A POINTWISE CHARACTERISATION OF THE PDE SYSTEM OF VECTORIAL CALCULUS OF VARIATIONS IN L^1

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Abstract. Let $n; N \ 2 \ N$ with R^n open. Given $H \ 2 \ C^2(\ R^N \ R^{Nn})$; we consider the functional

(1) $E_1 \ (u; O) := \text{ess sup} \, H(\ ; u; Du); \quad u \ 2 \ W^{1;} \quad H_P(\ ; u; Du) \quad H \ (\ ; u; Du) \ = 0;$

where $[\![A]\!]^{?} := \text{Proj}_{R(A)^?}$. Herein we establish that generalised solutions to (2) can be characterised as local minim1ll3r lof-4196(21ou 930(cal)r1(cte)-1(r)1(i193l(6i050)olutiono

of (regular) maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ are valued, whilst subscripts of H denotes derivatives with respect to the respective variables (x, η, P). We use the symbolisations $x = (x_1, ..., x_n)^\top$, $u = (u_1, ..., u_N)^\top$, $D_i \equiv \partial/\partial x_i$ whilst Latin indices $i, j, k, ...$ will run in $\{1, ..., n\}$ and Greek indices $\alpha, \beta, \gamma, ...$ will run in $\{1, ..., N\}$. Further, for any linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^N$, the notation $[\![A]\!]^\perp$ used above denotes the orthogonal projection onto the orthogonal complement of its range $R(A) \subseteq \mathbb{R}^N$:

$$[\![A]\!]^\perp := \mathrm{Proj}_{R(A)^\perp}. \tag{1.4}$$

Also, "$O \Subset \Omega$" means that O is open and $\overline{O} \subseteq \Omega$. In index form, F_∞ reads

$$
\begin{aligned}
F_{\infty\alpha}(x,\eta,P,X) := {} & \sum_i H_{P_{\alpha i}}(x,\eta,P) \left(\sum_{\beta,j} H_{P_{\beta j}}(x,\eta,P) X_{\beta ij} + \sum_\beta H_{\eta_\beta}(x,\eta,P) P_{\beta i} \right. \\
& \left. + H_{x_i}(x,\eta,P) \right) + H(x,\eta,P) \sum_\beta [\![H_P(x,\eta,P)]\!]^\perp_{\alpha\beta} \\
& \left(\sum_{\gamma, i,j} H_{P_{\beta i} P_{\gamma j}}(x,\eta,P) X_{\gamma ij} + \sum_{\gamma, i} H_{P_{\beta i} \eta_\gamma}(x,\eta,P) P_{\gamma i} \right. \\
& \left. + \sum_i H_{P_{\beta i} x_i}(x,\eta,P) - H_{\eta_\beta}(x,\eta,P) \right),
\end{aligned}
$$

where $\alpha = 1, ..., N$. Note that, although H is C^2, the projection map $[\![H_P(\cdot, u, Du)]\!]^\perp$ is discontinuous when the rank of $H_P(\cdot, u, Du)$ changes. Further, we remark that because of the perpendicularity of H_P and $[\![H_P]\!]^\perp$, the system can be decoupled into two independent systems which we write in a contracted fashion:

$$
\begin{cases}
H_P(\cdot, u, Du)\, D\big(H(\cdot, u, Du)\big) = 0, \\
H(\cdot, u, Du)\, [\![H_P(\cdot, u, Du)]\!]^\perp \left(\mathrm{Div}\big(H_P(\cdot, u, Du)\big) - H_\eta(\cdot, u, Du) \right) = 0.
\end{cases}
$$

When $H(x, \eta, P) = |P|^2$ (the Euclidean norm on $\mathbb{R}^{Nn}$ squared), the system (1.2)-(1.4) simplifies to the so-called ∞-Laplacian:

$$\Delta_\infty u := \Big(Du \otimes Du + |Du|^2 [\![Du]\!]^\perp \otimes I \Big) : D^2 u = 0. \tag{1.5}$$

The scalar case $N = 1$ first arose in the work of G. Aronsson in the 1960s [A1, A2] who initiated the area of Calculus of Variations in the space L^∞. The field is fairly well-developed today and the relevant bibliography is vast. For a pedagogical introduction to the topic accessible to non-experts, we refer to [K8]. We just mention that in the scalar case, generalised solutions to the respective PDE which is commonly referred to as the Aronsson equation and simplifies to

$$H_P(\cdot, u, Du) \cdot \Big(H_P(\cdot, u, Du)^\top D^2 u + H_\eta(\cdot, u, Du) Du + H_x(x, \eta, P) \Big) = 0$$

are understood in the viscosity sense (see [C, CIL, K8]). The study of the vectorial case $N \geq 2$ started much more recently and the full system (1.2)-(1.4) first appeared in the work [K1] of one of the authors in the early 2010s and it is being studied quite systematically ever since (see [K2]-[K7], [K9]-[K13], as well as the joint works of the second author with Abugirda, Pryer, Croce and Pisante [AK], [CKP], [KP, KP2]).

In this paper we are interested in the characterisation of appropriately defined generalised vectorial solutions $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ to (1.2)-(1.4) in terms of the

and

$$O(u) := \text{Argmax} \quad H(\ ; u; Du) \ : \ \overline{O} \ :$$

We conclude the introduction with some rudimentary facts about generalised

and hence

$$\frac{1}{} \; r(\;) \quad r(0) \qquad \frac{1}{} \; R(\;; y^0) \quad R(0; y^0) \;;$$

where $y^0 \; 2 \; \overline{O}$ is any point such that $R(0; y^0) = \max_{\overline{O}} R(0; \;)$. Hence, we have

$$\begin{aligned}
\underline{D}r(0^+) &= \liminf_{! \; 0^+} \frac{1}{} \; r(\;) \quad r(0) \\
&\quad \max_{y^0 2 \overline{O}} \; \liminf_{! \; 0^+} \frac{1}{} \; R(\;; y^0) \quad R(0; y^0) \\
&= \max_{y 2 O(u)} \; \liminf_{! \; 0^+} \frac{1}{} \; R(\;; y) \quad R(0; y) \\
&= \max_{O(u)} \; \liminf_{! \; 0^+} \frac{1}{} \; H \quad ; u + \; A; \; Du + \quad DA \qquad H \quad ; u; Du) \\
&= \max_{O(u)} \; \liminf_{! \; 0^+} \frac{1}{} \; H(\;; u; Du) + \quad H \quad ; u; Du \quad A + \quad H_P(\;; u; Du) : DA \\
&\quad + \; O \; j \; DAj^2 \\
&\quad)
\end{aligned}$$

Note that $F_1^?(x; ;P; X)$ 2 R^N, A

for any $(x; ; X) \ 2 \quad R^N \quad R_s^{Nn^2}$.

(B) In view of the mutual perpendicularity of the two components of F_1 (see (3.1)-(3.2)), (A) is a consequence of the following particular results:

$$H_P(; u; Du) F_1^k (; u; Du; D^2u) = 0 \quad \text{in} \quad ;$$

in the D-sense, if and only if

$$E_1 (u; O) \quad E_1 (u + A; O); \quad 8 \ O \ b \quad ; \ 8 \ A \ 2 \ A_O^{k;1} (u)$$

and also

$$H(; u; Du) \ H_P(; u; Du)^{?} \ F_1^{?} (; u; Du; D^2u) \quad H \ (; u; Du) \ = 0 \quad \text{in} \quad ;$$

in the D-sense, if and only if

$$E_1 (u; O) \quad E_1 (u + A; O); \quad 8 \ O \ b \quad ; \ 8 \ A \ 2 \ A_O^{?;1} (u):$$

We note that in the special case of C^2 solutions, Corollary 1 describes the way that classical solutions $u : R^n \quad ! \quad R^N$ to (1.2)-(1.4) are characterised.

Remark 8 (About pointwise properties of C^1 D-solutions). Let $u : R^n \quad ! \quad R^N$ be a D-solution to (1.2)-(1.4) in $C^1(\ ; R^N)$. By De nition 3, this means that for any $D^2u \ 2 \ Y \quad ; \overline{R}_s^{Nn^2}$,

$$F_1 \ x; u(x); Du(x); X_x \ = 0; \quad \text{a.e. } x \ 2 \ \text{ and all } \ X_x \ 2 \ \text{supp} \ D^2u(x) :$$

By De nition 2, every di use hessian of a putative solution is de ned a.e. on as

is clearly satisfied at x. If $H_P\big(x, u(x), Du(x)\big) \neq 0$, then we select any direction normal to the range of $H_P\big(x, u(x), Du(x)\big) \in \mathbb{R}^{N \times n}$, that is

$$n_x \in \big[R\big(H_P(x, u(x), Du(x))\big)\big]^{\perp} \subseteq \mathbb{R}^N$$

which means $n_x^{\top} H_P\big(x, u(x), Du(x)\big) = 0$. Of course it may happen that the linear map $H_P\big(x, u(x), Du(x)\big) : \mathbb{R}^n$

and by applying Lemma 5a), we have

$$0 \quad \max_{z \in \overline{O_\varepsilon(x)}} \Big\{ H_P\big(z;u(z);Du(z)\big) : DA(z) + H\big(z;u(z);Du(z)\big) \; A(z) \Big\}$$

$$\xrightarrow{\varepsilon \to 0} \quad H_P\big(x;u(x);Du(x)\big) : \quad F_1^k\big(x;u(x);Du(x);X_x\big)$$

and hence

$$H_P\big(x;u(x);Du(x)\big) \; F_1^k\big(x;u(x);Du(x);X_x\big) \qquad 0;$$

for any $\in \mathbb{R}^N$. By the arbitrariness of we deduce that

$$H_P\big(x;u(x);Du(x)\big) \; F_1^k\big(x;u(x);Du(x);X_x\big) = 0$$

for any $D^2u \in Y\big(\;;\overline{\mathbb{R}}_s^{Nn^2}\big)$, $x \in$ and $X_x \in \operatorname{supp}\big(D^2u(x)\big)$, as desired.

Conversely, we x O

Proof of Corollary 1. If $u \in C^2(\ ; R^N)$, then by Lemma 4 any di use hessian of u satis es $D^2u(x) = {}_{D^2u(x)}$ for a.e. $x \in$. By Remark 8, we may assume this happens for all $x \in$. Therefore, the reduced support of $D^2u(x)$ is the singleton set f ${}_{D^2u(x)}$