x$ and $[\![\quad]\!]_N := \quad_{j_{K_1}} n_{K_1}^x + \quad_{j_{K_2}} n_{K_2}^x$ on vertical faces.

In particular, the bilinear form $a_{TDG}(\cdot; \cdot)$ is coercive in the space $T(T_h)$ with respect to the DG norm, with coercivity constant equal to 1.

Proof. Using the elementwise integration by parts in time and space

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial F}{\partial t} dx dt &= \sum_{F_h^{hor}} \llbracket F \rrbracket_t dx + \sum_{F_h^T} F dx - \sum_{F_h^0} F dx \\ \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial F}{\partial x} dx dt &= \sum_{F_h^{ver}} \llbracket F \rrbracket_x dt + \sum_{F_h^R} F dt - \sum_{F_h^L} F dt = 8F^2 W^{1,1}(T_h); \end{aligned} \quad (14)$$

and the jump identity

$$v \llbracket v \rrbracket_t - \frac{1}{2} \llbracket v^2 \rrbracket_t = \frac{1}{2} \llbracket v \rrbracket_t^2 \quad \text{on } F_h^{hor}; \quad \forall v \in H^1(T_h); \quad (15)$$

we obtain the identity in the assertion:

$$\begin{aligned} a_{DG}(v_E; v_H; v_E; v_H) &\stackrel{(7)}{=} \sum_{K \in \mathcal{T}_h} \int_K \frac{1}{2} \frac{\partial}{\partial t} (v_E^2 + v_H^2) + \frac{\partial}{\partial x} (v_E v_H) dx dt \\ &\quad + \sum_{F_h^{hor}} (v_E \llbracket v_E \rrbracket_t + v_H \llbracket v_H \rrbracket_t) dt \end{aligned}$$

Proof. To prove uniqueness, assume that $E_L = E_R = E_0 = H_0 = 0$. Proposition 4.2 implies $E_{hp} = H_{hp} = 0$. Existence follows from uniqueness. For (16), the triangle inequality gives

$$||| (E; H) - (E_{hp}; H_{hp}) |||_{DG} \leq ||| (E; H) - (v_E; v_H) |||_{DG} + ||| (E_{hp}; H_{hp}) - (v_E; v_H) |||_{DG} \quad (17)$$

for all $(v_E; v_H) \in V_p(T_h)$. Since $(E_{hp}; H_{hp}) - (v_E; v_H) \in V_p(T_h) \cap T(T_h)$, Proposition 4.2, consistency (which follows by construction and from the consistency of the numerical fluxes), and Proposition 4.3 give

$$||| (E_{hp}; H_{hp}) - (v_E; v_H) |||_{DG}^2 = a_{TDG} (E - v_E; H - v_H; E_{hp} - v_E; H_{hp} - v_H) \\ - 2 ||| (E; H) - (v_E; v_H) |||_{DG}^2$$

for $(\mathbf{v}_E; \mathbf{v}_H) \in \mathbf{L}^2(Q)$. More precisely, we will need a bound on the $\mathbf{L}^2$ norm of the traces of $\mathbf{v}_E$ and $\mathbf{v}_H$ on horizontal and vertical segments in terms of the $\mathbf{L}^2(Q)$ norm of $(\mathbf{v}_E; \mathbf{v}_H)$:

$$\begin{aligned} & \| \mathbf{v}_E \|_{\mathbf{L}^2(Q)}^2 + \| \mathbf{v}_H \|_{\mathbf{L}^2(Q)}^2 \\ & + \| \mathbf{v}_E \|_{\mathbf{L}^2(Q)}^2 + \| \mathbf{v}_H \|_{\mathbf{L}^2(Q)}^2 \\ & C_{\text{stab}}^2 \| \mathbf{v}_E \|_{\mathbf{L}^2(Q)}^2 + \| \mathbf{v}_H \|_{\mathbf{L}^2(Q)}^2 \leq 8 \| (\mathbf{v}_E; \mathbf{v}_H) \|_{\mathbf{L}^2(Q)}^2; \end{aligned} \quad (19)$$

for some $C_{\text{stab}} > 0$. We have inserted the numerical flux parameters within the third and fourth term on the left-hand side of (19) because this is what we need in the proof of Proposition 4.7 below; then the constant C_{stab} will also depend on α and β .

Proposition 4.7. Assume that the estimate (19) holds true for $(\mathbf{v}_E; \mathbf{v}_H)$ solution of problem (18). Then, for any Tre tz function $(\mathbf{w}_E; \mathbf{w}_H) \in \mathbf{T}(T_h)$, the $\mathbf{L}^2(Q)$ norm is bounded by the DG norm:

$$\| \mathbf{w}_E \|_{\mathbf{L}^2(Q)}^2 + \| \mathbf{w}_H \|_{\mathbf{L}^2(Q)}^2 \leq \frac{1}{2} C_{\text{stab}} \| (\mathbf{w}_E; \mathbf{w}_H) \|_{\text{DG}}^2;$$

with C_{stab} as in (19).

Proof. Let $(\mathbf{v}_E; \mathbf{v}_H)$ be the solution of the auxiliary problem (18)

we obtain the desired estimate.

□

Recalling that the error $((E - E_{hp}); (H - H_{hp})) \leq T(T_h)$, and combining Proposition [4.7](#) and the quasi-optimality in DG norm proved in Theorem [4.10](#)

which in turns implies

$$E(t) = E(0) + \sum_{j=1}^Z \int_{(0;t)} v_E + v_H \, d$$

$$\begin{aligned}
& \stackrel{(18)}{=} \frac{2A}{h^x} \sum_{K \in \mathcal{T}_h} \int_K \left(\nabla_E^2 v_E + \nabla_H^2 v_H + 2(\mathbf{x} \cdot \nabla_K) \frac{\partial}{\partial t} (v_E v_H) + \nabla_E^2 v_E + \nabla_H^2 v_H \right) dx dt \\
& \quad + \frac{2A}{h^x} \sum_{K \in \mathcal{T}_h} \int_K \left(\nabla_E^2 v_E + 2 \nabla_H^2 v_H + (\mathbf{x} \cdot \nabla_K)^2 \nabla_E^2 v_E + \nabla_H^2 v_H \right) dx dt \\
& \quad + \frac{2A}{h^x} \sum_{K \in \mathcal{T}_h} \int_K 2j(\mathbf{x} \cdot \nabla_K) v_E v_H dx
\end{aligned}$$

Using $\nabla_E^2 v_E + \nabla_H^2 v_H$ with weight $\phi = (\phi^x)^{-1}$, we have the bound

$$\begin{aligned}
& \sum_{K \in \mathcal{T}_h} \int_K \left(\nabla_E^2 v_E + \nabla_H^2 v_H \right) \phi^2 dx \\
& = \frac{4A}{h^x} \sum_{K \in \mathcal{T}_h} \int_K \left(\nabla_E^2 v_E + \nabla_H^2 v_H \right) \phi^2 dx + \frac{4A}{h^x} \sum_{K \in \mathcal{T}_h} \int_K \left(\nabla_E^2 v_E + \nabla_H^2 v_H \right) \phi^2 dx \\
& \quad + 2A \sum_{K \in \mathcal{T}_h} \int_K \left(\nabla_E^2 v_E + \nabla_H^2 v_H \right) \phi^2 dx \\
& \stackrel{(27);(26)}{\leq} A \left(\frac{4T^2}{h^x} c_1^4 + \frac{h^x}{2} c_1^2 + 2c_1^3 N_{\text{hor}} e^T \right) \sum_{K \in \mathcal{T}_h} \int_K \left(\nabla_E^2 v_E + \nabla_H^2 v_H \right) \phi^2 dx;
\end{aligned}$$

recalling that $c_1 = k c_{L^1(Q)}$.

This, together with (26), gives the bound (19) with constant C_{stab}^2 as in (21). $\square$

In case of a tensor product mesh with all elements having horizontal edges of length h^x and vertical edges of length $h^t = h^x/c$, the constant C_{stab} is proportional to $(h^x)^{-1/2}$. We stress that we cannot expect a bound like (19) with C_{stab} independent of the meshwidth: indeed if the mesh is refined, say, uniformly, while the term in the brackets in the right-hand side of (19) is not modified, the left-hand side grows (consider e.g. the simple case $c = 1$, $\phi = 0$, $v_E = 0$, $v_H = t$).

One could attempt to derive the stability bound (19) by controlling with $(\cdot; \cdot)$ either the $H^1(Q)$ or the $L^x(\text{singgth})$

$$\frac{1}{2}z$$

for all $(E; H); (v_E; v_H) \in T(T$

and denote their length by

$$h_D := x_1 - x_0 + c(t_1 - t_0); \quad (32)$$

Their relevance is the following: the restriction to D of the solution of a Maxwell initial value problem posed in $\mathbb{R} \times \mathbb{R}^+$ will depend only on the initial conditions posed on $\bar{D} \cap \{t = 0\}$; see Figure 1.

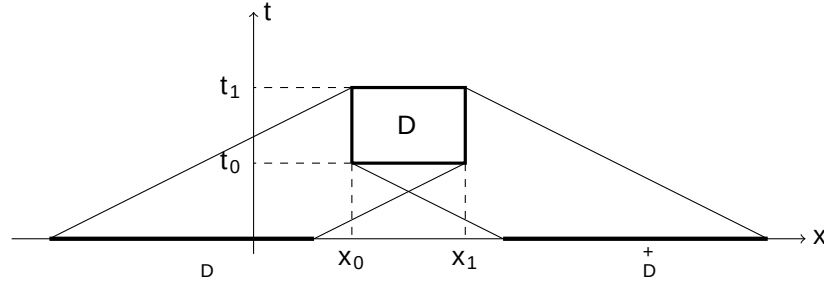


Figure 1: The intervals $\bar{D}$ in (31) corresponding to the space-time rectangle D .

Let $\alpha = (\alpha_x; \alpha_t) \in \mathbb{N}_0^2$ be a multi-index; for a sufficiently smooth function v , we define its anisotropic derivative $D_\alpha v$ as

$$D_\alpha v(x; t) := \frac{1}{c^{\alpha_t}} D^{\alpha_x} v(x; t) = \frac{1}{c^{\alpha_t}} \frac{\partial^{\alpha_x} v(x; t)}{\partial x^{\alpha_x} \partial t^{\alpha_t}}.$$

Note that, if u and w satisfy

$$u(x; t) = u_0(x - ct); \quad w(x; t) = w_0(x + ct); \quad (33)$$

with u_0 and w_0 defined in $\bar{D}$ and $\bar{D}^+$, respectively, then

$$D_\alpha u(x; t) = (-1)^{\alpha_t} u_0^{(\alpha)}(x - ct);$$

$$D_\alpha w(x; t) = w_0^{(\alpha)}(x + ct).$$

We define the Sobolev spaces $W_c^{j;1}(D)$ and $H_c^j(D)$ as the spaces of functions whose D_α derivatives, $0 \leq |\alpha| \leq j$, belong to $L^1(D)$ and $L^2(D)$, respectively. We define the following seminorms:

$$|v|_{W_c^{j;1}(D)} := \sup_{|\alpha| \leq j} |D_\alpha v|_{L^1(D)}; \quad |v|_{H_c^j(D)}^2 := \sum_{|\alpha| \leq j} |D_\alpha v|_{L^2(D)}^2.$$

Note that for j

$W_c^j(D)$ and $L^2(D)$ are

A similar result holds for $w(x;t) = w_0(x + ct)$, with $\mathbb{D}^+$ instead of $\mathbb{D}$.

Proof. For the $W_c^{j,1}(\mathbb{D})$ -seminorms in (i), we have

$$\|u\|_{W_c^{j,1}(\mathbb{D})} = \sup_{j=1}^j \|D_c u\|_{L^1(\mathbb{D})} = \sup_{j=1}^j \|u_0^{(j)}(x - ct)\|_{L^1(\mathbb{D})} = \|u_0\|_{W_c^{j,1}(\mathbb{D})} :$$

For the bound of $\|u_0\|_{H^1(\mathbb{D})}^2$ in (ii), we have

$$\begin{aligned} \|u_0\|_{H^1(\mathbb{D})}^2 &= \int_{\mathbb{D}} |u_0^{(j)}(z)|^2 dz = \int_{\mathbb{D}} \sup_{z \in \mathbb{D}} |u_0^{(j)}(z)|^2 = \int_{\mathbb{D}} \sup_{j=1}^j \sup_{(x;t) \in \mathbb{D}} |D_c u(x;t)|^2 \\ &= h_{\mathbb{D}} \|u\|_{W_c^{j,1}(\mathbb{D})}^2 : \end{aligned}$$

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This suggests a construction of discrete subspaces $\mathcal{T}(D)$: given $p \in \mathbb{N}_0$ and two sets of $p+1$ linearly independent functions $f'_0, \dots, f'_p \in C^m(\overline{D})$ and $f''_0, \dots, f''_p \in C^m(\overline{D}^+)$ we define the space

$$V_p(D) := \text{span} \left\{ \frac{f'_0(x-ct)}{2^{p+1-2}}, \frac{f'_0(x-ct)}{2^{p+1-2}}, \dots, \frac{f'_p(x-ct)}{2^{p+1-2}}, \frac{f'_p(x-ct)}{2^{p+1-2}}; \right. \\ \left. \frac{f''_0(x+ct)}{2^{p+1-2}}, \frac{f''_0(x+ct)}{2^{p+1-2}}, \dots, \frac{f''_p(x+ct)}{2^{p+1-2}}, \frac{f''_p(x+ct)}{2^{p+1-2}} \right\};$$

which is a subspace of $\mathcal{T}(D) \setminus C^m(\overline{D})^2$ with dimension $2(p+1)$.

By virtue of Proposition 5.1, the approximation properties of $V_p(D)$ in $\mathcal{T}(D)$ only depend on the approximation properties of the one-dimensional functions $f'_0, \dots, f'_p, g$ for all $(E; H) \in \mathcal{T}(D) \setminus W_{\mathcal{E}}^{j,1}(D)^2$, defining u, w, u_0 and w_0 from $(E; H)$ using (35) and (33),

$$(E_{hp}; H_{hp}) \in V_p(D) \quad \|^{1=2} (E - E_{hp})_{W_{\mathcal{E}}^{j,1}(D)} + \|^{1=2} (H - H_{hp})_{W_{\mathcal{E}}^{j,1}(D)} \quad (36)$$

$$\stackrel{(35)}{=} \inf_{\substack{u_{0,p} \in \text{span}\{f'_0, \dots, f'_p, g\}; \\ w_{0,p} \in \text{span}\{f''_0, \dots, f''_p, g\}}} \frac{1}{2} \|u(x; t) - u_{0,p}(x-ct) + w(x; t) - w_{0,p}(x+ct)\|_{W_{\mathcal{E}}^{j,1}(D)} \\ + \frac{1}{2} \|u(x; t) - u_{0,p}(x-ct) - w(x; t) + w_{0,p}(x+ct)\|_{W_{\mathcal{E}}^{j,1}(D)}$$

$$\text{Prop. 5.1 (i)} \quad \inf_{u_{0,p} \in \text{span}\{f'_0, \dots, f'_p, g\}} \|u_0 - u_{0,p}\|_{W_{\mathcal{E}}^{j,1}(D)} + \inf_{w_{0,p} \in \text{span}\{f''_0, \dots, f''_p, g\}} \|w_0 - w_{0,p}\|_{W_{\mathcal{E}}^{j,1}(D)};$$

while for all $(E; H) \in \mathcal{T}(D) \setminus H_{\mathcal{E}}^j(D)^2$

$$(E_{hp}; H_{hp}) \in V_p(D) \quad \|^{1=2} (E - E_{hp})_{W_{\mathcal{E}}^{j,1}(D)} + \|^{1=2} (H - H_{hp})_{W_{\mathcal{E}}^{j,1}(D)} \quad (36)$$

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Following the second route, we prove simple approximation bounds $\text{inH}_c^1(Q)$, for a general rectangle

6 Convergence rates

We now derive the convergence rates of the Tre tz-DG method with polynomial approximating spaces

$$V_p(T_h) = (v_E; v_H)_{L^2(Q)}^2 : (v_E; v_H)_{j_K} \text{ are as in (38) with } p = p_K : \quad (43)$$

The two main ingredients are the quasi-optimality results investigated in section 4 and the best approximation bounds proved in section 5. To combine them, we need to control the DG^+ norm (12) of the approximation error with its $H_c^1(Q)$ norm, weighted with ω and ω^{-1} , to be able to use the bound (40). To this purpose, we define the following parameters:

$$\kappa_K := \max_{K \in \mathcal{T}_h} \omega_K$$

If the bound (19) holds true for the solution of the auxiliary problem(18), we also have the following bound in $L^2(Q)$:

$$\begin{aligned} & \|E - E_{hp}\|_{L^2(Q)}^2 + \|H - H_{hp}\|_{L^2(Q)}^2 \\ & \leq \frac{12}{p} \frac{p-2}{c} C_{stab} \sum_{K \in \mathcal{T}_h} \left(6 \left(c + \frac{h_K^x}{h_K^t} \right) + 8 \right) \left(1 + c \frac{h_K^t}{h_K^x} \right)^{1-2} (e=2)^{\frac{s_K^2}{p_K}} \frac{h_K^x + c h_K^t s_K + \frac{3}{2}}{p_K^{s_K}} \\ & \quad + \|E\|_{W_c^{s_K+1;1}(K)}^2 + \|H\|_{W_c^{s_K+1;1}(K)}^2 : \end{aligned} \quad (47)$$

with C_{stab} from (19).

Proof. Given an element $K \in \mathcal{T}_h$, we denote by $@K^N$, $@K^S$, $@K^W$, and $@K^E$ its North, South, West and East sides, respectively, North pointing in the positive time direction, and set $@K^{WE} := @K^W \cup @K^E$.

For all $(v_E; v_H) \in H^1(T_h)^2$, expanding the DG^+ norm(a)-5@H98 -3.6 Td [(:=)-](+)-0.569-6.

where the last inequality follows noting that $f(x) = (1-x)^{\frac{1}{x}-1}(1+x)^{\frac{1}{x}-1}4^xe^{2-2x} - 1$ for all $0 < x < 1$ (which in turn can be verified by checking the convexity of $\log f$ and its limit values for $x \rightarrow 0$ and 1).

The estimate in

(What we actually need is only that $\mathbf{a}_0$ and $\mathbf{w}_0$ are analytic in a sufficiently large complex neighbourhood of the finite segments Γ_Q and Γ_Q^+ , respectively.)

For every mesh element K as above, we set $h_K := \text{length}(\Gamma_K)$. The complex ellipses with foci at the extrema of Γ_K and sum of the semiaxes equal to $\sigma + \rho$

Figure 3 shows convergence of the ϕ -version for degree zero through three. Solid lines correspond to results obtained with the Tre tz basis whereas the dashed lines were obtained using the non-Tre tz basis. Uniform mesh step sizes are applied by reducing h_x and h_t simultaneously. The Tre tz method exhibits optimal algebraic convergence rates h^{p+1} . However, in the non-Tre tz case, the results seem to suggest an odd-even pattern of the convergence rates, with convergence being suboptimal for odd degrees (by one order). Numerical odd-even effects in the convergence rates of DG methods have also been reported, e.g. in [18, section 6.5], although it has been shown in [17] that in some situations this might

$$:= \max_{n=1, \dots, N} \left(\| \cdot \|_{L^1(F_h^{ver})} + \| \cdot \|_{L^1(F_h^L[F_h^R])} \right); \quad \left(\| \cdot \|_{L^1(F_h^{ver})} + \| \cdot \|_{L^1(F_h^L[F_h^R])} \right)^0;$$

Proof. We assume that φ_E and φ_H are continuous in Q ; the general case will follow by a density argument.

First, we extend the initial problem to the entire space $\mathbb{R}$. Define $\varphi_E, \varphi_H, e, e$ in $\mathbb{R} \times \mathbb{R}^+$ as the $2(x_R - x_L)$ -periodic functions in x that satisfy $\varphi_E|_Q = \varphi_E; \varphi_H|_Q = \varphi_H; e|_Q = e; e|_Q = e$ and such that φ_E and e are odd around x_L (and consequently also around x_R), and φ_H and e are even around the same points, i.e.

$$\begin{aligned} \varphi_E(x_L + x; t) &= -\varphi_E(x_L - x; t); & \varphi_H(x_L + x; t) &= \varphi_H(x_L - x; t); \\ e(x_L + x; t) &= -e(x_L - x; t); & e(x_L + x; t) &= e(x_L - x; t); \end{aligned} \quad 8(x; t) \in \mathbb{R} \times \mathbb{R}^+;$$

(Note that the absolute values are $(x_R - x_L)$ -periodic in x .) Since time derivatives preserve parities and space derivatives swap them, the extended functions φ_E and φ_H are continuous and satisfy the extended initial problem

$$\begin{aligned} \frac{\partial \varphi_E}{\partial x} + \frac{\partial \varphi_H}{\partial t} &= e & \text{in } \mathbb{R} \times \mathbb{R}^+; \\ \frac{\partial \varphi_H}{\partial x} + \frac{\partial \varphi_E}{\partial t} &= e & \text{in } \mathbb{R} \times \mathbb{R}^+; \\ \varphi_E(\cdot; 0) &= 0; \quad \varphi_H(\cdot; 0) = 0 & \text{on } \mathbb{R}; \end{aligned}$$

Second, we split the right- and the left-propagating components. Define

$$u := \varphi_E^{1=2} + \varphi_H^{1=2}; \quad w := \varphi_E^{1=2} - \varphi_H^{1=2}; \quad \text{so that} \quad \varphi_E = \frac{u+w}{2^{1=2}}; \quad \varphi_H = \frac{u-w}{2^{1=2}};$$

They satisfy the inhomogeneous transport equations in $\mathbb{R} \times \mathbb{R}^+$

$$\frac{\partial u}{\partial x} + \frac{\partial c^{-1}u}{\partial t} = \varphi_E^{1=2} + \varphi_H^{1=2} =: f; \quad \frac{\partial w}{\partial x} + \frac{\partial c^{-1}w}{\partial t} = \varphi_E^{1=2} - \varphi_H^{1=2} =: g;$$

recalling that $(\cdot)^{1=2} = c^{-1}$, so they can be written explicitly with the following representation formula (e.g. [9, section 2.1.2, equation (5)], recall that from the assumptions made in the proof, f and g are piecewise continuous)

$$u(x; t) = \int_0^{Z_t} c f(x + c(s - t); s) ds; \quad w(x; t) = \int_0^{Z_t} c g(x - c(s - t); s) ds;$$

We first bound the L^2 norm of u and w on horizontal and vertical segments with the data f, g ; from the triangle inequality (φ_E, φ_H) will be bounded by e and e , and the bound for v_E and v_H will follow. For all $0 \leq t \leq T$

$$\begin{aligned} ku(\cdot; t)k_{L^2(\cdot)}^2 &= \int_{x_L}^{x_R} \int_0^{Z_t} c f(x + c(s - t); s) ds^2 dx \\ &\leq \int_{x_L}^{x_R} \int_0^{Z_t} c^2 f(x + c(s - t); s)^2 ds dx \\ &= \int_0^{Z_t} \int_{x_L + c(s - t)}^{x_R + c(s - t)} c f(y; s)^2 dy ds \\ &\leq \int_0^{Z_t} \int_{x_L + c(s - t)}^{x_R + c(s - t)} c^2 (|j e(y; s)|^2 + |j e(y; s)|^2) dy ds \\ &= 2tc^2 \left(\| \cdot \|_{L^2(\cdot(0; t))}^2 + \| \cdot \|_{L^2(\cdot(0; t))}^2 \right) \end{aligned}$$

(the last equality follows from the symmetries of e and e which ensure the equality of their L^2 norms on the rectangle $(x_L; x_R) \times (0; t)$ and on the parallelogram with vertices $(x_L - ct; 0); (x_R - ct; 0); (x_R; t); (x_L; t)$). Similarly, for all $x \in \mathbb{R}$

$$ku(x; \cdot)k_{L^2(0; T)}^2 = \int_0^T \int_0^{Z_t} c f(x + c(s - t); s) ds^2 dt$$

$$\begin{aligned}
& c^2 \int_t^T \int_{\gamma_t} f(x) + c(s-t); s^2 ds dt \\
&= c^2 \int_t^T \int_{\gamma_t} f(x) + c(s-t); s^2 dt ds \\
&= c^2 \int_0^T \int_{\gamma_s} f(x) + c(T-s); s^2 ds
\end{aligned}$$

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