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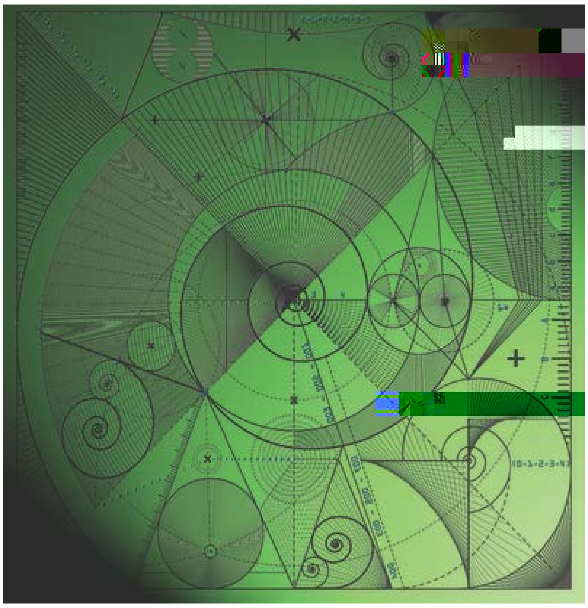

Solutions of the Fully Compressible Semi-Geostrophic System

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Abstract

The fully compressible semi-geostrophic system is widely used in the modelling of large-scale atmospheric flows. In this paper, we prove rigorously the existence of weak Lagrangian solutions of this system, formulated in the original physical coordinates. In addition, we provide an alternative proof of the earlier result on the existence of weak solutions of this system expressed in the so-called geostrophic, or dual, coordinates. The proofs are based on the optimal transport formulation of the problem and on recent general results concerning transport problems posed in the Wasserstein space of probability measures.

Keywords: geophysical fluid mechanics, nonlinear PDEs, meteorology.

MSC classification: 35Q35, 35Q86

1 Introduction

The behaviour of the atmosphere is governed by the compressible Navier-Stokes equations, together with the laws of thermodynamics, equations describing phase changes, source terms and boundary fluxes. These equations are too complex to be solved accurately in a large-scale atmospheric context and therefore reductions and approximations of the Navier-Stokes equations are often used to validate and understand the solutions that have been computed.

In [6], Benamou and Brenier assumed the fluid to be incompressible, the Coriolis parameter constant and the boundaries rigid. They then used a change of variables, introduced by Hoskins in [13], to derive the so-called dual formulation. In this formulation, the equations are interpreted as a Monge-Ampere equation coupled with a transport problem, to prove the existence of stable weak solutions.

In [9], Cullen and Gangbo relaxed the assumption of rigid boundaries with a more physically appropriate free boundary condition. However, they additionally assumed constant potential temperature to obtain the 2-D system known as the semi-geostrophic shallow water system. After passing to dual variables, they showed existence of stable weak solutions for this system of equations.

In [10], Cullen and Maroofi proved that stable weak solutions of the 3-D compressible semi-geostrophic system in its dual formulation exist, returning to the assumption of rigid boundaries.

The main problem posed by the existence results in [6], [9] and [10] is that they are all proved in dual space. It is difficult to relate these results directly to the Navier-Stokes equations, or indeed other reductions of them. For this reason, and in order to give the dual space results physical meaning, Cullen and Feldman [8] mapped the solutions back to the original, physical coordinates and extended the results of [6] and [9], proving existence of weak Lagrangian solutions in physical space, in the incompressible case. We mention that very recently results on the existence of Eulerian solutions have been proved in the case of a two-dimensional torus, and of a convex open subset in 3-dimensional space [2], [3].

In this paper, we make use of recent results in the analysis of ODEs in spaces of measures, in particular those of [4], [5], in order to provide an alternative proof of the dual space result in [10]. We also extend considerably the results of [10] to prove the existence of weak Lagrangian solutions of the fully compressible semi-geostrophic system in the original physical coordinates. As in the incompressible case studied in [8], the proof is based on the existence of an appropriate flow map with rather low regularity; however we also show that, if we could assume additional regularity, then the solutions derived would determine classical solutions.

The paper is organised as follows: in Section 2 we introduce various definitions, and the (*iii*) $u^g(t,x) = (u_1^g(t,x), u_2^g(t,x), 0)$ represents the geostrophic velocity;

(*iv*) $p(t,x)$ represents the pressure;

(*v*) $\rho(t,x)$ represents the density;

(*vi*) $\theta(t,x)$ represents the potential temperature. Given its physical meaning, we assume $\theta(t,x)$ to be strictly positive and bounded;

(*vii*) $\Phi(x)$ is the given geopotential representing gravitation and centrifugal forces. We assume that $\Phi \in C^2(\Omega)$ and that $\frac{\partial\Phi}{\partial x_3}(x) \neq 0$ for all $x \in \bar{\Omega}$;

(*viii*) f_{cor} denotes the Coriolis parameter, which we assume to be constant; indeed we will normalise this parameter to be equal to 1;

(*ix*) p_{ref} is the reference value of the pressure;

(*x*) c_v is a constant representing the specific heat at a constant pressure;

(*xi*) c_p is a constant representing the specific heat at a constant volume;

(*xii*) R represents the gas constant and satisfies $R = c_p - c_v$;

(*xiii*) $\gamma = c_p/c_v$ denotes the ratio of specific heats (this is approximately 1.4 for air).

Notations and other conventions (2.2)

Throughout, we will only consider measures that are absolutely continuous with respect to Lebesgue measure. Given an open set A in $\mathbb{R}^3$, we will denote by

- $\mathcal{P}_{ac}(A)$ - the set of probability measures in $\mathbb{R}^3$ with supports contained in A;
- χ_A - the characteristic function of A.

Unless otherwise specified, measurable means Lebesgue measurable and $a.e.$ means Lebesgue-$a.e.$

D_t denotes the Lagrangian derivative, defined as $D_t = \partial_t + u\cdot\nabla$, where u denotes the velocity of the flow.

e_3. denotes the unit vector $(0,0,1)$ in $\mathbb{R}^3$

For convenience, we will sometimes use the notation $F_{(t)}(\cdot) = F(t,\cdot)$ to denote the map F evaluated at fixed time t.

Important definitions

Definition 2.1. *We define*

$$\mathcal{P}_{ac}^2(\mathbb{R}^3) := \{\mu \in \mathcal{P}_{ac}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |x|^2 \, d\mu(x)\,dx < +\infty\},$$

with tangent space

$$T_\mu \mathcal{P}_{ac}^2(\mathbb{R}^3) = \overline{\{\nabla\varphi : \varphi \in C_c^\infty(\mathbb{R}^3)\}}^{L^2(\mu;\mathbb{R}^3)}. \tag{2.3}$$

De nition 2.2. *Given two Borel probability densities* $_1(\ )$ *and* $_2(\ )$ *in* R^3*, we de ne the* Wasserstein-2 distance*,* W_2*, between* $_1$ *and* $_2$ *as follows:*

$$W_2^2(\ _1;\ _2) := \inf_{\ 2\ (\ _1;\ _2)} \int_{\mathrm{R}^3\ \ \mathrm{R}^3} jx \quad yj^2\, d\ (x;y): \tag{2.4}$$

with

$$(\ _1;\ _2) = f \quad : \quad \textit{probability measure on } \mathrm{R}^3 \quad \mathrm{R}^3 \textit{ with marginals } \ _1 \textit{ and } \ _2 g:$$

We denote by $_0(\ _1;\ _2)$ *the set of minimisers of (2.4).*

The Wasserstein distance indeed de nes a distance in the space of probability measures on R^3 (or more generally, any complete separable metric space); it can be used as an optimal transport cost between these measures, see [17].

De nition 2.3. *Let* $F : \mathrm{R}^3\ !$

The unknowns in the above equations are $u^g = (u_1^g; u_2^g; 0)$, $u = (u_1; u_2; u_3)$, p, , .

Equation (3.1) is the momentum equation; (3.2) represents the adiabatic assumption; (3.3) is the continuity equation and (3.4) represents hydrostatic and geostrophic balance. The equation (3.5) is the equation of state which relates the thermodynamic quantities to each other, and (3.6) is the rigid boundary condition, where n is the outward normal to @ .

The semi-geostrophic equations are a valid approximation to the compressible Euler equations when UL << 1; and are accurate when $\frac{H}{L} < \frac{1}{N}$; where U is a typical scale for horizontal speed; L is a typical horizontal scale; H is a typical vertical scale; N is the buoyancy frequency.

The energy associated with the ow, known as the *geostrophic energy*, is de ned as

$$E(t) = \int \left[\frac{1}{2} |u^g|^2 (t;x) + (x) + c_v (t;x) \frac{p(t;x)}{p_{ref}} \right] (t;x)\, dx: \qquad (3.7)$$

In what follows, we set $f_{cor} = 1$.

3.1 Dual formulation

In [10], solutions were obtained using a transformation into the so-called *dual (geostrophic) coordinates* $y = (y_1; y_2; y_3)$. The coordinate transformation is given by:

$$T : \; ! \; R^3; \qquad T(t;x) = (T_1(t;x); T_2(t;x); T_3(t;x)) = (y_1; y_2; y_3);$$

with

$$y_1 = x_1 + u_2^g(t;x); \qquad y_2 = x_2 \quad u_1^g(t;x); \qquad y_3 = \; (t;x): \qquad (3.8)$$

Note that, by (2.1)(*vi*), $\bar{\ }$ R^2 $[\, ; \frac{1}{}]$ for some $0 < \; < 1$.

Using this transformation, as well as (3.5), it was shown in [10] that we can write the energy in (3.7) as

$$E(t) = E(t; \; ; T) = \int \tfrac{1}{2}\{ |x_1 \quad y_1|^2 + |x_2 \quad y_2|^2 \} + \; (x$$

where

$$E(; \quad) = \inf_{\overline{T} \# \quad =}$$

(ii) S *is optimal in the transport of to with cost* $\tilde{c}(y;x) = c(x;y)$,

(iii) S *and* T *are inverses, i.e.* S T (x) = x *for* a:e: x *and* T S(y) = y *for* a:e: y,

(iv) $_0 = (\mathrm{id};T)\#$ *is a minimiser of the relaxed optimal transport problem (3.22),*

(v) *For* $J_{(;)}(f;g)$ *de ned by (3.23), the following equality holds:*

$$\sup_{(f;g)\,2\,\mathrm{Lip}_c} J_{(;)}(f;g) = \inf_{2\,(;)} \bar{I}[\;] = \inf_{\bar{T}\# =} I\;[\bar{T}]:$$

4 The main existence result in dual space - an alternative proof

4.1 Statement of the theorem

Theorem 4.1. *Let* $1 < r < 1$ *and* $_0\ 2\ L^r(\ _0)$ *be an initial potential density with support in* $_0$*, where* $_0$ *is a bounded open set in* $\mathbb{R}^3$ *with* $\overline{\ _0}$ $\mathbb{R}^2$ $[\tilde{\ }; 1=\tilde{\ }]$ *for some* $0 < \tilde{\ } < 1$*. Let be an open bounded convex set in* $\mathbb{R}^3$*. Assume that* $c(;)$ *is given by (3.13) and that satis es (2.1)(* ix*). Then the system of semi-geostrophic equations in dual variables (3.14)-(3.19) has a stable weak solution* $(; T)$ *such that, with* $(t;) = T(t;)\#\ (t;)$ *and* w *as in (3.15),*

(i)

$$(;)\ 2\ L^r((0;) \quad); \qquad k\ (t;)k_{L^r( )}\ 6\ k\ _0( )k_{L^r( )}\ ; \qquad 8\,t\ 2\ [0;];$$

(ii)

$$(t;)\ 2\ W^{1;1}(\ ); \qquad k\ (t;)k_{W^{1;1}( )}\ 6\ C = C(\ ;\ ; c(;); ; K_1); \qquad 8\,t\ 2\ [0;];$$

(iii)

$$kw(t;)k_{L^1( )}\ 6\ C = C(\ ;\ ) \qquad 8\,t\ 2\ [0;];$$

where is a bounded open domain in $\mathbb{R}^3$ *containing* supp( )*, such that* $\overline{\ }$ $\mathbb{R}^2$ $;\ \frac{1}{\ }$ *for some* $0 < \ < 1$*.*

The proof of this theorem is given in [10, Theorem 5.5], using a time-approximation argument, similar in spirit to the original argument of the proof given by Benamou and[0

De nition 4.1. *Let* H : $P^2_{ac}(R^3)$! (1 ; +1] *be a proper, upper semicontinuous function and let* 2 D (

De nition 4.3. *Let* H : P

Let , be as in (3.20). For $2\,\mathcal{P}_{ac}(\ )$, de ne the Hamiltonian H by

$$H(\ ) = \inf_{2\,\mathcal{P}_{ac}(\ )} \quad E(\ ;\ ) + K_1 \int (\ (x))\ dx \quad ; \tag{4.7}$$

where

$$E(\ ;\ ) = \inf_{\overline{S}\#\ =} \int \tilde{c}(y;\overline{S}(y))\ (y)\,dy \tag{4.8}$$

with $\tilde{c}(y;x) = c(x;y)$ de ned by (3.13). We begin with the following:

Proposition 4.6. *Let and satisfy (3.20). Let the Hamiltonian* $H(\ )$ *on* $\mathcal{P}_{ac}(\ )$ *be de ned by (4.7). Then* H *is superdi erentiable, upper semicontinuous and* (2) *concave.*

Proof. Given $2\,\mathcal{P}_{ac}(\ )$, denote by the minimiser in (4.7). The existence and uniqueness of this minimiser follows from Theorem 3.3. For any $_h\ 2\,\mathcal{P}_{ac}(\ )$ we have

$$\begin{aligned} H(\ _h) &= \inf_{2\,\mathcal{P}_{ac}(\ )} \quad E(\ _h;\ ) + K_1 \int (\ (x))\ dx \\ &6\ E(\ _h;\ ) + K_1 \int (\ (x))\ dx: \end{aligned}$$

First, recall that we can guarantee that there exists a unique optimal transport map $R^{\ h}$ from to $_h$ with respect to the Wasserstein cost function $d(y;y_h) = \frac{1}{2}jy \quad y_h j^2$; see Remark 2.1.

We consider now transport with respect to the cost function $\tilde{c}(y;x) = c(x;y)$ given by (3.13). Let S be the optimal map in the transport of to and let $S_{\ h}$ be the optimal map in the transport of $_h$ to . Therefore, we have

$$\inf_{S\#\ =} \int \tilde{c}(y;S(y))\ (y)\,dy = \int \tilde{c}(y;S\ (y))\ (y)\,dy$$

and

$$\inf_{S\#\ _h=} \int \tilde{c}(y;S(y))\ _h(y)\,dy = \int \tilde{c}(y;S_{\ h}(y))\ _h(y)\,dy:$$

The existence of S and $S_{\ h}$ follows from Theorem 3.4. Note that, since (S (

$$ \tag{4.9} $$

where we have used (2.5).

Hence, using Definition 4.1, we conclude that $\nabla \tilde{c}(y; S(y)) \in \partial H(\ )$. Thus, $\partial H(\ )$ is non-empty, H is superdifferentiable and we can use [17, Proposition 10.12] to conclude that H is semi-concave, i.e.

$$ H \text{ is } (\ \ 2) \quad \text{concave:} \tag{4.10} $$

Also, from the narrow continuity of $E(\ ;\ )$ (see [10, Theorem 3.4]) and the uniform convergence of as the minimiser of (4.7) (see [10, Lemma 4.3]), we have that

$$ H \text{ is upper semicontinuous.} \tag{4.11} $$

From (4.10) and (4.11), we have that (H3) holds. □

The following proposition yields a proof of Theorem 4.1, alternative to the proof given in [10].

Proposition 4.7. *Let* $1 < r < \infty$ *and* $_0 \in L^r(\ _0)$

and

$$\int \ (y)\ \ (R^{\ s}(y) \quad y)\ \ (y)\,dy = \int \ (y)\ \ (g_s(y) \quad y)\ \ (y)\,dy = s \int \ (y)\ \ r\ '\ (y)\ \ (y)\,dy:$$

Combining this with (4.12), we therefore obtain

$$
\begin{aligned}
s \int \ (y)\ \ r\ '\ (y)\ \ (y)\,dy + s^2 \int jr\ '\ (y)j^2\ \ (y)\,dy \ 6\ H(\ ) \quad H(\ _s) \\
6\ E(\ ;\ \ _s)\ \ E\ (\ _s;\ \ _s) \\
= \int c(y; S^{\ s}(y))\ \ (y)\,dy \quad \int c(y; S^{\ s}_{\ s}(y))\ \ _s(y)\,dy \\
6 \int c(y; S^{\ s}_{\ s} \quad g_s(y))\ \ (y)\,dy \quad \int c(y; S^{\ s}_{\ s}(y))\ \ _s(y)\,dy
\end{aligned}
$$

$$= \int c(g_s^{\ 1}(y); S^{\ s}_{\ s}(y))\ \ _s(y)\,dy \quad \int c(y; S^{\ s}_{\ s}(y))\ \ _s(y)\,dy; \tag{4.13}$$

since $g_s\#\ \ =\ \ _s$. Here $S^{\ s}$ denotes the optimal transport map from to $_s$ and $S^{\ s}_{\ s}$ denotes the optimal transport map from $_s$ to $_s$ with respect to the cost function $c(\ ;\ )$. The existence of $S^{\ s}$ and $S^{\ s}_{\ s}$ follows from Theorem 3.4.

Note that

$$g_s^{\ 1}(y) = y \quad sr\ '\ (y) + s^2(\ ) = y \quad sy) + s).$$

existence ofS

Dividing both sides rst by $s > 0$, then by $s < 0$ and letting $|s| \to 0$ we obtain

$$\int (y) \quad r'(y) \quad (y)\,dy = \int \frac{S(y) \quad y}{y_3} \quad r'(y) \quad (y)\,dy$$

$$D_t T(t;x) = e_3 \quad [T(t;x) \quad x]; \tag{5.1}$$

$$@ \ (t;x) + r \quad (\ (t;x) u(t;x)) = 0; \tag{5.2}$$

$$r_x c(x;T(t;x)) + K_1 r \ ((\ (t;x)) \quad ^1) = 0; \tag{5.3}$$

$$u \quad n = 0 \quad \text{on } [0; \ ) \quad @ \ ; \tag{5.4}$$

$$(0;x) = \ _0(x); \quad T(0;x) = T_0; \quad T_0 \# \ _0 \ 2 \ L^r(\ _0); \quad _0 \quad R^3 \text{ compact}; \tag{5.5}$$

where r 2 (1; 1), $c(x;T(t;x))$ is de ned in (3.13) and K_1 is de ned as in (3.9).

Proposition 5.1. *A solution of (5.1)-(5.5) determines a solution of the original system (3.1)-(3.6).*

Proof. Given a solution $(u;T; \ )$ of (5.1)-(5.5), we set the function u in (3.1)-(3.6) to be equal to the function u in (5.1)-(5.5) and we de ne

$$(t;x) := T_3(t;x); \qquad (t;x) := \frac{(t;x)}{(t;x)}; \qquad u^g(t;x) := e_3 \quad [T(t;x) \quad x]: \tag{5.6}$$

Then (3.2) follows from (5.6) and equation $D_t T_3 = 0$ of (5.1). Using this together with (5.6) and the fact that (5.2) is satis ed, we see that (3.3) holds.

In order to show that (3.4) holds, we rst rearrange (3.5) to obtain $p = R \ p_{ref}^1 \ (\quad)$. Thus,

$$r \ p = R \ p_{ref}^1 \quad (\quad) \quad ^1 r \ (\quad): \tag{5.7}$$

Then, substituting (5.6) into (5.3) gives

$$e_3 \quad u^g + r \quad + \quad K_1 r \ (\quad) \quad ^1 \ (\quad) \ ^1 = 0:$$

e_3 u^g + r 2n) += e_3 u^g + x)(1) ref 8T1/F13 6. /F10 6.9738 T

Recalling that $K_1 = c_v \ \frac{R}{p_{ref}}$

(iii) There exists a Borel map

$$F : [0;\infty) \times \Omega \to \Lambda$$

such that, for every $t \in (0;\infty)$*, the map*

$$F_{(t)} = F(t;\cdot) : \Omega \to \Lambda$$

satisfies

$$F_{(t)} \circ F^*_{(t)}(x) = x \text{ and } F^*_{(t)} \circ F_{(t)}(x) = x;$$

for σ_0 *a.e.* $x \in \Omega$*, and*

$$F_{(t)\#}\,\nu(t;\cdot) = \sigma_0(\cdot):$$

(iv) The function

$$Z(t;x) := T(t;F_{(t)}(x)) \tag{5.12}$$

is a weak solution of

$$\begin{cases} \partial_t Z(t;x) = e_3 \wedge (Z(t;x) - F_{(t)}(x)) & \text{in } [0;\infty) \times \Omega; \\ Z(0;x) = T_0(x) & \text{in } \Omega; \end{cases} \tag{5.13}$$

in the following sense:
for any $\varphi \in C^1_c([0;\infty) \times \Omega;\mathbb{R}^3)$*,*

$$\int_{[0;\infty)\times\Omega} \{Z(t;x) \cdot \partial_t\varphi(t;x) + e_3 \wedge [Z(t;x) - F_{(t)}(x)] \cdot \varphi(t;x)\}\,\sigma_0(x)\,dtdx$$

$$+ \int \ldots$$

$t;x$

In order to justify Definition 5.2, we must show that a weak Lagrangian solution corresponds to a weak (Eulerian) solution of (5.1)-(5.5), as defined by Definition 5.1. Indeed, we prove that, with additional regularity property $\partial F \in L^1([0,))$, a weak Lagrangian solution $(F, T,)$ as defined in Definition 5.2 determines a weak (Eulerian) solution of (5.1)-(5.5) and, furthermore, that a smooth Lagrangian solution determines a classical solution of (5.1)-(5.5). This is the content of the following result:

Proposition 5.3. *Let be an open bounded convex set in* $\mathbb{R}^3$ *and let* > 0*. Let* $(F, T,)$ *be a weak Lagrangian solution of (5.1)-(5.5) in* $[0,)$ *.*

(i) If $\partial F \in L^1([0,) ; \mathbb{R}^3)$*, then the function*

$$u(t, x) := (\partial F)(t, F_{(t)}(x)) \tag{5.16}$$

satisfies $u \in L^1([0,) ; \mathbb{R}^3)$ *and* $(u, T,)$ *is a weak Eulerian solution of (5.1)-(5.5) in* $[0,)$ *in the sense of Definition 5.1.*

(ii) If $(F, F , T) \in C^2([0,] \bar{})$*, then the function (5.16) satisfies* $u \in C^1([0,] \bar{} ; \mathbb{R}^3)$*, and* $(u, T,)$ *is a classical solution of (5.1)-(5.5) in* $[0,)$ *.*

Proof. Let us first prove (i). Since F is a Borel map and, by our additional regularity

If we make the change of variables $X = F_{(t)}(x)$ in the second integral above, then, by (*ii*), (*iii*) in Definition 5.2 and by (2.5), we have that

$$\int_{\Omega} \psi(0;x)\rho_0(x)\,dx - \int_{[0;\infty)\times\Omega} (\partial F_{(t)})(F^{-1}_{(t)}(X))\cdot\nabla\psi(t;X)\rho(t;X)\,dtdX = -\int_{[0;\infty)\times\Omega} \partial\psi(t;X)\rho(t;X)\,dtdX.$$

Then, rearranging and using the definition of u in (5.16), we obtain

$$\int_{[0;\infty)\times\Omega} \{\partial\psi(t;X) + u(t;X)\cdot\nabla\psi(t;X)\}\rho(t;X)\,dtdX + \int_{\Omega}\psi(0;x)\rho_0(x)\,dx = 0.$$

Changing notations X to x gives us (5.9).

We now prove that (5.8) also holds. By the properties of F and T in Definition 5.2, we have that $Z(t;x)$ as defined in (5.12) satisfies $Z \in L^\infty([0;\tau]\times\bar{\Omega})$. Also, applying the definition of $Z(t;x)$ in (5.12) to equation (5.14) gives

$$\int_{[0;\infty)\times\Omega} \{T(t;F_{(t)}(x))\cdot\partial\varphi(t;x) + e_3\times[T(t;F_{(t)}(x)) - F_{(t)}(x)]\cdot\varphi(t;x)\}\rho_0(x)\,dtdx + \int_{\Omega}[T_0(x)\cdot\varphi(0;x)]\rho_0(x)\,dx = 0; \tag{5.17}$$

for any $\varphi \in C_c^1([0;\infty)\times\Omega)$. Now, since Ω is a bounded set and $F_{(t)}\#\rho_0(\cdot) = \rho(t;\cdot)$ for all $t\in[0;\infty)$, equation (2.5) allows us to make the change of variables $X = F_{(t)}(x)$ in the first integral of (5.17). Thus, by (*iii*) of Definition 5.2, we have that $x = F^{-1}_{(t)}(X)$ for ρ a.e. $x\in\Omega$ for every $t\in[0;\infty)$ and then, from (5.17), we obtain

$$\int_{[0;\infty)\times\Omega} \{T(t;X)\cdot\partial\varphi(t;F^{-1}_{(t)}(X)) + e_3\times[T(t;X) - X]\cdot\varphi(t;F^{-1}_{(t)}(X))\}\rho(t;X)\,dtdX + \int_{\Omega}[T_0(x)\cdot\varphi(0;x)]\rho_0(x)\,dx = 0; \tag{5.18}$$

for any $\varphi \in C_c^1([0;\infty)\times\Omega)$. We now show that (5.18) also holds for all φ such that

$$\begin{aligned} &\varphi \in L^\infty([0;\infty)\times\Omega);\\ &\partial\varphi \in L^\infty([0;\infty)\times\Omega);\\ &\operatorname{supp}(\varphi)\subset[0;\tau]\times\bar{\Omega}\ \text{ for some } \tau>0 \end{aligned}\tag{5.19}$$

In order to do this, we construct an approximating sequence for such φ. Let us extend φ to $(-\infty;\infty)\times\Omega$ by defining, for $x\in\Omega$, $\varphi(t;x) = \varphi(-t;x)$ for $t<0$ and $\varphi(t;\cdot)\equiv 0$ for $t>\tau$. Then, we let $h>0$ and define $\Omega_h = \{x\in\Omega : \operatorname{dist}(x;\partial\Omega) > h\}$, where $\partial\Omega$ denotes the boundary of Ω. Thus, $\varphi\chi_{\Omega_h}$ is now defined on $\mathbb{R}^1\times\mathbb{R}^3$, where χ_{Ω_h} denotes the characteristic function of the set Ω_h. Next, let $j_h(t;x) = \frac{1}{h^4}j(\frac{|(t;x)|}{h})$, where $j(\cdot)$ is a standard mollifier, and let $k>\frac{1}{\tau}$ be an integer. We then have that functions $\varphi_k = (\varphi\chi_{\Omega_{4h}}) * j_h$, with $h=\frac{1}{k}<\tau$, satisfy

$$\varphi_k \in C_c^1([0;\infty)\times\Omega) \quad\text{with}\quad \|\varphi_k;\partial\varphi_k\|_{L^\infty([0;\infty)\times\Omega)} \leqslant C;$$

where C does not depend on k, and

$$(\varphi_k;\partial\varphi_k)\to(\varphi;\partial_t\varphi)\quad\text{a.e. on } [0;\infty)\times\Omega \quad\text{as } k\to\infty.$$

Thus, by the dominated convergence theorem,

$$\lim_{k\to\infty}\int_{[0,\infty)\times\Omega} (\varphi_k, \partial_t\varphi_k)(t,x)\,\rho_0(x)\,dtdx = \int_{[0,\infty)\times\Omega} (\varphi, \partial_t\varphi)(t,x)\,\rho_0(x)\,dtdx.$$

Also, since $\varphi\in L^1([0,\infty)\times\Omega)$ by (5.19) and $F_{(t)}\#\rho(t,\cdot)=\rho_0(\cdot)$, we have from (2.5) that

$$\int_{[0,\infty)\times\Omega} (\varphi_k, \partial_t\varphi_k)(t,x)\,\rho_0(x)\,dtdx = \int_{[0,\infty)\times\Omega} (\varphi_k, \partial_t\varphi_k)(t,F_{(t)}(X))\,\rho(t,X)\,dtdX$$

and

$$\int_{[0,\infty)\times\Omega} (\varphi, \partial_t\varphi)(t,x)\,\rho_0(x)\,dtdx = \int_{[0,\infty)\times\Omega} (\varphi, \partial_t\varphi)(t,F_{(t)}(X))\,\rho(t,X)\,dtdX.$$

Hence,

$$\lim_{k\to\infty}\int_{[0,\infty)\times\Omega} (\varphi_k, \partial_t\varphi_k)(t,F_{(t)}(X))\,\rho(t,X)\,dtdX = \int_{[0,\infty)\times\Omega} (\varphi, \partial_t\varphi)(t,F_{(t)}(X))\,\rho(t,X)\,dtdX.$$

We now state the main result, which we will prove in Sections 6 and 7:

Theorem 5.4. *Let* $\mathbb{R}^3$

by Theorem 4.1 we have that w 2 L^1 ([0;)),

by (6.1), (6.2) and (6.3) we have that jDw(t;)j( ) 2 L^1_{loc}(0;).

Since has compact support in [0;] R^3, we can modify w away from so that the modi ed function w̃ satis es

$$\tilde{w}\ 2\ L^1\ ([0;\ )\ \ R^3); \qquad \tilde{w}(t;\ )\ 2\ BV_{loc}(R^3)\ \text{for a:e: } t\ 2\ (0;\ ); \tag{6.4}$$

$$jD\tilde{w}(t;\ )j(R^3)\ 2\ L^1_{loc}(0;\ ); \qquad r\ \ \tilde{w}(t;\ ) = 0\ \text{in } R^3\ \text{for every } t\ 2\ [0;\ ) \tag{6.5}$$

and

$$w = \tilde{w} \quad \text{in} \ : \tag{6.6}$$

We construct such a modi cation as follows. Following [10, Section 5], de ne := B(0; R) (; $\frac{1}{}$), where 0 < < 1 and B(0; R) represents the open ball of radius R centered at the origin in R^2. De ning , R as in [10, (55), (56)] ensures that supp( ) is contained in . De ne 2 C^1 (R) as = 1 on fj sj < R g, = 0 on fj sj > R g and 0 6 6 1 on R. De ne 2 C^1 (R) as = 1 on f < s < $\frac{1}{}$g, = 0 when s 6 $\frac{}{2}$ or when s > $\frac{2}{}$ and 0 6 6 1 on R. Then, de ne for y 2 R^3

$$M(y) = (M_1(y); M_2(y); M_3(y)) = (\ (jy_1j)y_1;\ (jy_2j)y_2;\ (y_3)y_3) \tag{6.7}$$

and de ne the modi ed velocity as

$$\tilde{w} = e_3 \quad (M(y) \quad S(t; M(y))) = \ (y_3)y_3e_3 \ \ r \ \ g_0(M(y)): \tag{6.8}$$

Then w̃ satis es (6.4)-(6.6). These conditions enable us to apply the theory of [1] to the transport equation (3.14) with w replaced by our modi ed velocity w̃:

Lemma 6.1. *There exists a unique locally bounded Borel measurable map* : [0;) R^3 ! R^3 *satisfying*

(i) (; y) 2 $W^{1;1}$ ([0;)) *for* $_0$ a:e: y 2 R^3*;*

(ii) (0; y) = y *for* $_0$ a:e: y 2 R^3*;*

(iii) for $_0$ a:e: y 2 R^3*,*

$$@ \ (t; y) = \tilde{w}(t; \ (t; y)); \tag{6.9}$$

(iv) there exists a Borel map : [0;) R^3 ! R^3 *such that, for every* t 2 (0;)*, the map* $_{(t)}$: R^3 ! R^3 *is Lebesgue-measure preserving, and such that* $_{(t)}$ $_{(t)}$(y) = $_{(t)}$ $_{(t)}$(y) = y *for* a:e: y 2 R^3*;*

(v) (t;) : R^3 ! R^3 *is a Lebesgue-measure preserving map for every* t 2 [0;)*.*

Proof. The proof is essentially identical to that of [8, Lemma 2.8]. □

We now show that the image of the ow map is contained in , and therefore corresponds to the velocity eld w.

Lemma 6.2. *Let be as in Theorem 4.1. Let be the map de ned in Lemma 6.1 and let* w *be de ned as in (3.15). Then*

$$(t; y) \qquad \text{for} \ _0 \ \ \text{a:e:} \ \ y\ 2\ T_0(\ ) \ \text{and every } t\ 2\ [0;\ ): \tag{6.10}$$

In particular,

$$@ \ (t; y) = w(t; \ (t; y)) \quad \text{for} \ _0 \ \ \text{a:e:} \ \ y\ 2\ T_0(\ ) \ \text{and every } t\ 2\ [0;\ ): \tag{6.11}$$

Proof. For h, k, j_h, g_h^k as in [10, Section 5] define

$$\tilde{w}_h^k(y) := \ (y_3)y_3 e_3 \ \nabla \ (j_h \ g_h^k)(M \ (y)); \qquad (6.19)$$

where M , are defined in (6.7). Define functions $\tilde{w}_h$ on $[0; \ ] \ \mathbb{R}^3$ by setting them equal to $\tilde{w}_h^k$ on the time-interval $t \in [kh; (k+1)h)$. Following [10, Lemma 5.3], the corresponding potential density $_h$ is a weak solution of

$$\begin{aligned} &\partial \ _h + \nabla \ (\ _h \tilde{w}_h) = 0 \quad \text{in } (0; \ ) \ \mathbb{R}^3; \\ &_h(0; y) = \ _h^0(y): \end{aligned} \qquad (6.20)$$

The construction of $\tilde{w}_h$ implies that $\tilde{w}_h$ is a divergence-free vector field satisfying (6.4)-(6.6) and

$$(i) \qquad \int_{[0; \ ) \ \mathbb{R}^3} |\tilde{w}_h| \ _h \, dtdy < +\infty \ ;$$

$$(ii) \qquad \int$$

Passing to the limit $j \to \infty$ in the last equality, using (6.24), the fact that $\,^{0}_{h_j} \to \,_0$ as $j \to \infty$ in $L^r(\mathbb{R}^3)$ and the dominated convergence theorem in the left-hand side, and using (6.22) in the right-hand side, we obtain

$$\int_{\mathbb{R}^3} \varphi(\;(t;y))\;_0(y)\,dy = \int_{\mathbb{R}^3} \varphi(Y)\;(t;Y)\,dY \tag{6.26}$$

for any $\varphi \in C_c(\mathbb{R}^3)$. This implies (6.17).

Since $\;_{(t)}$ is a measure preserving map, we use Lemma 6.1 (*iv*) to conclude that the left-hand side of (6.26) is equal to

$$\int_{\mathbb{R}^3} \varphi(Y)\;_0(\;_{(t)}(Y))\,dY;$$

and now (6.26) implies (6.18).

□

Remark 6.5. It would be desirable to be able to avoid using the approximating solutions in dual space when showing that is a weak Lagrangian solution of the transport equation. However, we have not been able to approximate in L^1 the velocity $\tilde{w}$ directly. We appeal instead to the sequence of solutions of the approximating equations in dual space constructed in [10], as was done for the proof of the analogous result for the incompressible case given in [8].

7 Lagrangian flow in physical space

Throughout this section we will assume that , , r, T_0, T, , w, $\tilde{w}$, are as in Proposition 6.4. Note that, by Theorem 4.1, we can apply (2.5) to $\,_0$, , $\,_0$, throughout this section.

We now perform the last step of the analysis and prove the existence of a Lagrangian flow $F : [0;\;) \;\; \to$ in the physical space. Indeed, we define $F_{(t)} : \;\to$ for $t \in [0;\;)$ as

$$F_{(t)} := S_{(t)} \;\;_{(t)}\;\; T_0; \tag{7.1}$$

where T_0 is as in (5.5), $S_{(t)}$ is the inverse of $T_{(t)}$ (see Theorem 3.4) and $\;_{(t)}$ is the Lagrangian flow in dual space constructed in Lemma 6.1. To justify this definition, we prove the following lemma:

Lemma 7.1. *For any* $t \in [0;\;)$*, the right hand side of (7.1) is defined* $\,_0$ a:e: *in* . *The map* $F : [0;\;) \;\; \to$ *defined by (7.1) is Borel.*

Proof. Since T_0 exists and is unique $\,_0$ a:e: in , we have that T_0 exists and is unique on $\setminus N_0^1$ where N_0^1 is a Borel subset of with $\,_0[N_0^1] = 0$. Also, since S exists and is unique a:e: in for every $t \in [0;\;)$, we have that S exists and is unique on $\setminus N^2$ for every $t \in [0;\;)$, where N^2 is a Borel subset of with $[N^2] = 0$. Then, the right-hand side

Fix t 2 $[0;\ )$. Then, using that $T_0\#\ _0 = \ _0$ and thus $T_0\#\ _0 = \ _0$ for all x 2 $\ n N_0^1$, and using (6.17) as well as Lemma 6.1 (*iv*), we can apply (2.5) and compute

$$
\begin{aligned}
{}_0\ M_{(t)} &= {}_0\ \mathrm{hn}\ x\ 2\ \ n N_0^1 : T_0(x)\ 2\ \ {}_{(t)}^{1}(N^2)\ \mathrm{oi} \\
&= Z_{T_0{}^1(\ {}_{(t)}^{1}(N^2))}\ {}_0(x)\,dx = Z_{{}_{(t)}^{1}(N^2)}\ {}_0(y)\,dy \\
&= Z_{N^2}\ (y)\,dy = 0:
\end{aligned}
$$

Thus, we can de ne $F : [0;\ )\ \ !\ \ $ by (7.1). Then, by Lemma 6.1, F is a Borel mapping. □

It remains to prove that, if F is de ned by (7.1), then $(F; T;\ )$ is a weak Lagrangian solution of (5.1)-(5.5) in the sense of De nition 5.2. We begin by showing that the initial condition for the ow in De nition 5.2 (*i*) is satis ed.

Proposition 7.2. *Let* F *be de ned as in (7.1). Then,* $F(0; x) = x$ *for* $\ _0\ $ a:e: x 2 $\ $.

Proof. By (7.1) we have that $F_{(0)}(x) = S_{(0)}\ \ {}_{(0)}\ \ T_0(x)$ for all x 2 $\ n N_0$ where N_0 is a Borel set with $\ _0[N_0] = 0$.

By Lemma 3.4, there exist Borel sets $N_1\ \ $, $N_2\ \ $ with $\ _0[N_1] = \ _0[N_2] = 0$ such that T_0 exists and is unique in $\ n N_1$ and $S_{(0)}$ exists and is unique in $\ n N_2$. Moreover, if x 2 $\ n [N_1 [\ T_0{}^1(N_2)]$, then $S_{(0)}\ \ T_0(x) = x$. Also, by Lemma 6.1 (*ii*), we have that ${}_{(0)}(y) = y$ in $\ n N_3$, where N_3 is a Borel set with $\ _0[N_3] = 0$.

Therefore, $F_{(0)}(x) = x$ for all x 2 $\ n [N_0 [\ N_1 [\ T_0{}^1(N_2 [\ N_3)]$. We must now show that $\ _0\ \ \backslash\ T_0{}^1(N_2 [\ N_3)\ = 0$.

We have that $\ _0[N_2 [\ N_3] = 0$. Then, since $T_0\#\ _0 = \ _0$, we obtain

$$
{}_0\ \ \backslash\ T_0{}^1(N_2 [\ N_3)\ = Z_{T_0{}^1(N_2 [\ N_3)}\ {}_0(x)\,dx = Z_{N_2 [\ N_3}\ {}_0(y)\,dy = \ {}_0[N_2 [\ N_3] = 0:\ )
$$

9.9626 Tf 17.196 0 Td [(F

since $\varphi \quad S_{(t)} \quad {}_{(t)} \in L^1(\ )$. Then, since $\varphi \quad S_{(t)} \in L^1(\ )$ we can use (6.17) and apply (2.5) to get

$$\int \varphi \quad S_{(t)} \quad {}_{(t)}(y) \ {}_0(y)\,dy = \int \varphi \quad S_{(t)}(Y) \ (Y)\,dY:$$

Finally, since $S_{(t)}$ satisfies $S_{(t)}\# \ = \ $, we have that

$$\int \varphi \quad S_{(t)}(Y) \ (Y)\,dY = \int \varphi(X) \ (X)\,dX:$$

Thus, we have shown that

$$\begin{aligned} \int \varphi(F_{(t)}(x)) \ {}_0(x)\,dx &= \int \varphi \quad S_{(t)} \quad {}_{(t)} \quad T_0(x) \ {}_0(x)\,dx \\ &= \int \varphi \quad S_{(t)} \quad {}_{(t)}(y) \ {}_0(y)\,dy \\ &= \int \varphi \quad S_{(t)}(Y) \ (Y)\,dY \\ &= \int \varphi(X) \ (X)\,dX; \end{aligned}$$

as required. □

We now prove that (5.10) holds for all $q \in [1;\infty)$.

Proposition 7.4. *For any* $t_0 \in [0;\ )$ *and any* $q \in [1;\infty)$*,*

$$\lim_{t \to t_0; t \in [0;\ )} \int |F_{(t)}(x) \quad F_{(t_0)}(x)|^q \ {}_0(x)\,dx = 0:$$

Proof. By Lemma 7.1 we have that, for any $t \in [0;\ )$, (7.1) holds ${}_0$ a:e: in . Thus, since $T_0\# \ {}_0 = \ {}_0$, we see that, for any $t; t_0 \in [0;\ )$,

$$\int F_{(t)}(x) \quad F_{(t_0)}($$

(6.17) and Holder's inequality to estimate

$$
\begin{aligned}
I_1 &= \int S_{(t)} \quad {}_{(t)}(y) \quad S_{(t_0)} \quad {}_{(t)}(y)^{\,q} \; {}_0(y)\, dy \\
&= \int S_{(t)}(y) \quad S_{(t_0)}(y)^{\,q} \; {}_{(t)}(y)\, dy \\
&6 \left(\int S_{(t)}(y) \quad S_{(t_0)}(y)^{\,qr^0} dy \right)^{\frac{1}{r^0}} \left(\int {}_{(t)}(y)^{\,r} dy \right)^{\frac{1}{r}} \\
&= S_{(t)} \quad S_{(t_0)} {}^{q}_{qr^0} \quad {}_{(t)} \; {}_{r} \\
&= S_{(t)} \quad S_{(t_0)} {}^{q}_{qr^0} k \; {}_0 k_r \; ! \; 0 \text{ as } t \; ! \; t_0 :
\end{aligned}
$$

Next, we show that $I_2 \; ! \; 0$ as $t \; ! \; t_0$. Since $S_{(t)}$ 2 for each t and for a:e: y, then, by the dominated convergence theorem, it remains to prove that for every t_0,

$$
S_{(t_0)} \quad {}_{(t)}(y) \quad S_{(t_0)} \quad {}_{(t_0)}(y) \; ! \; 0 \quad \text{as } t \; ! \; t_0 \tag{7.2}
$$

for a:e: y 2 . First we note that, since ${}_{(t)}$ is measure preserving, then it follows from Lemma 6.1 (*i*), and the fact that 2 L^1 ([0;) R^3) by (6.4), that

$$
{}_{(t)}(y) \; ! \quad {}_{(t_0)}(y) \quad \text{as } t \; ! \; t_0
$$

in [0;] for a:e: y 2 . If y is such a point and if, in addition, ${}_{(t_0)}(y)$ is a point of continuity for $S_{(t_0)}$, then convergence in (7.2) holds at y. Since ${}_{(t_0)}$ is measure preserving, it follows that ${}_{(t_0)}(y)$ is a point of continuity for $S_{(t_0)}$ for a:e: y. Thus, (7.2) holds for a:e: y 2 . □

Lemma 7.5. *Let* Z *be de ned as in (5.12) with* F *de ned as in (7.1). Then, for all* t 2 [0;),

$$
Z_{(t)}(x) = \quad {}_{(t)} \quad T_0(x)
$$

for ${}_0$ a:e: x 2 .

Proof. Using (5.12), we have that $Z_{(t)} = T_{(t)} \quad F_{(t)}$. Therefore, we need to justify the following formal computation:

$$
T_{(t)} \quad F_{(t)} = T_{(t)}
$$

Let $\widetilde{M} = \{y \in \ \ : T(t; S(t; y)) \neq y\}$, Then $\widetilde{M}$ is a Borel set. Now, the proof of the lemma will be completed if we show that

$$ {}_0\left[\{x \in \ \ \setminus \widetilde{N} : \ {}_{(t)} \ T_0(x) \in \widetilde{M}\}\right] = 0. \tag{7.3}$$

From Theorem 3.4 (*iii*) we have that, for any $t \in [0; \ )$,

$$T_{(t)} \ \ S_{(t)}(y) = y \quad \text{for} \quad a.e. \ y \in \ \ .$$

Thus, we have

$$\int_{\widetilde{M}} \ (y)\, dy = 0$$

for any $t \in [0; \ )$. Therefore, using that ${}_0[\widetilde{N}] = 0$, which implies that thus $T_{(t)}\#\quad = \quad$ for all x 2 $\quad n N^1$, and using (6.17), we can apply (2.5) and compute

$$
\begin{aligned}
[M] &= \quad x \; 2 \quad n N^1 : T_{(t)}(x) \; 2 \quad {}_{(t)}(N_0^2) \\
&= \int_{T_{(t)}^{1}(\;{}_{(t)}(N_0^2))} (x)\, dx = \int_{{}_{(t)}(N_0^2)} (y)\, dy \\
&= \int_{N_0^2} {}_0(y)\, dy = 0:
\end{aligned}
$$

Thus, we can de ne $F_{(t)} = S_{(0)} \quad {}_{(t)} \quad T_{(t)}$ and F is a Borel mapping.

We can now prove that property (*iii*) of De nition 5.2 holds. Since $F_{(t)}\# \;{}_0 = \;$, we have that $F_{(t)} \quad F_{(t)}(x) = S_{(0)} \quad {}_{(t)} \quad T_{(t)} \quad F_{(t)}(x)$ for a:e: x 2 . Then, using Lemma 7.5, we get $F_{(t)} \quad F_{(t)}(x) = S_{(0)} \quad {}_{(t)} \quad {}_{(t)} \quad T_0(x) \quad {}_0$ a:e: in . Since, by Lemma 6.1 (*iv*), ${}_{(t)} \quad {}_{(t)}(y) = y$ for a:e: y and thus for ${}_0$ a:e: y 2 , and since $T_0\# \;{}_0 = \;{}_0$, we have ${}_{(t)} \quad {}_{(t)} \quad T_0(x) = T_0(x)$ for ${}_0$ a:e: x 2 . Thus, $F_{(t)} \quad F_{(t)}(x) = S_{(0)} \quad T_0(x) = x$ for a:e: x 2 by Lemma 3.4 (*iii*).

By a similar argument, we we have that $F_{(t)} \quad F_{(t)} = x$ for a:e: x 2 . □

Finally, we show that property (*iv*) of De nition 5.2 holds for F de ned in (7.1).

Proposition 7.7. *Let* F *be de ned as in (7.1). Then, equality (5.14) holds for any* ' 2 $C_c^1((0; \;) \quad ; R^3)$. *Moreover, we have that* $Z(\;; x)$ 2 $W^{1;1}$ $([0; \;))$ *for* ${}_0$ a:e: x 2 *, and (5.21) holds.*

Proof. From the de nition of the Lagrangian ow in Lemma 6.1, we have that

$$
(t; y) = y + \int_0^t w(s; \;{}_{(s)}(y))\, ds
$$

for ${}_0$ a:e: y 2 and every t 2 $[0; \;)$. Thus, this equality holds for all y 2 $\quad n N$ where ${}_0[N] = 0$. Since $T_0\# \;{}_0 = \;{}_0$ it follows that

$$
{}_0 \quad \backslash \; T_0^{1}(N) = \int_{T_0^{1}(N)} (0)\; dy \; 2 \;\text{equal}\; 0{:}0
$$

0 0 69.9626 Tf 17.157 0 Td [(y)]TJ/F3J/F8 9.9626 Tf 4.469 1.49

$$\int_{[0,\infty)\times\Omega} \partial_t \varphi(t,x)\,\psi(t, T_0(x))\,\rho_0(x)\,dtdx = \int_{\Omega} \varphi(0,x) T_0(x)\,\rho_0(x)\,dtdx - \int_{[0,\infty)\times\Omega} \varphi(t,x) w(t, \Phi_{(t)}(T_0(x)))\,\rho_0(x)\,dtdx; \tag{7.5}$$

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