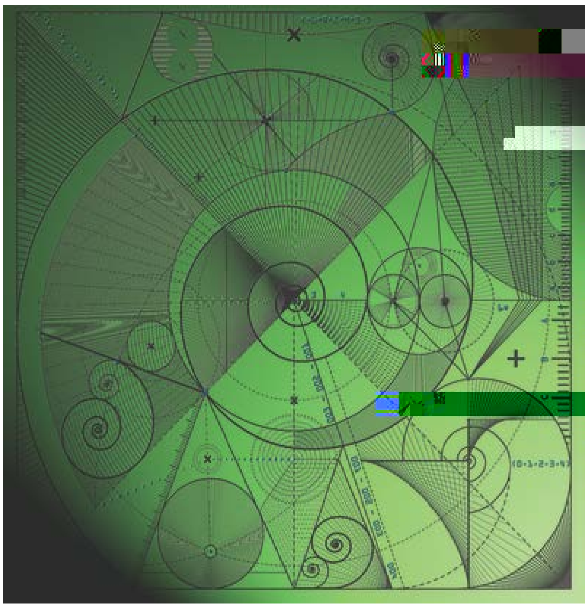

A SHAPE CALCULUS BASED METHOD FOR A TRANSMISSION PROBLEM WITH RANDOM INTERFACE

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Abstract. The present work is devoted to approximation of the statisti cal moments of the unknown solution of a class of elliptic transmission problems in R^3 with uncertainly located transmission interfaces. Within this model, the di usion coe cient has a jump discontinuity acro ss the random transmission interface which models linear di u6.12487(n)2.80666(e)5.25645(a)633.88 -9.36016 Td TJ 219.6 0 Td [(i)-6.1264(c)5.25339(a)4.03117(l)-352.4(f)-7.23528(r)-7.5753(a)4.03117(n

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This work has been funded by BMBF and the Group of Eight Austra lia within the DAAD-Go8 Project \Nu-

In this article we develop a deterministic method for numerical solution for a class of transmission problems with randomly perturbed interfaces. The equation to be solved is of the form

$$r \quad (\; r \; u) = f \quad \text{in } D \;;$$

where D is a random bounded domain in R^3 and $D_+ = R^3 \, n \, \overline{D}$ is its complement. The domains share a common random surface , and the coefficient function takes (in general) distinct constant values in D and D_+, respectively. The solution u is subject to jump conditions across . A precise description of the model problem is deferred until Section 2.3, where a probabilistic perturbation model for the surface (and thus D) will be rigorously introduced. Within this model, the

with the natural inner product satisfying $\langle v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_k\rangle_{X^{(k)}} = \langle v_1, w_1\rangle_X \cdots \langle v_k, w_k\rangle_X$.

Definition 2.1. For a random field $v \in L^k(\Omega; X)$, its k-order moment $\mathcal{M}^k[v]$ is an element of $X^{(k)}$ defined by

$$\mathcal{M}^k[v] := \int_\Omega \underbrace{v(\omega) \otimes \cdots \otimes v(\omega)}_{k\text{-times}} \, dP(\omega). \tag{2.4}$$

In the case $k = 1$, the statistical moment $\mathcal{M}^1[v]$ coincides with the mean value of v and is denoted by $\mathbb{E}[v]$. If $k \geq 2$, the statistical moment $\mathcal{M}^k[v]$ is the k

$D_+(\omega) := \mathbb{R}^3 \setminus \overline{D}$. The shape calculus in Section 3 requires a somewhat stronger smoothness assumption on , namely that the realizations of belong to $C^1(\ ^0)$. From (2.7) we observe that the mean random interface is represented by

$$\mathbb{E}[\ ] = \{ x + \mathbb{E}[\ (x;\cdot)]n^0(x);\ x \in \ ^0 \}.$$

Without loss of generality, we may assume that the random perturbation amplitude $(x;\omega)$ is centered, i.e.,

$$\mathbb{E}[\ (x;\cdot)] = 0 \qquad \forall x \in \ ^0. \tag{2.9}$$

$s \in \mathbb{R}$, the Sobolev space $H^s_{mix}(G^k)$ is defined to be the space of all distributions $v(y_1;\dots;y_k)$ with $y_1;\dots;y_k \in G$ satisfying

$$\|v\|_{H^s_{mix}(G^k)} := \langle v,v\rangle^{1/2}_{H^s_{mix}(G^k)} < \infty;$$
$$\langle v,w\rangle_{H^s_{mix}(G^k)} := \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\prod_{i=1}^{k}(1+\ell_i)^{2s}\right)\hat{v}_{\ell,m}\,\hat{w}_{\ell,m} \tag{2.18}$$

with the Fourier coefficients

$$\hat{v}_{\ell,m} := \int_{x_1\in S}\cdots\int_{x_k\in S} v(\gamma(x_1);\dots;\gamma(x_k))\left(\prod_{i=1}^{k} Y_{\ell_i,m_i}(x_i)\right) d\sigma_{x_1}\dots d\sigma_{x_k} \tag{2.19}$$

Recalling definition (2.3) we observe that $H^s_{mix}(G^k)$ is isometrically isomorphic to the tensor product space $H^s(G)^{(k)}$. These spaces will be identified in what follows. We also use the notation $H^s_{mix}(K^k)$ for the tensor product $H^s(K)^{(k)}$ where K is a compact subset of $\mathbb{R}^3$.

Sobolev spaces on bounded domains in $\mathbb{R}^3$ are defined, as usual, as spaces of all distributions whose partial derivatives are square integrable. Proper treatment of the transmission problem (2.11a)–(2.11d) in unbounded domains in $\mathbb{R}^3$ requires a special care. Following [17], for an unbounded domain $U \subset \mathbb{R}^3$ we introduce the space

$$H^1_w(U) := \left\{ v \in \mathcal{D}'(U) : \|v\|_{H^1_w(U)} = \left(\int_U |\nabla v|^2 + \frac{|v(x)|^2}{1+|x|^2}\,dx\right)^{1/2} < +\infty \right\}. \tag{2.20}$$

Specifically, for a given partition R_m

) :

3.1. **Perturbation of deterministic interfaces.** In this subsection we collect several properties of perturbed interfaces which are important for the subsequent analysis. Assume that the perturbation function is a fixed deterministic function in $W^{1;1}(\ ^0)$, in particular is independent of $!$. Then is defined by

$$(3.1) \qquad := f x + \ (x) n^0(x) : x \ 2 \ ^0 g; \quad > 0:$$

As already noticed in Section 2.2, is a closed Lipschitz manifold in R^3 provided $0 \quad _0$ and $_0$ is sufficiently small. In this case introduces a decomposition of R^3 into the interior and exterior subdomains D and D_+, respectively.

Following [20], we define a mapping $T : R^3 \, ! \, R^3$ which transforms ^{0} into and D^0 into D, respectively, by

$$(3.2) \qquad T \ (x) := x + \ \tilde{\ }(x) \tilde{n}^0(x); \quad x \ 2 \ R^3;$$

where $\tilde{\ }$ and $\tilde{n}^0$ are any smoothness-preserving extensions of and n^0 into R^3. We require in particular that $\tilde{\ } \ 2 \ W^{1;1}(R^3)$. Without loss of generality we assume that the extension $\tilde{\ }$ vanishes outside a sufficiently large ball $B_R := f x \ 2 \ R^3 : jxj < R g$ containing for any $0 \quad _0$. This implies that the perturbation mapping $T \ (x)$ is an identity in the complement B^c_R

where $_1$, $_2$, $_3$ are de ned by (3.9) and $_4$:= 0 for notational convenience later. In particular, for su ciently small > 0, there holds

$$(3.12) \qquad (; x) = 1 + \ _1(x) + \ ^2 \ _2(x) + \ ^3 \ _3(x) \quad c > 0 \quad 8x \ 2 \ R^3:$$

Consider from now on su ciently small > 0. It follows from (3.10) and (3.12) that the ij -entry of the matrix $A(; x) \quad I$ is

$$(3.13) \qquad A_{ij} (;) \quad _{ij} = \quad (;) \ ^1 \sum_{n=1}^{4} \ ^{n \ 1} h_{ijn} \quad _{ij} \ _n :$$

Hence, (3.11) yields

$$kA_{ij} (;) \quad _{ij} k_{L^1 (R^3)} ! \ 0 \ \text{ as } \ ! \ 0;$$

proving (3.6).

From (3.13), we have

$$(3.14) \qquad \frac{A_{ij} (;) \quad _{ij}}{} = \quad (;) \ ^1 \sum_{n=1}^{4} \ ^{n \ 1} h_{ijn} \quad _{ij} \ _n :$$

Taking the limit when goes to 0, noting that $(;) ! \ 1$, we obtain

(3.15) $A^0_{ij} (0;) = h_{ij\,1} \quad _{ij}$ i4 Tf 4.43984 -4.2 Td [( )-0.936115]TJ /R17 6.97385 Tf 5.16016

Using the change of variables $y = T_\varepsilon(x)$ and noting (3.11), we have

$$\|v\circ T_\varepsilon\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |v(y)|^2 \,\mathcal{J}(\varepsilon; (T_\varepsilon)^{-1}(y))^{-1}\,dy \le C\|v\|_{L^2(\mathbb{R}^3)}.$$

Therefore,

$$\lim_{\varepsilon\to 0}\left\|\sqrt{1+|\xi|^2}\,\big(\mathcal{J}(\varepsilon;x)-1\big)(v\circ T_\varepsilon)\right\|_{L^2(\mathbb{R}^3)} = 0. \tag{3.18}$$

Furthermore, (3.3) also gives

$$\left\|\sqrt{1+|\xi|^2}\,\big(v\circ T_\varepsilon - v\big)\right\|_{L^2(\mathbb{R}^3)} = \left\|\sqrt{1+|\xi|^2}\,\big(v\circ T_\varepsilon - v\big)\right\|_{L^2(B_R)} \le \sqrt{1+R^2}\,\|v\circ T_\varepsilon - v\|_{L^2(B_R)}.$$

Note that $\lim_{\varepsilon\to 0}\|v\circ T_\varepsilon - v\|_{L^2(B_R)} = 0$ if v is continuous. By using a density argument we deduce that $\lim_{\varepsilon\to 0}\|v\circ T_\varepsilon - v\|_{L^2(B_R)} = 0$ for $v\in L^2(B_R)$. Hence,

$$\lim_{\varepsilon\to 0}\left\|\sqrt{1+|\xi|^2}\,\big(v\circ T_\varepsilon - v\big)\right\|_{L^2(\mathbb{R}^3)} = 0.$$

The above identity and (3.18) together with the triangle inequality give the required result. □

Lemma 3.4. For any function $v\in H^1(\mathbb{R}^3)$, there holds

$$\lim_{\varepsilon\to 0}\left\|\sqrt{1+|\xi|^2}\left(\frac{\mathcal{J}(\varepsilon;\cdot)(v\circ T_\varepsilon) - v}{\varepsilon} - \operatorname{div}(vV)\right)\right\|_{L^2(\mathbb{R}^3)} = 0.$$

Proof. Noting (3.3), Lemma 3.1, (3.17) and the triangle inequality, we obtain

$$\begin{aligned}
&\left\|\sqrt{1+|\xi|^2}\left(\frac{\mathcal{J}(\varepsilon;\cdot)(v\circ T_\varepsilon) - v}{\varepsilon} - \operatorname{div}(vV)\right)\right\|_{L^2(\mathbb{R}^3)}\\
&\quad= \left\|\sqrt{1+|\xi|^2}\left(\frac{\mathcal{J}(\varepsilon;\cdot)(v\circ T_\varepsilon) - v}{\varepsilon} - \operatorname{div}(vV)\right)\right\|_{L^2(B_R)}\\
&\quad\lesssim \left\|\frac{\mathcal{J}(\varepsilon;\cdot)(v\circ T_\varepsilon) - v}{\varepsilon} - \operatorname{div}(vV)\right\|_{L^2(B_R)}\\
&\quad\le \left\|\frac{\mathcal{J}(\varepsilon;\cdot)-1}{\varepsilon}(v\circ T_\varepsilon) - v\operatorname{div}V\right\|_{L^2(B_R)} + \left\|\frac{v\circ T_\varepsilon - v}{\varepsilon} - V\cdot\nabla v\right\|_{L^2(B_R)}.
\end{aligned} \tag{3.19}$$

Recall from (3.9) that $\mathcal{J}_1 = \operatorname{div}V$. It follows from (3.12) that

$$\frac{\mathcal{J}(\varepsilon;\cdot)-1}{\varepsilon}(v\circ T_\varepsilon) - v\operatorname{div}V = \mathcal{J}_1(v\circ T_\varepsilon - v) + (\varepsilon\mathcal{J}_2 + \varepsilon^2\mathcal{J}_3)(v\circ T_\varepsilon).$$

Employing the density argument as in proof of Lemma 3.3, we obtain

$$\lim_{\varepsilon\to 0}\|\mathcal{J}_1(v\circ T_\varepsilon - v)\|_{L^2(B_R)} = 0 \quad\text{and}\quad \lim_{\varepsilon\to 0}\|(\varepsilon\mathcal{J}_2 + \varepsilon^2\mathcal{J}_3)(v\circ T_\varepsilon)\|_{L^2(B_R)} = 0,$$

so that

lim T

3.2. Material and shape derivatives. In this subsection, for notational convenience we use the notation D for D or D_+, and $H^1(D)$ for $H^1(D)$ or $H^1_w(D_+)$.

Definition 3.5. For any sufficiently small , let v be an element in $H^1(D)$ or $H^{1/2}(\;)$. The material derivative of v, denoted by $\underline{v}$, is defined by

$$\underline{v} := \lim_{\to 0} \frac{v \quad T \quad v^0}{}; \tag{3.20}$$

if the limit exists in the corresponding space $H^1(D^0)$ or $H^{1/2}(\;^0)$. The shape derivative of v is defined by

$$v^0 = \begin{cases} \underline{v} \;\; \nabla \;\; v^0 \;\; V & \text{if } v \in H^1(D); \\ \underline{v} \;\; \nabla_{0} v^0 \;\; V & \text{if } v \in H^{1/2}(\;); \end{cases} \tag{3.21}$$

where ∇_{0} denotes the surface gradient.

Lemma 3.6. If

(i) The material and shape derivatives of the product $v\ w$ are $\underline{v}w^0 + v^0\underline{w}$ and $v^0w^0 + v^0w^0$, respectively.

(ii) The material and shape derivatives of the quotient $v\ =w$ are $(\underline{v}w^0 \quad v^0\underline{w})=(w^0)^2$ and $(v^0w^0$ $v^0w^0)=(w^0)^2$, respectively, provided that all the fractions are well-defined.

(iii) If $v = v$ for all 0, then $\underline{v} = r\ v^0 \quad V = r\ v \quad V$ and $v^0 = 0$.

(iv) If

$$J_1(D\ ) := \int_D v\ dx\,;\ J_2(D\ ) := \int v\ d\,;\ \text{ and } dJ_i(D\ )j_{=0} := \lim_{!\ 0} \frac{J_i(D\ ) \quad J_i(D^0)}{};\ i = 1;2;$$

then

$$dJ_1(D\ )j_{=0} = \int_{D^0} v^0 dx + \int_0 v^0 \quad V;n^0 \quad d$$

and

$$dJ_2(D\ )j_{=0} = \int_0 v^0 d \ + \int_0 \frac{@v^0}{@n} + \mathrm{div}_0(n^0)\,v^0 \quad V;n^0 \quad d\,:$$

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Proof. Statements (i){(iii) can be obtained by using elementary calculations. Statement (iv) is proved in [20, pages 113, 116]. □

Lemma 3.9. The material and shape derivatives of the normal field n are given by

$$n\ ) \qquad \underline{n} = n^0 = r \quad {}_0:$$

v
) =

Proof. nWe start by recalling that the materi4(a)-5.8887(t)-419(t)2.56084(h)2 noting from (3.8) that

$$\lim_{\varepsilon\to 0} J_{T_\varepsilon}^{\top} = \lim_{\varepsilon\to 0} J_{T_\varepsilon} = I.$$

Since $I = J_{T_\varepsilon}^{-1}(T_\varepsilon(x))\, J_{T_\varepsilon}(x)$ for all $x \in \mathbb{R}^3$, we have $0 = \frac{d}{d\varepsilon}\big(J_{T_\varepsilon}^{-1} J_{T_\varepsilon}\big)\big|_{\varepsilon=0}$, which together with the product rule and (3.8) yields

$$\frac{d}{d\varepsilon} J_{T_\varepsilon}^{-\top}(T_\varepsilon(x))\Big|_{\varepsilon=0} = -(J_{T_0})^{-\top}\,\frac{d}{d\varepsilon}(J_{T_\varepsilon}^{\top})\Big|_{\varepsilon=0}(J_{T_0})^{-1} = -\frac{d}{d\varepsilon}(J_{T_\varepsilon}^{\top})\Big|_{\varepsilon=0} = -J_V^{\top}; \tag{3.24}$$

We also have, using the fact that $|J_{T_0}^{-\top} n^0| = 1$,

$$\begin{aligned}\frac{d}{d\varepsilon}\big|J_{T_\varepsilon}^{-\top} n^0\big|\Big|_{\varepsilon=0} &= \big|J_{T_0}^{-\top} n^0\big|\,\frac{d}{d\varepsilon}\big|J_{T_\varepsilon}^{-\top} n^0\big|\Big|_{\varepsilon=0} = \frac{1}{2}\frac{d}{d\varepsilon}\big|J_{T_\varepsilon}^{-\top} n^0\big|^2\Big|_{\varepsilon=0} \\ &= \frac{1}{2}\frac{d}{d\varepsilon}\big\langle J_{T_\varepsilon}^{-1}J_{T_\varepsilon}^{-\top} n^0; n^0\big\rangle = -\frac{1}{2}\big\langle (J_V^{\top}+J_V)\, n^0; n^0\big\rangle.\end{aligned} \tag{3.25}$$

Simple calculation reveals that

$$J_V^{\top} = \nabla V\,(n^0)^{\top} \quad\text{and}\quad (J_V^{\top}+J_V)\,n^0 = \nabla V + \langle \nabla V; n^0\rangle\, n^0. \tag{3.26}$$

Inserting (3.24)–(3.26) into (3.23), we obtain

$$\dot{n} = -J_V^{\top} n^0 + \frac{1}{2}\big\langle (J_V^{\top}+J_V)\,n^0; n^0\big\rangle\, n^0 = -\nabla V + \langle \nabla V; n^0\rangle\, n^0 = -\nabla_{\Gamma^0} V;$$

finishing the proof of the lemma. □

3.3. **Shape derivative of solutions of transmission problem.** In this subsection, we shall discuss the existence of material and shape derivatives of the solutions of transmission problems on perturbed interfaces. Consider a deterministic problem with respect to the reference interface Γ^0:

$$-\Delta u^0 = f \quad \text{in } D_-^0 \cup D_+^0; \tag{3.27a}$$

$$[u^0] = 0 \quad \text{on } \Gamma^0; \tag{3.27b}$$

$$\Big[\frac{\partial u^0}{\partial n}\Big] = 0 \quad \text{on } \Gamma^0; \tag{3.27c}$$

$$u^0(x) = O(|x|^{-1}) \quad \text{when } |x| \to \infty. \tag{3.27d}$$

The perturbed problem corresponding to the perturbed interface Γ_ε is given by

$$-\Delta u_\varepsilon = f \quad \text{in } D_-^\varepsilon \cup D_+^\varepsilon; \tag{3.28a}$$

$$[u_\varepsilon] = 0 \quad \text{on } \Gamma_\varepsilon; \tag{3.28b}$$

$$\Big[\frac{\partial u_\varepsilon}{\partial n}\Big] = 0 \quad \text{on } \Gamma_\varepsilon; \tag{3.28c}$$

$$u_\varepsilon(x) = O(|x|^{-1}) \quad \text{when } |x| \to \infty; \tag{3.28d}$$

where (cf. (2.10))

$$(x) = \begin{cases} \quad ; & x \; 2 \; D \\ {}_+ ; & x \; 2 \; D_+ : \end{cases}$$

Lemma 3.10. Suppose f 2 $L^2(R^3) \setminus W_0$ and 2 $C^1($ $^0)$, then

$$\lim_{! \; 0} \quad u \quad T \quad u^0 \quad {}_{W_0} = 0 : \tag{3.29}$$

Here, W_0 denotes the dual space of W_0 with respect to the L^2-inner product.

Proof. By multiplying both sides of (3.28a) with an arbitrary function v 2 C_0^1 (R^3) and integrating over D [D_+, we obtain

$$Z_{R^3} f v \, dx = Z_{D} \; 4 \; u \; (x) \, v(x) \, dx \quad {}_+ \; Z_{D_+} \; 4 \; u \; (x) \, v(x) \, dx : \tag{3.30}$$

Applying Green's identity and noting (3.28c), we obtain

$$Z_{D_+ [\; D} \; (x) \, r \; u \; (x) \quad r \; v(x) = hf; v \, i_{L^2(R^3)} \quad 8v \; 2 \; C_0^1 \; (R^3) : \tag{3.31}$$

Since the space C_0^1 (R^3) is dense in W (see [17, Remark 2.9.3]), there holds

$$Z_{D_+ [\; D} \; (x) \, r \; u \; (x) \quad r \; v \; (x) = hf; v \; i_{L^2(R^3)} \quad 8v \; 2 \; W : \tag{3.32}$$

Choosing v = u gives

$$j u \; j^2_W \; ' \; h \; f; u \; i_{L^2(R^3)} \quad k \; f \; k_W \quad k u \; k_W :$$

It follows from Lemma 2.2 that

$$k u \; k_W \; . \; k f \; k_W \; ' \; k \; f \; k_{W_0} : \tag{3.33}$$

On the other hand, using the change of variables x = T (y) in (3.32), we have (noting that (T (y)) = (y))

$$Z_{D^0_+ [\; D^0} (y) \, (r \; w(y))^> \; A(\; ; y) \, r \; (u \quad T \;)(y) \, dy = Z_{D^0_+ [\; D^0} f \, (T \; (y)) \, w(y) \; (\; ; y) \, dy ; \tag{3.34}$$

for any w 2 W_0. We also obtain from problem (3.27a){(3.27d)

$$Z_{D^0_+ [\; D^0} (y) \, (r \; w(y))^> \; r \; u^0(y) \, dy = Z_{D^0_+ [\; D^0} f \, (y) \, w(y) \, dy ; \tag{3.35}$$

for any w 2 W_0. Subtracting (3.35) from (3.34) we deduce

$$Z_{D^0_+ [\; D^0} (y) r \; w(y)^> \; r \quad (u \quad T \;)(y) \quad u^0$$

Subtracting (3.37) from (3.38) yields

$$
\begin{aligned}
&\int_{D'_+ \cup D'_-} (y)\,\nabla\,(z\,(y) \quad z(y)) \quad \nabla\, w(y)\,dy \\
&\quad = \int_{D'_+ \cup D'_-} (y) \quad \nabla\,(u \quad T\,)(y)\, \frac{A(\;;y) \quad I}{} \quad \nabla \quad u^0(y) A^0(0;y) \quad \nabla\, w(y)\,dy \\
&\quad + \int_{D'_+ \cup D'_-} \frac{(\;;y) f\,(T\,(y)) \quad f\,(y)}{} \quad \mathrm{div} \quad V(y) f\,(y) \quad w(y)\,dy
\end{aligned}
$$

(3.39) $\qquad =: I_1(w) + I_2(w):$

The rst integral in the right hand side of (3.39) can be written as

$$
\begin{aligned}
I_1(w) = &\int_{D'_+ \cup D'_-} (y)\,\nabla\,(u \quad T\,)(y)\, \frac{A(\;;y) \quad I}{} \quad A^0(0;y) \quad \nabla\, w(y)\,dy \\
&+ \int_{D'_+ \cup D'_-} (y)\,\nabla \quad y
\end{aligned}
$$

and

$$dJ_2(D)|_{=0} = \int_{@} \left(u^0 \frac{@v}{@n} - u^0 \, r_0 v \, r_0 V;n^0 \right) d + \int_0 \frac{@}{@n}\left(u^0 \frac{@v}{@n} \right) V;n^0 \, d$$

$$= \int_0 + \int \mathrm{div}_0(n^0) u^0 \frac{@v}{@n} V;n^0 \, d ;$$

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Zon

since u^0 @v u^0_+ on the interface ^{0} by (3.27b). Therefore, di erentiating by both sides of (3.46), us-
ing Green's formula, the jump condition (3.27c)9
00 Td [(u)-6.15949]TJ76]TJ /R51 9.9626475977 3.6 Td4929933030

Di erentiating by both sides, applying Lemma 3.8 we have

$$0 = d \int [u\ ] v\, d \ = d \left(\int u \ \ v\, d \quad \int_{+} u_{+} v\, d \right)$$

$$= \int_{0} (u^{0} v)^{0} + \int_{0} \frac{\partial(u^{0} v)}{\partial n}$$

and (3.55) follows. An analogous estimate holds for

$$M^{k}[u \quad E[u]] \quad {}^{k}M^{k}[u^{0}] = \quad {}^{k} \; M^{k}[u^{0} + (h \quad E[h$$

These equalities together with (4.7) imply

$$u^0(x) = \begin{cases} (\nabla S_- \; W_-)(u^0_-)(x) =: E_-(u^0_-)(x), & x \in D^0_- \\ (W_+ \; \nabla S_+)(u^0_+)(x) =: E_+(u^0_+)(x), & x \in D^0_+. \end{cases} \tag{4.10}$$

The randomness of the interface (ω) which is given via the randomness of the vector field $V(\cdot, x, \omega)$ implies the randomness in the solution u. From (4.10), we have

$$u^0(x,\omega) = \begin{cases} E_-(u^0_-(\omega)|_{0})(x), & x \in D^0_-, \\ E_+(u^0_+(\omega)|_{0})(x), & x \in D^0_+. \end{cases}$$

Tensorizing and integrating both sides of the above equation, we deduce

$$\mathrm{Cov}[u^0](x_1, x_2) = \begin{cases} (E_{-,x_1} \; E_{-,x_2})\mathrm{Cor}[u^0_- |_{0}](x_1, x_2), & x_1, x_2 \in D^0_-, \\ (E_{+,x_1} \; E_{+,x_2})\mathrm{Cor}[u^0_+ |_{0}](x_1, x_2), & x_1, x_2 \in D^0_+, \end{cases} \tag{4.11}$$

and in general

$$M^k[u^0](x_1, \dots, x_k) = \begin{cases} (E_{-,x_1} \; \cdots \; E_{-,x_k})M^k[u^0_- |_{0}](x_1, \dots, x_k), & x_1, \dots, x_k \in D^0_-, \\ (E_{+,x_1} \; \cdots \; E_{+,x_k})M^k[u^0_+ |_{0}](x_1, \dots, x_k), & x_1, \dots, x_k \in D^0_+. \end{cases} \tag{4.12}$$

Equation (4.11) suggests that the covariance of the solution u^0 in D^0 can be computed from the correlation function of the Dirichlet data $u^0|_{0}$ on the transmission interface.

The jump conditions in (4.1) gives

$$u^0_-(\omega) = u^0_+(\omega) + g_D(\omega) \quad \text{on} \quad {}^0, \tag{4.13}$$

and

$$\underbrace{(\; S_- \; + \; S_+)}_{=:[\; S]} u^0_+(\omega) = g_N(\omega) - (\; S_-)g_D(\omega) \quad \text{on} \quad {}^0. \tag{4.14}$$

We note that for a fixed $\omega \in$, the right hand side $g_N(\omega) - (\; S_-)g_D(\omega) \in H^{-1/2}(\,^0)$. The solution $u^0_+(\omega)$ of (4.14) belongs to $H^{1/2}(\,^0)$ ...

and $H^{1/2}(\Gamma)$-elliptic, i.e.

$$B(v,v) \geq C_2 \|v\|^2_{H^{1/2}(\Gamma')} \quad \forall v \in H^{1/2}(\Gamma'); \tag{4.18}$$

where the positive constants C_1 and C_2 are independent of v.

Proof. The boundedness of the bilinear form B is derived directly from the boundedness of V^{-1} and K. To prove ellipticity we first note that the hypersingular operator D is $H^{1/2}(\Gamma')$-semi-elliptic for all closed interface Γ', i.e.,

$$\langle Dv, v\rangle_{L^2(\Gamma')} \geq C |v|_{H^{1/2}(\Gamma')} \quad \forall v \in H^{1/2}(\Gamma'); \tag{4.19}$$

see e.g. [21, Corollary 6.25]. The Cauchy data $(u, \frac{\partial u}{\partial n})$ on Γ' satisfy

$$\begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix}. \tag{4.20}$$

Substituting (4.8) into the second equation of (4.20) gives

$$\frac{\partial u}{\partial n} = D u + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)u \quad \text{on } \Gamma'.$$

This equation and (4.8) yield

$$S = D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K).$$

Noting that K' is the adjoint operator of K, we have

$$\langle S v, v\rangle = \langle Dv, v\rangle + \langle V^{-1}(\frac{1}{2}I + K)v, (\frac{1}{2}I + K)v\rangle \quad \forall v \in H^{1/2}(\Gamma'). \tag{4.21}$$

Similarly, the exterior Dirichlet-to-Neumann operator S_+ satisfies

$$S_+ = D - (\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K)$$

and

$$\langle S_+ v, v\rangle = \langle D v, v\rangle - \langle V^{-1}(\frac{1}{2}I - K)v, (\frac{1}{2}I - K)v\rangle \quad \forall v \in H^{1/2}(\Gamma'). \tag{4.22}$$

From (4.21), (4.22), (4.19) and noting the $H^{1/2}$-ellipticity of the inverse operator of V, we derive

$$\langle [\alpha S]v, v\rangle = (\alpha_- + \alpha_+)\langle Dv, v\rangle + \alpha_- \langle V^{-1}(\frac{1}{2}I + K)v, (\frac{1}{2}$$

hS

implies

Lemma 4.2. The bilinear form $B(\cdot\,;\cdot) : H^{1/2}_{mix}(\Gamma^0 \times \Gamma^0) \times H^{1/2}_{mix}(\Gamma^0 \times \Gamma^0) \to \mathbb{R}$ is bounded and $H^{1/2}_{mix}(\Gamma^0 \times \Gamma^0)$-elliptic, i.e.,

$$B(v;w) \leq C_1 \|v\|_{H^{1/2}_{mix}(\Gamma^0\times\Gamma^0)} \|w\|_{H^{1/2}_{mix}(\Gamma^0\times\Gamma^0)}; \tag{4.28}$$

and

$$C_2 \|v\|^2_{H^{1/2}_{mix}(\Gamma^0\times\Gamma^0)} \leq B(v;v) \tag{4.29}$$

for all $v;w \in H^{1/2}_{mix}(\Gamma^0 \times \Gamma^0)$. By Lemma 4.2 there exists a unique solution of (4.27).

5. Examples. In this section, we consider the transmission problem (2.11a)–(2.11d) where the random interface $\Gamma(\omega)$ is given by

$$\Gamma(\omega) = \{x + \varepsilon\kappa(x;\omega)n(x) : x \in \mathbb{S}\}.$$

Here, the reference interface Γ^0 is the unit sphere $\mathbb{S}$. The perturbation parameter $\kappa(x;\omega) = a(\omega)$, where $a(\omega)$ is uniformly distributed in $[-1;1]$. The mean value $\mathbb{E}[\kappa] = 0$ and the covariance $\mathrm{Cov}[\kappa](x;y) = \mathrm{Cor}[\kappa](x;y) = 1/3$. The interface $\Gamma(\omega)$ is a sphere of radius $R(\omega) = 1 + \varepsilon a(\omega)$.

5.1. Analytic example. Firstly, we choose the right hand side f to be

$$f(x) = \begin{cases} (4r_x^2 - 1)^2 & \text{if } 0 \leq r_x \leq 1/2; \\ 0 & \text{if } 1/2 \leq r_x; \end{cases}$$

where $r_x = |x|$. Then solution of the transmission problem with respect to the random interface $\Gamma(\omega)$ can be analytically computed as follows:

$$u(x;\omega) = \begin{cases} \frac{1}{\alpha^-}\left(\frac{8}{21}r_x^6 - \frac{2}{5}r_x^4 + \frac{r_x^2}{6}\right) - \frac{3}{105 r_x} - \frac{23}{840} + \frac{\alpha^- + \alpha^+}{105\alpha^-\alpha^+ R(\omega)} & \text{if } 0 \leq r_x \leq \frac{1}{2}; \\ \frac{1}{105\alpha^- r_x} + \frac{\alpha^- + \alpha^+}{105\alpha^-\alpha^+ R(\omega)} & \text{if } \frac{1}{2} \leq r_x \leq R(\omega); \\ \frac{1}{105\alpha^+ r_x} & \text{if } R(\omega) \leq r_x. \end{cases} \tag{5.1}$$

In particular, the exact solution u^0 of the transmission problem on the reference interface Γ^0 ...

0 6 rx 1 6 r [( )-5.92546TJ ET Q 1 q 40.8 0 0 -4.8 2459.88 3711.88 cm28f +

We then compute the covariance of the solution u by elementary calculations, noting (5.1), to obtain

(5.4) $$\mathrm{Cov}_u(x;y) = \Big(\tfrac{1}{3} \frac{[\ ]^2}{(105}$$

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