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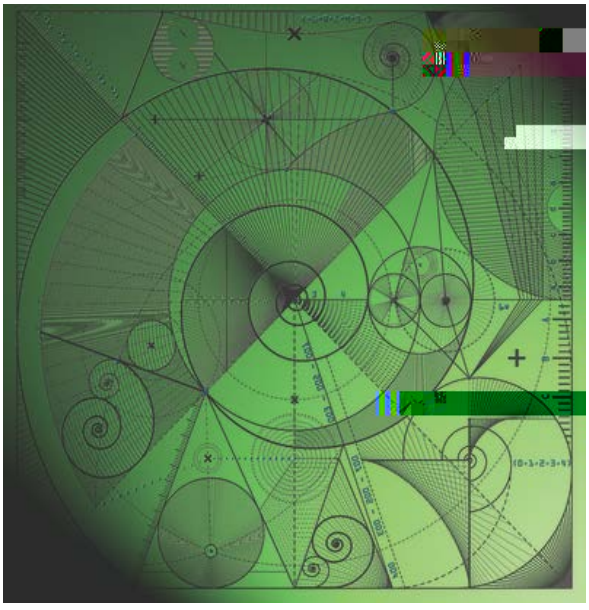

WELL-POSED PDE AND INTEGRAL EQUATION FORMULATIONS FOR SCATTERING BY FRACTAL SCREENS

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Abstract. We consider time-harmonic acoustic scattering by planar sound-soft (Dirichlet) and sound-hard (Neumann) screens embedded in R^n for $n = 2$ or 3. In contrast to previous studies in which the screen is assumed to be a bounded Lipschitz (or smoother) relatively open subset of the plane, we consider screens occupying arbitrary bounded subsets. Thus our study includes cases where the screen is a relatively open set with a fractal boundary, and cases where the screen is fractal with empty interior. We elucidate for which screen geometries the classical formulations of screen scattering are well-posed, showing that the classical formulation for sound-hard scattering is not well-posed if the screen boundary has Hausdor dimension greater than n 2. Our main contribution is to propose novel well-posed boundary integral equation and boundary value problem formulations, valid for arbitrary bounded screens. In fact, we show that for su ciently irregular screens there exist whole families of well-posed formulations, with in nitely many distinct solutions, the distinct formulations distinguished by the sense in which the boundary conditions are understood. To select the physically correct solution we propose limiting geometry principles, taking the limit of solutions for a sequence of more regular screens converging to the screen we are interested in, this a natural procedure for those fractal screens for which there exists a standard sequence of prefractal approximations. We present examples exhibiting interesting physical behaviours, including penetration of waves through screens with \holes" in them, where the \holes" have no interior points, so that the screen and its closure scatter di erently. Our results depend on subtle and interesting properties of fractional Sobolev spaces on non-Lipschitz sets.

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1. Introduction. This paper is concerned with the mathematical analysis of time-harmonic acoustic scattering problems modelled by the Helmholtz equation

$$u + k^2 u = 0; \tag{1.1}$$

(or its inhomogeneous variant (3.1) below) where $k > 0$ is the wavenumber. Our focus is on scattering by thin planar screens in R^n ($n = 2$ or 3), so that the domain in which (1.1) holds is $D := R^n \setminus \overline{\Gamma}$, where Γ, the screen, is a bounded subset of $\{x = (x_1, \ldots, x_n) : x_n = 0\}$, and the compact set $\overline{\Gamma}$ is its closure. As usual, the complex-valued function u is to be interpreted physically as either the (total) complex acoustic pressure eld or the velocity potential, and we write u as $u = u^i + u^s$, where u^i is a given incident eld and $u^s := u \quad u^i$ is the scattered eld which is to be determined and is assumed to satisfy (1.1) and the standard Sommerfeld radiation condition (equation (2.19) below). We suppose that either sound-soft or sound-hard boundary conditions hold, respectively that either

$$u = 0 \text{ or } \frac{\partial u}{\partial n} = 0 \tag{1.2}$$

on the screen in some appropriate sense, where n

These are long-standing scattering problems, their mathematical study dating back at least to [49, p. 139], and it is well-known (e.g., [45, 56], and see §3.1 for more detail) that, for arbitrary bounded D, these problems are well-posed (and the solutions depend only on the closure $\overline{D}$) if the boundary conditions are understood in the standard weak senses that $u \in W_0^{1;\mathrm{loc}}(D)$ in the sound-soft case, that $u \in W^{1;\mathrm{loc}}(D)$ and

$$\int_D (v\,\Delta u + \nabla v \cdot \nabla u)\,dx = 0; \quad \text{for all } v \in W^{1;\mathrm{comp}}(D); \tag{1.3}$$

in the sound-hard case. We spell out these weak formulations more fully in Definitions 3.1 and 3.2 below using standard Sobolev space notations defined in §2.

In the well-studied case where D is a relatively open[1]

A main aim of this paper is to derive the correct mathematical formulations for screens that are fractal or have fractal boundary, and to understand the convergence of solutions as sequences of prefractals (as in Figures 8.1, 8.2, and 8.4) converge to a fractal limit. One motivation for such a study is that fractal screen problems are of relevance to a number of areas of current engineering research, for example in the design of antennas for electromagnetic wave transmission/reception (see e.g. [48, 51]), and in piezoelectric ultrasound transducers (see e.g. [44, 43]). The attraction of using fractal structures (in practice, high order prefractal approximations to fractal structures) for such applications is in their potential for wideband performance. Indeed, a key property of fractals is that they possess structure on every length scale, and the idea is to exploit this to achieve e cient transmission/reception of waves over a broad range of frequencies simultaneously. (In this direction, much earlier, Berry [8] urged the study of waves di racted by fractal structures (termed di ractals), as a situation where distinctive high frequency asymptotics can be expected.) Although this is a mature engineering technology (at least for electromagnetic antennas), as far as we are aware no analytical framework is currently available for such problems. Understanding well-posedness and convergence of prefractal solutions for the simpler acoustic case considered in the current paper can be regarded as a signi cant rst step towards these applications.

Regarding related mathematical work, there is a substantial literature studying trace spaces on fractal boundaries (see [35] and the references therein). This is an important ingredient in formulating and analysing BVPs, and this theory has been applied to the study of elliptic PDEs in domains with fractal boundaries for example in [36]. However, a key assumption in these results is that the boundary satis es a so-called Markov inequality [35, 36]. This assumption on the boundary@ of a domain R^n (in the language of numerical analysis, a requirement that a type of inverse

we introduce new in nite families of BVP and BIE formulations distinguished by the sense in which the boundary condition is to be enforced, prove their well-posedness, and study their relationship to the standard weak and classical formulations. If the screen is su ciently regular these formulations collapse to single formulations, equivalent to the standard BVPs and BIEs, but generically these formulations have in nitely many distinct solutions.

When has empty interior the incident eld may not `see' the screen; the scattered eld may be zero. Section 4 studies when this does and does not happen: a key consideration is whether a set S_1 is or is not 1=2-null (a set S_1 is s-null if there are no $2 H^s(\ ^1)$ supported in S), which we study using results from [32]. In x5 we establish the size (cardinality) of our sets of novel formulations, and prove that distinct formulations have distinct solutions, at least for plane wave incidence and almost all incident directions. In x5.1 we investigate, mainly using recent results from [19], a key criterion in answering many of our questions, namely: when is $\tilde{H}^{1=2}(\ ) = H_{-}{}^{1=2}$? In x6 we elucidate precisely for which screens the classical formulations of De nitions 3.10 and 3.11 are or are not well-posed. Inx7 we study dependence of the screen scattering problems on , establishing continuous dependence results for weak notions of set convergence, and use these results to select, from the in nite set of solutions arising from the formulations of x3.2 and x3.3, physically relevant solutions by studying a general screen as the limit of a sequence of more regular screens. Finally, in x8 we illustrate the results of the previous sections by a number of concrete examples, mainly examples where is fractal or has fractal boundary.

We remark that some of the results on the BVP and BIE formulations in this paper appeared previously in the conference papers [31, 14] and the unpublished report [16], and that elements of some of the results of x7 and part of Example 8.2

let $I : H^{s}(R^{n}) \to (H^{s}(R^{n}))$ be the unitary isomorphism implied by the duality pairing (2.2), i.e. $I u(v) := \langle u; v \rangle_{H^{s}(R^{n}) \; H^{s}(R^{n})}$, let $R : H^{s}(R^{n}) \to (H^{s}(R^{n}))$ denote the standard Riesz isomorphism, and let $j : H^{s}(R^{n}) \to H^{s}(R^{n})$ be the unitary isomorphism de ned by $j := I^{1}R$ (j can be expressed explicitly as a Bessel potential operator, see [19

For wave scattering problems, in which functions decay only slowly at in nity, it is convenient to de ne also

$$W^{1;loc}(\ ) := \quad f\, u\ 2\ L^2_{loc}(\ ) : \quad r\ u\ 2\ L^2_{loc}(\ )\ g;$$
$$W^{1;loc}(\ ;\ ) := \quad f\, u\ 2\ W^{1;loc}(\ ) : \quad u\ 2\ L^2_{loc}(\ )\ g;$$

and

$$W^{1;loc}_0(\ ) := \quad f\, u\ 2\ W^{1;loc}(\ ) : \quad j\ u\ 2\ W^1_0(\ )\ ;\ \text{for every}\quad 2\ D(R^n)g;$$

where $L^2_{loc}(\ )$ is the set of locally integrable functions u on for which $\int_G ju(x)j^2dx < 1$ for every bounded measurable G. Analogously, for $s \quad 0$,

$$H^{s;loc}(\ ) := \quad f\, u\ 2\ L^2_{loc}(\ ) : \quad j\ u\ 2\ H^s(\ )\ ;\ \text{for every}\quad 2\ D(R^n)g:$$

Clearly $H^{0;loc}(\ ) = \quad L^2_{loc}(\ )$ and $H^{1;loc}(R^n) = W^{1;loc}(R^n)$. It holds moreover (since $n = 2$ or 3) that

$$W^{1;loc}(R^n;\ ) = \quad H^{2;loc}(R^n) \quad C(R^n); \tag{2.13}$$

since if $2\ D(R^n)$ and $u\ 2\ W^{1;loc}(R^n;\ )$, then $v := \ u\ 2\ H^1(R^n)$ and $v\ 2\ L^2(R^n)$, from which it follows, by elliptic regularity (e.g., [25]), that $v\ 2\ H^2(R^n)$, so that $v\ 2\ C(R^n)$ by the Sobolev imbedding theorem (e.g., [41, Theorem 3.26]).

2.1. Function spaces on $_1$ and trace operators. Recall that the propagation domain in which the scattered eld is assumed to satisfy (1.1) is $D := R^n\ n\ \overline{\ }\ )$ (n = 2 or 3), where (the screen) is a bounded subset of the hyperplane $_1 := f\, x\ R^n$ as;988(erplane)11erplane2

For compact $K \quad _1$, let

$$D_{1;K} := f \quad 2 D(R^n) : \quad = 1 \text{ in some neighbourhood of} K\ g:$$

For $u\ 2\ W^{1;loc}(D\ ;\ )$, we de ne, where is any element of $D_{1;-}$, the jumps

$$[u] := \quad ^{+}(\ u\ )$$

noting that layer potentials on $\overline{\Omega}$ can be thought of as layer potentials on any larger bounded open set $\overline{\Omega'}$; for more details see [16, Theorem 3.1].

Theorem 2.1. (i) For any $\phi \in H_-^{-1/2}$ and $\psi \in H_-^{1/2}$ the potentials $S\phi$ and $D\psi$ are infinitely differentiable in Ω, and satisfy the Helmholtz equation (1.1) in Ω and the Sommerfeld radiation condition (2.19);

(ii) for any $\chi \in D(\mathbb{R}^n)$ the following mappings are bounded:

$$S\chi : H_-^{-1/2} \to W^1(\mathbb{R}^n); \qquad D\chi : H_-^{1/2} \to W^1(\Omega);$$

(iii) the following jump relations hold for all $\phi \in H_-^{-1/2}$, $\psi \in H_-^{1/2}$, and $\chi \in D_{1,-}$:

$$[S\phi] = 0; \tag{2.23}$$

$$\partial_\nu(S\phi) = \tfrac{1}{2}\phi; \quad \text{so that } [\partial_\nu S\phi] = \phi; \tag{2.24}$$

$$(D\psi) = \tfrac{1}{2}\psi; \quad \text{so that } [D\psi] = \psi; \tag{2.25}$$

$$[\partial_\nu D\psi] = 0. \tag{2.26}$$

We will obtain BIE formulations for our scattering problems that can be expressed with the help of single-layer and hypersingular operators, S_1 and T_1 respectively, defined as mappings from $D(\Gamma_1)$ to $C^1(\Gamma_1)$ by the standard formulae

$$S_1\phi(x) = \int_{\Gamma_1} \Phi(x,y)\,\phi(y)\,ds(y); \quad T_1\psi(x) = \frac{\partial}{\partial\nu(x)} \int_{\Gamma_1} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\,\psi(y)\,ds(y); \tag{2.27}$$

for $x \in \Gamma_1$. Fix $\chi \in D_{1,-}$, in which case $\chi = 1$ in some bounded open neighbourhood Ω_Y of $\overline{\Omega}$. It is standard (e.g. [41]) that, if $\phi, \psi \in D(\Gamma_1)$, then

$$S_1\phi = S(\chi\phi) \text{ and } T_1\psi = \partial_\nu(D(\chi\psi)) \text{ in } \Omega_Y; \tag{2.28}$$

so that we can define mappings S_Y and T_Y from $D(\Gamma_Y)$ to $D(\overline{\Omega}_Y)$ by

$$S_Y\phi := S_1(\phi|_{\Gamma_Y}) = S(\chi\phi)|_{\Omega_Y}; \quad T_Y\psi := T_1(\psi|_{\Gamma_Y}) = \partial_\nu(D(\chi\psi))|_{\Omega_Y}; \tag{2.29}$$

for $\phi, \psi \in D(\Gamma_Y)$. It is clear from the mapping properties of the trace operators, those of S and D in Theorem 2.1(ii), and the density of $D(\Gamma_Y)$ in $H^s(\Gamma_Y)$, for $s \in \mathbb{R}$, that the representations (2.29) extend the definitions of S_Y and T_Y to bounded operators $S_Y : H^{-1/2}(\Gamma_Y) \to H^{1/2}(\Gamma_Y)$ and $T_Y : H^{1/2}(\Gamma_Y) \to H^{-1/2}(\Gamma_Y)$. (These mapping properties are well-known in the case that Γ_Y is Lipschitz or smoother - see e.g. [28, 29], where Γ is assumed C^1.)

Recall that $H^{-1/2}(\Gamma_Y)$ can be identified with the dual space $H^{1/2}(\Gamma_Y)'$ via the unitary mapping implied by the duality pairing (2.11). As noted in §2, an alternative natural unitary realisation of $H^{1/2}(\Gamma_Y)'$, via the duality pairing (2.8), is $\widetilde{H}^{1/2}_{\overline{\Omega}_Y}$ $'$ $H^{1/2}(\Gamma_1)$. We can define versions of S_Y and T_Y, $S_Z : H^{1/2}(\Gamma_Y)' \to \widetilde{H}^{1/2}_{\overline{\Omega}_Y}$ $'$ and $T_Z : H^{1/2}(\Gamma_Y)' \to \widetilde{H}^{-1/2}_{\overline{\Omega}_Y}$ $'$, which map to these alternate realisations of the dual spaces, by

$$S_Z := P_+ S(\chi\phi); \quad T_Z := P\,\partial_\nu(D(\chi\psi)); \tag{2.30}$$

for $2 H^{1/2}(\Gamma)$, $2 H^{-1/2}(\Gamma)$, where P denotes orthogonal projection onto $H^{1/2}_{c}(\Omega)^{\perp}$ in $H^{1/2}(\partial\Omega)$. Since $\varphi|_{\Omega} = 0$ for $\varphi \in H^{1/2}_{c}(\Omega)$, we see from (2.29) and (2.30) that

$$S_\Omega = j_\Omega S_\Gamma, \quad T_\Omega = j_\Omega T_\Gamma; \tag{2.31}$$

with the mapping $j_\Omega : H^{1/2}_{c}(\Omega)^{\perp} \to H^{1/2}(\Omega)$ a unitary isomorphism, as noted in §2, whose inverse (2.6) takes $\psi \in H^{1/2}(\Omega)$ to its unique extension in $H^{1/2}(\partial\Omega)$ with minimum norm.

Thus S_Ω is simply the restriction of S_Γ to Ω, and S_Γ the minimum norm extension of S_Ω; and the same relationship holds between T_Ω and T_Γ. Moreover, it is immediate from (2.27), (2.28), (2.29), and (2.31), that, for $\psi \in D(\Omega)$ and $x \in \Omega$,

$$S_\Omega \psi(x) = S_\Gamma \psi(x) = \int_\Gamma \Phi(x,y)\, \psi(y)\, ds(y); \tag{2.32}$$

$$T_\Omega \psi(x) = T_\Gamma \psi(x) = \frac{\partial}{\partial n(x)} \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n(y)}\, \psi(y)\, ds(y). \tag{2.33}$$

To write down weak forms of BIEs, we introduce sesquilinear forms associated to these BIOs, defined by

$$a_S(\psi, \varphi) := \langle S_\Omega \psi, \bar{\varphi} \rangle_{H^{1/2}(\Omega) \times H^{-1/2}(\Omega)} = \langle S_\Gamma \psi, \bar{\varphi} \rangle_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)}; \tag{2.34}$$

for $\psi, \varphi \in H^{1/2}(\Omega)$, and

$$a_T(\psi, \varphi) := \langle T_\Gamma \psi, \bar{\varphi} \rangle_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)}$$

.9626 Tf 4.151 1.494 Td [(:)]TJ/F8 9.9626 Tf 107.701 0 Td[M3t8 [(2)]TJ
.9626 Tf 4.151 1.494 Td [(·8,)-167( )]TJ/F14 9.9626 Tf 17.211 0 Td [(f)]TJ/F10 6.9738 Tf 3.874 -3.868 Td [(H)]TJ/F6 4.9813 Tf 7.059 1.996 Td [(1)]T

for all $\varphi, \psi \in H^{1/2}(\Gamma_y)$.

Since $H^{1/2}(\Gamma_y)$ is a unitary realisation of $(H^{-1/2}(\Gamma_y))'$ through the duality pairing (2.11), the upper bounds in this theorem are equivalent to the bounds

$$\|S_y\| = \|S_z\| \le C_S, \quad \|T_y\| = \|T_z\| \le C_S. \tag{2.39}$$

Further, by Lax-Milgram, the above theorem implies that these operators are invertible, with

$$\|S_y^{-1}\| = \|S_z^{-1}\| \le c_S^{-1}, \quad \|T_y^{-1}\| = \|T_z^{-1}\| \le c_T^{-1}. \tag{2.40}$$

Our BIEs will be expressed in terms of single-layer and hypersingular operators associated to the screen Γ, defined analogously to (2.30). Specifically, let V denote any closed subspace of $H^{-1/2}$ (so that $V' \subset H^{1/2}(\Gamma_y)$ by (2.1)), and let $V^\# := ((V')^a)^?$ denote the natural unitary realisation of $(V')'$ implied by the duality pairing (2.8), where $(V')^a$ denotes the annihilator of V' in $H^{1/2}(\Gamma_1)$, defined by (2.7). We

3. Formulating screen scattering problems.

3.1. Standard BVP formulations and their interrelation. In this section we study the standard formulations for screen scattering from the literature. We will see that these formulations are equivalent for screens that occupy open sets in $\mathbb{R}^1$ with Lipschitz boundaries (this is well-known), but that some of these standard formulations (Problems SD-cl, SN

case that the domain D satisfies a local compactness condition, which it does and ∂D satisfies Wilcox's finite tiling property - see [56, Theorem 4.3 and p.62].

Theorem 3.5. Problems SD-w and SN-w have exactly one solution for every $u^i \in W^{1,loc}(D;\mathbb{R})$.

The following lemma uses the notations introduced above (2.42), so that V is any closed subspace of $H_-^{1/2}$ and P_V orthogonal projection onto the realisation $V \subset H^{1/2}(\Gamma_1)$ of its dual space. This lemma will allow us to make connections between SD-w, SN-w and the other formulations we introduce below.

Lemma 3.6. If u satisfies SD-w or SN-w then $u \in C(D)$, $u^s \in C^1(D)$, $[u] \in H_-^{1/2}$, $[\partial_n u] \in H_-^{-1/2}$, and

$$[u^s] = [u] = 0 \text{ if } u \text{ satisfies SD-w, while } [\partial_n u^s] = [\partial_n u] = 0 \text{ if } u \text{ satisfies SN-w.} \tag{3.3}$$

Further, for all $\Gamma \subset \partial D_{1,-}$,

$$P_{V^+}(u) = 0 \text{ with } V = H_-^{1/2}; \text{ which implies that } (u)|_\Gamma = 0; \tag{3.4}$$

if u satisfies SD-w, while

$$P_V \partial_n(u) = 0 \text{ with } V^+ = H_-^{1/2}; \text{ which implies that } \partial_n(u)|_\Gamma = 0; \tag{3.5}$$

if u satisfies SN-w.

Proof. If u satisfies SD-w or SN-w then $u^s \in C^1(D)$ by (

so that $\partial_n^+(u)$, and also $\partial_n(u)$ as $[\partial_n u] = 0$, is in the annihilator of $H^{1/2}_-$, i.e. $P_{V^+}\partial_n(v) = 0$ with $V^+ = H^{1/2}_-$. This implies, arguing as above for the Dirichlet case, that $\partial_n(u)|_\Gamma = 0$. □

The next lemma is immediate from standard elliptic regularity results up to the boundary (e.g., [25]).

Lemma 3.7. Suppose that $v \in W^{1,\mathrm{loc}}(D;\Delta)$ satisfies the Helmholtz equation (1.1) in D and that, for some $\Delta \in \mathcal{D}_{1,-}$, either $\gamma(v)|_\Gamma \in C^1(\Gamma)$ or $\partial_n(v) \in C^1(\Gamma)$. Then v is smooth up to the boundary away from Γ, i.e., $v \in C^1_\Gamma(\overline{D})$ in the notation of (2.18).

From Lemma 3.7 and Lemma 3.6 we have the following corollary.

Corollary 3.8. If u satisfies SD-w or SN-w and $u^i + k^2 u^i = 0$ in a neighbourhood of $\overline{\Gamma}$ (so that u^i is C^1 in a neighbourhood of $\overline{\Gamma}$), then $u^s \in C^1_\Gamma(\overline{D})$.

Remark 3.9. Related to Corollary 3.8, common choices for the incident field in Problems SD-w and SN-w are the plane wave

$$u^i(x) = \exp(ik\,d\cdot x), \quad x \in \mathbb{R}^n, \tag{3.6}$$

for some unit vector $d \in \mathbb{R}^n$, i and C the incident field

and Neumann BVPs with more general boundary data. The following is a standard Sobolev space formulation (e.g., [52]).

Definition 3.11 (

(3.12). Similarly, if $V^+ = H^{1=2}(\ )$, then $N(V^+)$ is the standard formulation N-st augmented by the additional constraints (3.14) and (3.15). We will show in Theorem 6.1 that, if is sufficiently regular, these additional constraints are superfluous, which in turn will imply (Theorem 6.2) that the standard formulations D-st and N-st are then well-posed.

Corollary 3.23. If (3.18) holds and v satisfies $D(V\ )$ ($N(V^+)$), then v satisfies D-st with $g_D = g_D|\ $ (N-st with $g_N = g_N|\ $).

Remark 3.24. Arguing as in the proof of Lemma 3.21, we have also that if (3.18) holds and g_D and g_N are given as in Problems $SD(V\ )$ and $SN(V^+)$, then, for every $2 D_{1;-}$,

$$g_D| = (u^i)| \quad \text{and} \quad g_N| = \partial_n(u^i)| :$$

Corollary 3.23, Remark 3.24, and Lemma 3.12 imply:

Corollary 3.25. If (3.18) holds, u^s satisfies $SD(V\ )$ ($SN(V^+)$), and $(+k^2)u^i = 0$ in a neighbourhood of $\bar{-}$, then u^s satisfies SD-cl (SN-cl).

The following result is one half of a proof of well-posedness of $D(V\ )$ and $N(V^+)$ that we will complete in Theorems 3.29 and 3.30 below.

Theorem 3.26. Problems $D(V\ )$ and $N(V^+)$ (and hence also $SD(V\ )$ and $SN(V^+)$) have at most one solution.

Proof. Suppose that v satisfies $D(V\ )$ with $g_D = 0$, and choose real-valued $;\ 2 D_{1;-}$ such that $= 1$ in a neighbourhood of the support of

Combining this result with Theorems 3.5 and 3.26, we obtain:

Theorem 3.28. For $V = H_-^{1/2}$, Problem SD(V) has exactly one solution which is the unique solution of SD-w. Similarly, for $V^+ = H_-^{1/2}$, Problem SN(V^+) has exactly one solution which is the unique solution of SN-w.

3.3. BIEs, well-posedness, and equivalence of formulations. We now study the reformulation as BIEs of the various BVPs we have introduced above. We will also use these BIEs to complete proofs of well-posedness and to complete our study of the connections between the various formulations.

The operators in these BIEs will be the single layer and hypersingular operators, $S : V \to V^+$ and $T : V^+ \to V$, that we introduced in (2.42), where, as above, V is some closed subspace of $H_-^{(-1+2)}(\Gamma_1)$, and V

with $[v] \in V^+$ the unique solution of the BIE

$$T[u] = g_N. \tag{3.25}$$

Further, (3.25) is equivalent to the variational problem: find $[v] \in V^+$ such that

$$a_T([v], \psi) = \langle g_N, \psi \rangle_{H^{1/2}(\Gamma_1) \times H^{1/2}(\Gamma_1)}, \quad \text{for all} \quad \psi \in V^+. \tag{3.26}$$

For every V^+ satisfying (3.18), the solution of $N(V^+)$ is a solution of N-st with $g_N = g_N|_{\Gamma_1}$.

The following corollary is immediate from Theorem 3.29, noting that $[\partial_n u^i] = 0$ as observed in the proof of Lemma 3.6. The exception is the penultimate sentence which is a restatement of Theorem 3.28, and the last sentence which follows from Lemma 3.25.

Corollary 3.31. Problem $SD(V^-)$ has a unique solution, which satisfies (where $u := u^i + u^s$)

$$u(x) = u^i(x) - S[\partial_n u](x), \qquad x \in D, \tag{3.27}$$

with $[\partial_n u] \in V^-$ the unique solution of the BIE

$$S[\partial_n u] = P_{V^+}(-u^i), \tag{3.28}$$

where $\psi \in D_{1,-}$ is arbitrary. Further, (3.28) is equivalent to the variational problem: find $[\partial_n u] \in V^-$ such that

$$a_S([\partial_n u], \psi) = \langle -(u^i), \psi \rangle_{H^{1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_1)}, \quad \text{for all} \quad \psi \in V^-. \tag{3.29}$$

If $V^- = H_-^{-1/2}$ then $SD(V^-)$ and SD-w have the same unique solution. For every V^- satisfying (3.18), the solution of $SD(V^-)$ is a solution of SD-cl if $(\Delta + k^2)u^i = 0$ in a neighbourhood of $\overline{D}$.

Similarly, the following corollary follows from Theorem 3.30, with the penultimate sentence a consequence of Theorem 3.28, and the last sentence a consequence of Lemma 3.25.

Corollary 3.32. Problem $SN(V^+)$ has a unique solution, and this solution satisfies (where $u := u^i + u^s$)

$$u(x) = u^i(x) + D[u](x), \qquad x \in D, \tag{3.30}$$

with $[u] \in V^+$ the unique solution of the BIE

$$T[u] = -P_{V^-}(\partial_n(-u^i)), \tag{3.31}$$

where $\psi \in D_{1,-}$ is arbitrary. Further, (3.31) is equivalent to the variational problem: find $[u] \in V^+$ such that

$$a_T([u], \psi) = -\langle \partial_n(-u^i), \psi \rangle_{H^{1/2}(\Gamma_1) \times H^{1/2}(\Gamma_1)}, \quad \text{for all} \quad \psi \in V^+. \tag{3.32}$$

If $V^+ = H_-^{1/2}$ then $SN(V^+)$ and SN-w have the same unique solution. For every V^+ satisfying (3.18), the solution of $SN(V^+)$ is a solution of SN-cl if $(\Delta + k^2)u^i = 0$ in a neighbourhood of $\overline{D}$.

4. When is the scattered field just $u^s = 0$?. From a variety of perspectives, including that of inverse scattering, a fundamental question is: does the incident field 'see' the screen, by which we mean simply: is $u^s \neq 0$?

We first note from Corollaries 3.31 and 3.32 that a necessary condition for the solution of SD(V) (SN(V^+)) to be non-zero is $V \neq \{0\}$ ($V^+ \neq \{0\}$). And, trivially, there exists a subspace V of $H^{-1/2}$ with $V \neq \{0\}$ if and only if $H^{-1/2} \neq \{0\}$. So one relevant question is: for which compact sets $K \subset \mathbb{R}^1$ is $H_K^{-1/2} = \{0\}$? That is, using the terminology introduced below (2.3): for which compact sets K is K 1/2-null? We address this question in Theorem 4.1, which will be a key tool in much of our later analysis, using results from [32].

Before stating the theorem we note that, as will be of no surprise to readers familiar with potential theory (e.g., [37], and [6, Theorem 2.7.4]), for the Dirichlet problem a key role is played by the capacity, defined for a compact set $F \subset \mathbb{R}^n$ by $\mathrm{cap}(F) := \inf\{\|u\|^2_{H^1(\mathbb{R}^n)}\}$, where the infimum is over all $u \in D(\mathbb{R}^n)$ such that $u \geq 1$ in a neighbourhood of F. For an open set $\subset \mathbb{R}^n$, and for an arbitrary Borel set $E \subset \mathbb{R}^n$,

$$\mathrm{cap}(\ ) := \sup_{\substack{F \subset \\ F \text{ compact}}} \mathrm{cap}(F); \quad \mathrm{cap}(E) := \inf_{\substack{E \subset \\ \text{open}}} \mathrm{cap}(\ ) :$$

This last definition for arbitrary Borel sets applies, in particular, in the cases E compact and E open, for which it coincides with the immediately preceding definitions for these cases (as shown e.g. in [32, 33]). Also, we note that in Theorem 4.1 and the rest of the paper we use the notation $m(E)$ to denote the $(n-1)$-dimensional Lebesgue measure of E, for measurable $E \subset \Gamma$. Finally, we remark that illustrations of the last sentence of the theorem are given in Examples 8.1-8.3.

Theorem 4.1. Let E be a Borel subset of Γ. Then:

(a) E is 1/2-null if and only if $\mathrm{cap}(E) = 0$;

(b) if E is closed then E is 1/2-null if and only if $W_0^1(\mathbb{R}^n \setminus E) = W^1(\mathbb{R}^n)$;

(c) if E is non-empty then E is not 1/2-null;

(d) if $\dim_H(E) < n-2$ then E is 1/2-null, and if $\dim_H(E) > n-2$ then E is not 1/2-null;

(e) if $m(E) = 0$ or E is 1/2-null, then E is 1/2-null;

(f) if $E = \Omega$ and Ω is in the algebra of subsets of $\mathbb{R}^n$ generated by all C^0 open sets, then E

Another approach to the question `Is $u^s = 0$?' is to observe, since $SD($

in nitely many $d \in S^n$. □

Remark 4.4. Regarding the application of the above lemmas, it is important to note that any closed subspace $V \subset H^s(\Gamma_1)$

to SN(V^+) is $u^s = 0$. If $\overline{\ }$ is not 1=2-null, in particular if $\overline{\ }$ is non-empty (though this is not necessary, see Example 8.3), then: (i) if u^i satisfies the conditions of Theorem 4.6 for the Neumann case, then the solution u^s to SN-w does not vanish, and nor does the solution to SN(V^+) as long as V^+ satisfies (3.18) and is non-empty; (ii) if u^i is the plane wave (3.6) and $V^+ \neq f0g$ then the solution u^s to SN(V^+) is non-zero for almost all incident directions d (all but finitely many directions if $n = 2$).

5. Do all our formulations have the same solution?. We focus in this section on the formulations D(V), N(V^+), SD(V), and SN(V^+) introduced in x3.2, with V satisfying the physical selection principle (3.18), and on the standard formulations introduced in x3.1, addressing the question of the section title. We will show that either the formulations SD(V) and SN(V^+) satisfying (3.18) coincide for all choices of V^+ and V satisfying (3.18), or, in each case, the cardinality of the set of distinct formulations is that of the continuum. Further we show that, in the specific case of plane wave incidence, for almost all directions of incidence, there are, in both the sound-soft and sound-hard cases, infinitely many distinct solutions u^s to these formulations.

The main results of the section are Theorems 5.5 and 5.6. For their proof we shall appeal to a number of preliminary results. The first, an approximation lemma, is used to prove Lemma 5.2, and is a special case of [19, Lemma 3.22].

Lemma 5.1. Suppose that $N \in \mathbb{N}$ and $x_1, \ldots, x_N \in \mathbb{R}^n$ are distinct. Then there exists a family $(v_j)_{j \in \mathbb{N}} \subset C^1(\mathbb{R}^n)$ such that: for all $j \in \mathbb{N}$, $v_j(x) = 0$, if $|x - x_i| < 1/j$ for some $i \in \{1, \ldots, N\}$; for all $\in H^s(\mathbb{R}^n)$ with $|s| \leq 1/2$, $\|v_j\|_{H^s(\mathbb{R}^n)} \to 0$ as $j \to \infty$.

Lemma 5.2. If $|s| \leq 1/2$ and $\neq H^s(\ ) \neq H^{\overline{s}}$, then the quotient space $H^{\overline{s}}/H^s(\ )$ is an infinite-dimensional separable Hilbert space.

Proof. It is standard that the quotient space $H^{\overline{s}}/H^s(\ )$ is a Hilbert space (e.g., [19, x2.1]), and it is separable as $H^s(\mathbb{R}^n) \subset H^{\overline{s}}$ is separable.

Suppose now that $|s| \leq 1/2$ and $H^s(\ ) \neq H^{\overline{s}}$. We show first that for every $v \in H^{\overline{s}} \setminus H^s(\ )$ there exists a set $K \subset \overline{\ }$ with at least countably many points such that each point $x \in K$ has the following property:

$$\text{for all } > 0 \text{ there exists } \in D(B(x)) \text{ with } v \in H^{\overline{s}} \setminus H^s(\ ). \tag{5.1}$$

For suppose this is not true. Then, for some $v \in H^{\overline{s}} \setminus H^s(\ )$ the set X of points $x \in \overline{\ }$ with this property is finite. Choose a sequence $(v_\ell)_{\ell=1}^{\infty} \subset C^1(\mathbb{R}^n)$ as follows: set $v_\ell = 1$ for $\ell \in \mathbb{N}$ if $X = \ ;$ if $X = \{x_1, \ldots, x_N\}$, for distinct $x_1, \ldots, x_N \in \overline{\ }$, choose (v_ℓ) to have the properties of Lemma 5.1. For every $x \in K := \overline{\ } \setminus X$ there exists $(x) > 0$ [illegible]

Then W is a linear subspace of $H^s = H^s(\ )$. Further, for every $N \in \mathbb{N}$, choosing $\varepsilon > 0$ such that $\varepsilon < |x_j - x_\ell|/3$, for $j, \ell = 1, \ldots, N$, and $\varphi_j \in D(B_\varepsilon(x_j))$ with $\varphi_j v \in H^s_- \cap H^s(\ )$ for $j = 1, \ldots, N$, it is clear that

$$\{\varphi_1 v + H^s(\ ), \ldots, \varphi_N v + H^s(\ )\} \subset W$$

is linearly independent, for if $w := \sum_{j=1}^N \alpha_j \varphi_j v \in H^s(\ )$, for some $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$, then, for $\ell \in \{1, \ldots, N\}$, choosing $\psi \in D(\mathbb{R}^n)$ such that $\psi = 1$ in a neighbourhood of $B_\varepsilon(x_\ell)$ and $\mathrm{supp}(\psi) \subset B_{2\varepsilon}(x_\ell)$, we see that $\psi w \in H^s(\ ) \cap w = \alpha_\ell \psi \varphi_\ell v = \alpha_\ell \varphi_\ell v$, so $\alpha_\ell = 0$ since $\varphi_\ell v \notin H^s(\ )$. Thus $\dim(W) \geq N$, for each N, so that $H^s_-/H^s(\ )$ is in nite-dimensional. □

The following lemma uses cardinal arithmetic from ZFC set theory [30].

Lemma 5.3. Let H be a separable Hilbert space of dimension at least two. Then the cardinality of the set of closed subspaces of H is $\mathfrak{c}$, the cardinality of $\mathbb{R}$.

Proof. Suppose H is a separable Hilbert space. Then its cardinality is at least $\mathfrak{c}$ if its dimension is at least two, for if $v_1, v_2 \in H$ are orthogonal, $\{\cos(\theta) v_1 + \sin(\theta) v_2) : \theta \in [0, 2\pi)\}$ is a distinct closed subspace of H for each $0 \leq \theta < \pi$. Further, H has a countable orthonormal basis, so that the cardinality of H itself is no larger than that of the set of sequences of complex numbers, which is, as $|\mathbb{C}^{\mathbb{N}}| = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. Finally, since there is an injection $H \to \mathbb{R}$, and each closed subspace V is characterised by a countable set of orthonormal basis vectors in H, the cardinality of the set of closed subspaces of H is no larger than $|\mathbb{R}^{\mathbb{N}}| = \mathfrak{c}$. □

Lemma 5.4. If $H^{1/2}(\ ) \subset H^{1/2}_-$, then the set of closed subspaces V satisfying (3.18) has cardinality $\mathfrak{c}$.

Proof. $H^{1/2}_-/H^{1/2}(\ )$ is unitarily isomorphic to W, the orthogonal complement of $H^{1/2}(\ )$ in $H^{1/2}_-$, so the set of closed subspaces of W has cardinality $\mathfrak{c}$ by Lemmas 5.2 and 5.3. The result follows as there is a one-to-one correspondence between the set of closed subspaces of W and the closed subspaces V of $H^{1/2}_-$ that satisfy (3.18), given by $L \mapsto H^{1/2}(\ ) \oplus L$, for L a closed subspace of W. □

We are now ready to state and prove the main theorems in this section.

Theorem 5.5. If $H^{1/2}(\ ) = H^{1/2}_-$ and V satisfies (3.18) then $V = H^{1/2}_-$, so that there is only one formulation $D(V\ )$, and one formulation $SD(V\ )$, with V satisfying (3.18). Further, in this case $SD(V\ )$ and SD-w have the same unique solution. If $H^{1/2}(\ ) \neq H^{1/2}_-$ then the set of subspaces V satisfying (3.18) has cardinality $\mathfrak{c}$. Further, if u^I is the incident plane wave (3.6), then:

(a) if V_1 & V_2 are any two subspaces

denote S, V^+, and P_{V^+}, respectively, when $V = V_j$. Now, for $j = 1;2$, $S_j : V_j \to V_j$ is an isomorphism, so that, if $V_1 \subsetneq V_2$, $V_2 = S_2(V_1) \oplus W$, where W is the non-empty orthogonal complement of $S_2(V_1)$ in V_2. Let P denote orthogonal projection onto W in $H^{1=2}(\Gamma_1)$. Then, by Lemma 4.3 and Remark 4.4, $P g \neq 0$ for almost all $d \in S^n$ (all but finitely many d if $n = 2$), which implies that $\Gamma_2 \notin V_1$, so that $\Gamma_1 \neq \Gamma_2$. In the general case that $V_1 \neq V_2$, it holds for $j = 1;2$ that $V_j = V \oplus W_j$, for some orthogonal subspaces V, W_1, and W_2 of $H^{1=2}(\Gamma_1)$, with at least one of W_1 and W_2 non-trivial. Lemma

The next theorem provides su cient conditions on ensuring that $\widetilde{H}^{1/2}(\ ) = H_{-}^{1/2}$. Here, and henceforth, when we say that the open set $\mathbb{R}^n$ is C^0 except at countably many points $P \subset \partial$, we mean that its boundary ∂ can at each point in $\partial \setminus P$ be locally represented as the graph (suitably rotated) of a C^0 function from $\mathbb{R}^{n-1}$ to $\mathbb{R}$, with lying only on one side of ∂. (In more detail we mean that satis es the conditions of [26, De nition 1.2.1.1], but for every $x \in \partial \setminus P$ rather than for every $x \in \partial$.) Examples of such sets include prefractal appoximations to the Sierpinski triangle (Figure 8.1). That the theorem holds when is C^0 is well known (e.g. [41, Theorem 3.29]), and that it holds in the generality stated here follows from [19, Theorem 3.24].

Theorem 5.10. If is C^0, or is C^0 except at countably many points $P \subset \partial$, with P having only a nite set of limit points, then $\widetilde{H}^{1/2}(\ ) = H_{-}^{1/2}$.

By combining the results in Corollary 5.9, Theorem 5.10 and Theorem 4.1, we can derive the following corollary, which explores the case where $\overline{\ }$ is C^0, except perhaps at countably many points, but where $\subset \overline{\ }$. We illustrate this corollary with examples where is open, so that $= \overline{\ }$ and $\overline{\ } = \overline{\ }$, but $\subset \overline{\ }$, in Examples 8.5 and 8.6.

Corollary 5.11. Suppose that $\subset \overline{\ }$ and that $\overline{\ }$ is C^0, or C^0 except at a countable set of points P that has only nitely many limit points. Then:

(i) if $\overline{\ } \setminus$ has interior points then $\widetilde{H}^{1/2}(\ ) \subset H_{-}^{1/2}$;

(ii) if $\overline{\ } \setminus$ is countable, or $\overline{\ } \setminus \subset \partial$, where $_1$ is a Lipschitz open set, then $\widetilde{H}^{1/2}(\ ) = H_{-}^{1/2}$;

(iii) if $\widetilde{H}^{1/2}(\ ) = H_{-}^{1/2}$ or $m(\overline{\ } \setminus\ ) = 0$, then $\widetilde{H}^{1/2}(\ ) = H_{-}^{1/2}$;

(iv) $\widetilde{H}^{1/2}(\ ) = H_{-}^{1/2}$ if and only if $\mathrm{cap}(\overline{\ } \setminus\ ) = 0$;

(v) if $d := \dim_H(\overline{\ } \setminus\ ) < n - 2$ then $\widetilde{H}^{1/2}(\ ) = H_{-}^{1/2}$, while $\widetilde{H}^{1/2}(\ ) \subset H_{-}^{1/2}$ for $d > n - 2$.

6. Well-posedness of standard formulations. In this section we investigate for which screens the standard formulations SD-cl, SN-cl, D-st, and N-st are well-posed. Since we know already that these formulations have at least one solution (Corollary 3.15 and Theorems 3.29 and 3.30), we need only consider the question of uniqueness. By Corollary 3.13 it is enough to consider this question for D-st and N-st.

We noted in Remark 3.22 that $D(V\ )$ with $V = \widetilde{H}^{1/2}(\ )$ is equivalent to D-st, augmented by the additional constraints (3.11) and (3.12), and that $N(V^+)$ with $V^+ = \widetilde{H}^{1/2}(\ )$ is equivalent to N-st augmented by (3.14) and (3.15). Since $D(V\ )$ and $N(V^+)$ are well-posed (Theorem 3.29), asking whether D-st has more than one solution is equivalent to asking whether (3.11) and (3.12) are super uous, i.e. whether $D(V\ )$ remains well-posed if (3.11) and (3.12) are deleted. Similarly, asking whether N

3.11) and 3ivalend 0 J va

Proof. If $v \in W^{1,\mathrm{loc}}(D)$ and satisfies (1.1) in D, so that $v \in W^{1,\mathrm{loc}}(D;\Delta)$, then $[v] \in H_-^{1/2}$ and $[\partial_\nu v] \in H_-^{-1/2}$. Thus (3.12) and (3.15) are superfluous if $V = H_-^{-1/2}$. If also v satisfies (3.10) then $P_{V^+}[v] = 0$, i.e. $[v] \in (V^+)^\perp$, so that $[v] \in (V^+)^\perp \cap H_-^{1/2}$. Thus (3.11) is superfluous if $(V^+)^\perp \cap H_-^{1/2} = \{0\}$. Similarly, if v satisfies (3.13), then $[\partial_\nu v] \in (V^-)^\perp \cap H_-^{-1/2}$, so (3.14) is superfluous if $(V^-)^\perp \cap H_-^{-1/2} = \{0\}$.

Conversely, suppose that $V_1 \neq V_2$, where $V_1 = V^-$ and $V_2 = H_-^{-1/2}$. Then, by Theorem 5.5(a), there exists an incident wave direction d such that $u_1^s \neq u_2^s$, where u_j^s is the solution to SD(V_j) for $j = 1, 2$. Since $V_1 \subset V_2$, u_2^s satisfies all the conditions of SD(V^-) except for (3.12). So $v := u_1^s - u_2^s \neq 0$ is a solution to D(V^-) with $g_D = 0$ and (3.12) deleted. Thus uniqueness fails for D(V^-) if (3.12) is deleted and $V^- \neq H_-^{-1/2}$. Similarly, arguing using Theorem 5.6(a), uniqueness fails for N(V^+) if (3.15) is deleted and $V^+ \neq H_-^{1/2}$.

Next, suppose that $(V^+)^\perp \cap H_-^{1/2} \neq \{0\}$, choose a non-zero $\varphi \in (V^+)^\perp \cap H_-^{1/2}$, and set $v := D\varphi$. Then, by Theorem 2.1, except that (3.11) is not satisfied as $[v] = \varphi$, v satisfies D(V^+) with $g_D = 0$. Thus uniqueness fails for D(V^+) if (3.11) is deleted and $(V^+)^\perp \cap H_-^{1/2} \neq \{0\}$. Similarly, uniqueness fails for N(V^+) if (3.14) is deleted and $(V^-)^\perp \cap H_-^{-1/2} \neq \{0\}$, arguing in this case by defining $v := S\varphi$, where $0 \neq \varphi \in (V^-)^\perp \cap H_-^{-1/2}$.

If $V = H_{(\Gamma)}^{1/2}(\Gamma)$, then (recall (2.41)) $V^\perp = H_{(\Gamma^c)}^{1/2}{}^*$ and $(V^+)^\perp \cap H_-^{1/2} = H_{\partial\Omega}^{1/2}$, so that $\{0\} = (V^+)^\perp \cap H_-^{1/2}$ if and only if $\partial\Omega$ is $1/2$-null. If V satisfies (3.18), then $V \supset H_{(\Gamma)}^{1/2}(\Gamma)$ and, as noted below (2.8), $V^\perp \subset H_{(\Gamma^c)}^{1/2}{}^*$, so that $(V^+)^\perp \cap H_-^{1/2} \subset H_{\partial\Omega}^{1/2}$, so that $\{0\} = (V^+)^\perp \cap H_-^{1/2}$ if $\partial\Omega$ is $1/2$-null. □

By applying Theorem 6.1 with $V = H_{(\Gamma)}^{1/2}(\Gamma)$, and recalling Theorem 4.1, and the results of §5.1, we can now clarify when the standard scattering and BVP formulations are well-posed.

Theorem 6.2. (a) Problems SD-cl and D-st are well-posed if and only if $H_{(\Gamma)}^{1/2}(\Gamma) = H_-^{1/2}$ and $\partial\Omega$ is $1/2$-null. In particular, any of the following conditions is sufficient to ensure that SD-cl and D-st

One approach to selecting the \physically correct" solution from this multitude is to think of as the limit of a sequence of screens $(\Gamma_j)_{j\in\mathbb{N}}$, where each screen Γ_j is su ciently regular so that the correct choice of solution is clear. This is a natural approach for recursively generated fractal structures. For example the open set whose boundary is the Koch snow ake is usually generated as the limit of a sequence $\Omega_1 \quad \Omega_2 \quad \cdots$ (see Figure 8.4), where each Ω_j is a Lipschitz open set. Likewise the closed set which is the Sierpinski triangle is usually generated as the limit of a sequence of closed sets $K_1 \quad K_2 \quad \cdots$ (see Figure 8.1), where, for each j, Γ_j is C^0 (indeed Lipschitz) except at a nite set of points. Our rst theorem (cf. [19, Theorem 4.3 and Proposition 4.5]) deals with these and related cases.

Theorem 7.1. (a) Suppose that $\Omega_j \quad \Omega_1$ is open for $j \in \mathbb{N}$, that $\Omega_1 \quad \Omega_2 \quad \cdots$, and that $\Omega = [_{j=1}^{1} \Omega_j$ is bounded. Further (as usual) let $D := \mathbb{R}^n \setminus \overline{\Omega}$. Let u^s denote the solution to SD(V) with $V = H^{1/2}(\Gamma)$, and u_j^s the solution to SD(V) with $V = H^{1/2}(\Gamma_j)$. Then, for every $\in D(\mathbb{R}^n)$,

$$\| (u^s - u_j^s)\|_{W^1(\mathbb{R}^n)} \to 0 \quad \text{as} \quad j \to 1 \quad : \tag{7.1}$$

Similarly, if u^s denotes the solution to SN(V^+) with $V^+ = H^{1/2}(\Gamma)$, and u_j^s the solution to SN(V^+) with $V^+ = H^{1/2}(\Gamma_j)$, then, for every $\in D(\mathbb{R}^n)$, $\| (u^s - u_j^s)\|_{W^1(D)} \to 0$ as $j \to 1$.

(b) Suppose that $\Omega_j \quad \Omega_1$ is compact for $j \in \mathbb{N}$, that $\Omega_1 \quad \Omega_2 \quad \cdots$, and that is given by $= \setminus_{j=1}^{1} \Omega_j$. Let u^s denote the solution to SD(V) with $V = H^{1/2}$, and u_j^s the solution to SD(V) with $V = H^{1/2}_{\Omega_j}$. Then (7.1) holds for every $\in D(\mathbb{R}^n)$. Similarly, let u^s denote the solution to SN(V^+) with $V^+ = H^{1/2}$, and u_j^s the solution to SN(V^+) with $V^+ = H^{1/2}_{\Omega_j}$. Then, for every $\in D(\mathbb{R}^n)$ and every open $\supset \Omega_1$ with , $\| (u^s - u_j^s)\|_{W^1(D)} \to 0$ as $j \to 1$, where $D := \mathbb{R}^n \setminus \overline{\Omega}$.

Proof. Part (a). In the rst case, by Corollary 3.31, $u^s = S\,[\partial_n u]$ and $u_j^s = S\,[\partial_n u_j]$, where $[\partial_n u] \in V$ is the unique solution of (3.29) with $V = H^{1/2}(\Gamma)$, and $[\partial_n u_j]$ the unique solution with $V = H^{1/2}(\Gamma_j)$. Since, by Theorem 2.2, a_S is coercive and, by [19, Proposition 3.33],

$$H^{1/2}(\Gamma) = \overline{[_{j=1}^{1} H^{1/2}(\Gamma_j)}^{\,};$$

it follows from Cea's lemma (cf. [19, (8)]) that $\|$

Corollary 3.32 and the coercivity of a_T, and noting that, if $\quad_1$ is open and , then $\quad_j$ for all j su ciently large. □

Remark 7.2. We note that, if $u_j^s \to u^s$ in any of the senses indicated in the above theorem, then also, by elliptic regularity arguments, $u_j^s \to u^s$ uniformly on compact subsets of D. To see this, let $F \quad D$ be any such compact subset, choose $2 D(D)$ with $= 1$ in a neighbourhood of F, and let $v_j := (u^s \quad u_j^s)$. Then $(+ k^2)v_j = f_j \; 2 \; L^2_{comp}(R^n)$, which implies that (3.2) holds with u^i and f replaced by v_j and f_j. From this, noting also that $\|f_j\|_{L^2(R^n)} \to 0$ as $j \to 0$ and $\mathrm{supp}(f_j) \quad \mathrm{supp}( )$, we see that, uniformly for $x \; 2 \; F$, $|u^s(x) \quad u_j^s(x)|$

by Theorem 5.10.

Similarly, part (b) follows from Theorem 7.1(b) and Remark 7.2, provided there exists a sequence $(_j)$ satisfying the conditions of De nition 7.4. One such sequence can be constructed as follows. Let$J_j := f` 2 Z^{n\ 1} : \ \overline{`;j}$ 6 ;g. Then $_j :=$ $[_{`2J_j} \overline{`;j}$ satis es the required conditions. □

Remark 7.6. The limiting geometry principles in De nitions 7.3 and 7.4 provide criteria for selecting physically relevant solutions when is either compact, or bounded

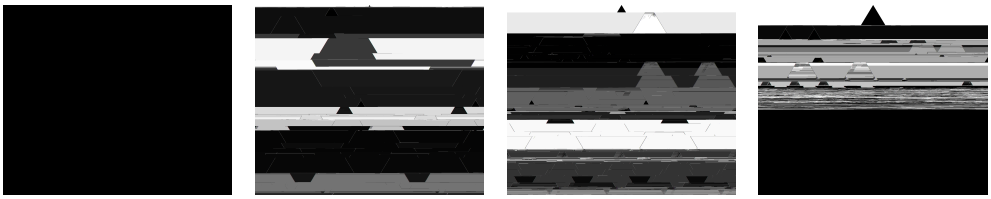

Fig. 8.1 . The rst four prefractal approximations to the Sierpinksi triangle.

number of examples where the screen is fractal or has a fractal boundary.

Our rst three examples consider scattering by screens that are compact sets with empty interior. While the standard formulations SD-w and SN-w are physically relevant in this case, in particular are limiting geometry solutions in the sense of De nition 7.4, the formulations SD-cl and SN-cl lack boundary conditions: equations (3.8) are empty. In Examples 8.1 and 8.2 the screen has zero surface measure and the incident eld fails to see the screen for sound-hard scattering, while Example 8.3 is a screen with empty interior but positive surface measure where the scattered eld, de ned as a limiting geometry solution by De nition 7.4, is non-zero for both sound-soft and sound-hard scattering.

Example 8.1 (Scattering by a Sierpinski triangle and its prefractal approximations). Suppose thatn = 3 and

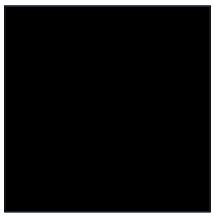

Fig. 8.2 . The rst four prefractal approximations to the standard two-dimensional middle-third Cantor set (or Cantor dust).

for $V = H_j^{1=2} = H^{1=2}(\ _j)$, and (assuming $u^i + k^2 u^i = 0$ in a neighbourhood of) u_j^s also satis es SD-cl by Lemma 3.14; further SD-cl is well-posed by Theorem 6.2(a). By Theorem 7.1 and Remark 7.2, $u_j^s \to u^s$ as $j \to 1$ uniformly on compact subsets of $D := R^3 \setminus$, and locally in W^1 norm.

In the case of sound-soft scattering and $n = 3$, some of the results of the following example can be found in [19

Our next example is a screen with empty interior but positive surface measure which is not 1=2-null. Our \Swiss cheese" construction follows that of Polking, who used it in [47, Theorem 4] to construct explicitly a compact set $F \quad R^n$ with empty interior that is not n=2-null.

Example 8.3 (Scattering by a \Swiss cheese" screen). By a Swiss cheese screen we mean, for $n = 2;3$, a compact subset of $_1$ constructed as follows: take a bounded open set $_1$, and sequences $(x_j)_{j=1}^1$ and $(r_j)_{j=1}^1$ $(0;1)$, and de ne $_j := F_j$ for $j \ 2 \ N$, where

$$F_j := \overline{\ } n \bigcup_{m=1}^{j} B_{r_m}(x_m);$$

and set $:= \backslash_{j=1}^1 \ _j$. If the sequence $(x_m)_{m=1}^1$ is dense in , then is empty. But need not be empty: indeed, if the radii r_m are su ciently small and decrease su ciently rapidly, then has positive measure, since

$$m(\ ) \quad m(\overline{\ }) \quad 2(\ = 2)^n \ {}_1 \overset{1}{\underset{j=1}{\times}} r_m^n \ {}^1; \quad \text{for } n = 2;3:$$

The condition $m(\ ) > 0$ is necessary (Theorem 4.1(e)), but is not su cient to ensure that is not 1=2-null. But if the radii are small enough and decrease su ciently rapidly then, indeed, is not 1=2-null. It is shown in [32, Theorem 4.6] that for every open

$_1$ r

>eBut 1=2 $_m$ $_{m1}$

Fig. 8.3 . The irregular screen, consisting of a countable number of circles, of Example 8.4, this an example of a non-Lipschitz (indeed non- C^0) screen for which the classical formulations remain well-posed and equivalent to the weak formulations and to the standard BIEs (1.4).

in the sound-soft case, and SN-w, SN(V^+) subject to (3.18), and SN-cl all have the same unique solution in the sound-hard case. In Examples 8.5 and 8.6 the classical formulations SD-cl and SN-cl both fail to be well-posed. In each case $\overline{\ }$ is an interval for $n = 2$, a square for $n = 3$, and the sound-hard scattered field for $\overline{\ }$ in the limiting geometry sense, equivalently the solution to our new formulation SN(V^+) with $V^+ = I + K^{1=2}(\ )$, is different from the scattered field for $\overline{\ }$, which satisfies SN-w. In Example 8.6 the same effects are seen for the sound-soft case.

Example 8.4 (An irregular screen where all formulations are well-posed and coincide). Suppose that $n = 3$ and, for $j \ 2 \ N$, let $s_j := (2j + 1)=(2j(j + 1))$, $r_j := 1=(2j(j + 1))$, $\overline{\ }$, let

null if $n = 3$ and $= (1 \quad)=2 \quad 1=4$, so that $\overline{\ } \; n \quad C$ is $1=2$-null, and $\mathcal{H}^{1=2}(\quad) = H_{-}^{1=2}$ by Corollary 5.9. As also noted in Example 8.2, if $n = 2$ then $\dim_H(C) > 0$ so that $\dim_H(\overline{\ } \; n \quad) > \dim_H(C \; n @_0) > 0$ since $\dim_H(@_0) = 0$, while if $n = 3$ and $> \quad 1=4$ then $\dim_H(C) > 1$ so that $\dim_H(\overline{\ } \; n \quad) > \dim_H(C \; n @_0) > 1$ since $\dim_H(@_0) = 1$. Thus, by Corollary 5.11(iii), $\mathcal{H}^{1=2}(\quad) \; 6 \; H_{-}^{1=2}$ if $n = 2$, or if $n = 3$ and $> \quad 1=4$.

Since $\mathcal{H}^{1=2}(\quad) = H_{-}^{1=2}$, the formulations SD(V) for sound-soft scattering by that satisfy (3.18) collapse onto a single formulation with a single unique solution u^s, and this solution, by Corollary 7.5(a), is also the unique solution of SD-w, and the limiting geometry solution in the sense of De nition 7.3. In particular, if u_j^s denotes the solution, when is replaced by $_j$, to SD-w or SD-cl (all formulations have the same solution for the screen $_j$, as in the other examples above), $u_j^s \; ! \quad u^s$, uniformly on compact subsets of D and locally in W^1 norm as $j \; ! \; 1$, by Theorem 7.1(a).

Since $_0$ is Lipschitz, $_0 = \overline{\ }$, and $\mathcal{H}^{1=2}(\overline{\ }) = H_{-}^{1=2}$, u^s is also the unique solution to (any of the formulations) for sound-soft scattering by $_0$, including SD-w and SD-cl. Thus the limiting geometry solution for sound-soft scattering by the screen $= \quad _0 \; n \; C$ is the same as the solution for the screen $_0$: the fractal \hole" C in does not have any e ect.

51(e6.9738 Tf 3.527 -11.374 Td [(e)]TJ/F11 9.9626 Tf -2.332 -2.518 Td [(H)]TJ/F7 6.9738 Tf 9

Similar remarks apply in the sound-hard case if $n = 3$ and $\quad 1=4$, for then $\mathcal{H}^{1=2}(\quad) = H_{-}^{1=2}$, and the limiting geometry solution of De nition 7.3 for is just the solution for scattering by the square screen $_0$. But if n

s 7.3pf
SD
) = H)30ni37 /F7 6.9738 379 1-a
mulations ns)-401(ha(c)51(493651(lapse)-379(onto)-379(a)5single)-379(formulation)-379(with)-379(a)-380(si)1(ngle)-380(unique)-379(solution)]T

Fig. 8.4 . The rst four prefractal approximations to the Koch snow ake.

that (x_j) is dense in $_0$

in this case (Theorem 6.2(b)). These formulations do have solutions, namely the solutions of the well-posed formulations $SN(V^+)$ and $N(V^+)$, respectively, for any V^+ satisfying (3.18), in particular for $V^+ = \widetilde{H}^{1/2}(\Gamma)$ and $V^+ = H^{1/2}_{\overline{\Gamma}}$ (Theorem 3.30 and Corollary 3.32), but uniqueness does not hold.

An open problem in this case is whether or not $\widetilde{H}^{1/2}(\Gamma) = H^{1/2}_{\overline{\Gamma}}$. If this does hold then the formulations for sound-hard scattering, $SN(V^+)$ with V^+ satisfying (3.18), collapse to a single formulation, but if this does not hold then there are infinitely many of these formulations (with cardinality c), with infinitely many distinct solutions (Theorem 5.6). In particular the solution to $SN(V^+)$ with $V^+ = \widetilde{H}^{1/2}(\Gamma)$ is the limiting geometry solution in the sense of Definition 7.3, in particular is the limit as $j \to \infty$ of u^s_j, the solution to SN-w, or equivalently SN-cl, for the Lipschitz open set that is the jth prefractal Ω_j. The solution for $V^+ = H^{1/2}_{\overline{\Gamma}}$ is the limiting geometry solution for the closed von Koch snowflake $\overline{\Omega}$ in the sense of Definition 7.4, equivalently the solution to SN-w (Corollary 7.5). It is an open question whether these solutions are the same: all we can currently say is that they are the same if $\widetilde{H}^{1/2}(\Gamma) = H^{1/2}_{\overline{\Gamma}}$, and that if $\widetilde{H}^{1/2}(\Gamma) \subsetneq H^{1/2}_{\overline{\Gamma}}$ then these solutions are different for plane wave incidence for almost all incident wave directions (Theorem 5.6).

Similar remarks apply for sound-soft scattering. It is also an open problem whether or not $\widetilde{H}^{-1/2}(\Gamma) = H^{-1/2}_{\Gamma}$. SD-cl and D-st are well-posed if and only if equality holds (Theorem 6.2), and this also determines whether the formulations $SD(V^-)$ and $D(V^-)$ with V^- satisfying (3.18) collapse to a single formulation, or whether there are infinitely many formulations with cardinality c (Theorem 5.5). Similarly to the sound-hard case the solution to $SD(V^-)$ with $V^- = \widetilde{H}^{-1/2}(\Gamma)$ is the limiting geometry solution in the sense of Definition 7.3, in particular is the limit as $j \to \infty$ of u^s_j, the solution to SD-w, or equivalently SD-cl, for the Lipschitz open set that is the jth prefractal Ω_j. The solution for $V^- = H^{-1/2}_{\Gamma}$ is the limiting geometry solution for the screen $\overline{\Omega}$ in the sense of Definition 7.4, equivalently the solution to SN-w (Corollary 7.5). We do not know whether these solutions are the same: this depends on whether or not $\widetilde{H}^{-1/2}(\Gamma) = H^{-1/2}_{\Gamma}$ (Theorem 5.5).

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REFERENCES

[1] Y. Achdou and T. Deheuveis , A transmission problem across a fractal self-similar interface , Multiscale Model. Simul., 14 (2016), pp. 708{736.
[2] Y. Achdou, C. Sabot, and N. Tchou , Di usion and propagation problems in some rami ed domains with a fractal boundary , ESAIM: Mathematical Modelling and Numerical Analysis, 40 (2006), pp. 623{652.
[3] ———, A multiscale numerical method for Poisson problems in some rami ed domains with a fractal boundary , Multiscale Model. Simul., 5 (2006), pp. 828{860.
[4] ———, Transparent boundary conditions for the Helmholtz equation in some rami ed domains with a fractal boundary , J. Comput. Phys., 220 (2007), pp. 712{739.
[5] Y. Achdou and N. Tchou , Trace theorems for a class of rami ed domains with self-similar fractal boundaries , SIAM J. Math. Anal., 42 (2010), pp. 1449{1482.
[6] D. R. Adams and L. I. Hedberg , Function Spaces and Potential Theory , Springer, 1999.
[7] P. Bagnerini, A. Buffa, and E. Vacca , Mesh generation and numerical analysis of a Galerkin method for highly conductive prefractal layers , Appl. Numer. Math., 65 (2013), pp. 63{78.
[8] M. V. Berry , Di ractals , J. Phys. A: Math. Gen., 12 (1979), p. 781.

[9] ———, Distribution of modes in fractal resonators , in Structural Stability in Physics, W. Güttinger and H. Eikemeier, eds., Springer, 1979, pp. 51{53.
[10] ———, Some geometrical aspects of wave motion: Wavefront dislocations, di raction catastrophes, di ractals , in Geometry of the Laplace Operator, R. Osserman and A. Weinstein, eds., AMS, 1980, pp. 13{38.
[11] A. Bj orn and J. Bj orn , Approximations by regular sets and Wiener solutions in metric spaces, Comment. Math. Univ. Carolin., 48 (2007), pp. 343{355.
[12] C. J. Bouwkamp , Di raction theory , Rep. Prog. Phys., 17 (1954), pp. 35{100.
[13] R. Capitanelli and M. A. Vivaldi , Weighted estimates on fractal domains , Mathematika, 61 (2015), pp. 370{384.
[14] S. N. Chandler-Wilde , Scattering by arbitrary planar screens , in Computational Electromagnetism and Acoustics, Oberwolfach Report No. 03/2013, R. Hiptmair, R. H. W. Hoppe, P. Joly, and U. Langer, eds., 2013, pp. 154{157.
[15] S. N. Chandler-Wilde, I. G. Graham, S. Langdon, and E. A. Spence , Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering , Acta Numer., 21 (2012), pp. 89{305.
[16] S. N. Chandler-Wilde and D. P. Hewett , Acoustic scattering by fractal screens: mathematical formulations and wavenumber-explicit continuity and coercivity estimates . Technical report. University of Reading preprint MPS-2013-17.
[17] ———, Wavenumber-explicit continuity and coercivity estimates in acoustic scattering by planar screens, Integr. Equat. Oper. Th., 82 (2015), pp. 423{449.
[18] S. N. Chandler-Wilde, D. P. Hewett, and A. Moiola , Interpolation of Hilbert and Sobolev spaces: quantitative estimates and counterexamples , Mathematika, 61 (2015), pp. 414{443.
[19] ———, Sobolev spaces on non-Lipschitz subsets of R^n with application to boundary integral equations on fractal screens , Integr. Equat. Operat. Th., 87 (2017), pp. 179{224.
[20] D. L. Colton and R. Kress , Integral Equation Methods in Scattering Theory , Wiley, 1983.
[21] M. Costabel , Time-dependent problems with the boundary integral equation method , Encyc. Comput. Mech., 1:25 (2004).
[22] C. P. Davis and W. C. Chew , Frequency-independent scattering from a at strip with TE_z-polarized elds , IEEE Trans. Ant. Prop., 56 (2008), pp. 1008{1016.
[23] L. A. De Lacerda, L. C. Wrobel, H. Power, and W. J. Mansur , A novel boundary integral formulation for three-dimensional analysis of thin acoustic barriers over an impedance plane, J. Acoust. Soc. Am., 104 (1998), pp. 671{678.
[24] K. Falconer , Fractal Geometry: Mathematical Foundations and Applications , Wiley, 3rd ed., 2014.
[25] D. Gilbarg and N. S. Trudinger , Elliptic Partial Di erential Equations of Second Order , Springer, 1998.
[26] P. Grisvard , Elliptic Problems in Nonsmooth Domains , Pitman, 1985.
[27] G. Grubb , Distributions and Operators , Springer, 2009.
[28] T. Ha-Duong , On the transient acoustic scattering by a at object , Japan J. Indust. Appl. Math., 7 (1990), pp. 489{513.
[29] ———, On the boundary integral equations for the crack opening displacement of at cracks , Integr. Equat. Oper. Th., 15 (1992), pp. 427{453.
[30] P. R. Halmos , Naive Set Theory , Springer, 1974.
[31] D. P. Hewett and S. N. Chandler-Wilde , Wavenumber-explicit coercivity estimates in scattering by screens, in Proc. 11th Int. Conf. on Mathematical and Numerical Aspects of Waves, Gammarth, Tunisia, 2013, pp. 253{254.
[32] D. P. Hewett and A. Moiola , On the maximal Sobolev regularity of distributions supported by subsets of Euclidean space, Anal. Appl., published online, doi:10.1142/S021953051650024X, (2016).
[33] C. Hua and B. D. Sleeman , Fractal drums and the n-dimensional modi ed Weyl-Berry conjecture , Commun. Math. Phys., 168 (1995), pp. 581{607.
[34] ———, High-frequency estimates for the Neumann scattering phase in non-smooth obstacle scattering , Inverse Prob., 17 (2001), pp. 1{12.
[35] A. Jonsson and H. Wallin , Function Spaces on Subsets of R^n , Math. Rep., 2 (1984).
[36] ———, Boundary value problems and brownian motion on fractals , Chaos, Solitons and Fractals, 8 (1997), pp. 191{205.
[37] O. D. Kellogg , Foundations of Potential Theory , Springer, 1929.
[38] M. L. Lapidus , Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture , Trans. Am. Math. Soc., 325 (1991), pp. 465{529.
[39] M. Levitin and D. Vassiliev , Spectral asymptotics, renewal theorem, and the Berry conjecture for a class of fractals , Proc. Lond. Math. Soc., s3-72 (1996), pp. 188{214.

[40] V. G. Maz'ya , Sobolev Spaces with Applications to Elliptic Partial Di erential Equations , Springer, 2nd ed., 2011.
[41] W. McLean , Strongly Elliptic Systems and Boundary Integral Equations , CUP, 2000.
[42] J. Meixner , Die Kantenbedingung in der Theorie der Beugung electromagnetischer Wellen an vollkommen leitenden ebenen Schirmen , Ann. Phys., 6 (1949), pp. 1{9.
[43] A. J. Mulholland, J. W. Mackersie, R. L. O'Leary, A. Gachagan, A. J. Walker, and N. Ramadas , The use of fractal geometry in the design of piezoelectric ultrasonic transducers, in 2011 IEEE International Ultrasonics Symposium (IUS), IEEE, 2011, pp. 1559{1562.
[44] A. J. Mulholland and A. J. Walker , Piezoelectric ultrasonic transducers with fractal geometry , Fractals, 19 (2011), pp. 469{479.
[45] P. Neittaanmaki and G. F. Roach , Weighted Sobolev spaces and exterior problems for the Helmholtz equation , Proc. R. Soc. Lond. A, 410 (1987), pp. 373{383.
[46] J. M. Neuberger, N. Sieben, and J. W. Swift , Computing eigenfunctions on the Koch snow ake: a new grid and symmetry , J. Comp. Appl. Math., 191 (2006), pp. 126{142.
[47] J. C. Polking , Leibniz formula for some di erentiation operators of fractional order , Indiana U. Math. J., 21 (1972), pp. 1019{1029.
[48] C. Puente-Baliarda, J. Romeu, R. Pous, and A. Cardama , On the behavior of the Sierpinski multiband fractal antenna , IEEE T. Antenn. Propag., 46 (1998), pp. 517{524.
[49] J. W. S. Rayleigh , The Theory of Sound, Volume II , Macmillan, 2nd Ed., 1896.
[50] B. D. Sleeman and C. Hua , An analogue of Berry's conjecture for the phase in fractal obstacle scattering , IMA J. Appl. Math., 49 (1992), pp. 193{202.
[51] G. Srivatsun, S. S. Rani, and G. S. Krishnan , A self-similar fractal Cantor antenna for MICS band wireless applications , Wireless Eng. Tech., 2 (2011), pp. 107{111.
[52] E. P. Stephan , Boundary integral equations for screen problems in R^3, Integr. Equat. Oper. Th., 10 (1987), pp. 236{257.
[53] E. P. Stephan and W. L. Wendland , An augmented Galerkin procedure for the boundary integral method applied to two-dimensional screen and crack problems , Appl. Anal., 18 (1984), pp. 183{219.
[54] W. L. Wendland and E. P. Stephan , A hypersingular boundary integral method for two-dimensional screen and crack problems , Arch. Rational Mech. Anal., 112 (1990), pp. 363{390.
[55] N. Wiener , Certain notions in potential theory , J. Math. Phys., 3 (1924), pp. 24{51.
[56] C. H. Wilcox , Scattering Theory for the d'Alembert Equation in Exterior Domains , Springer, 1975.