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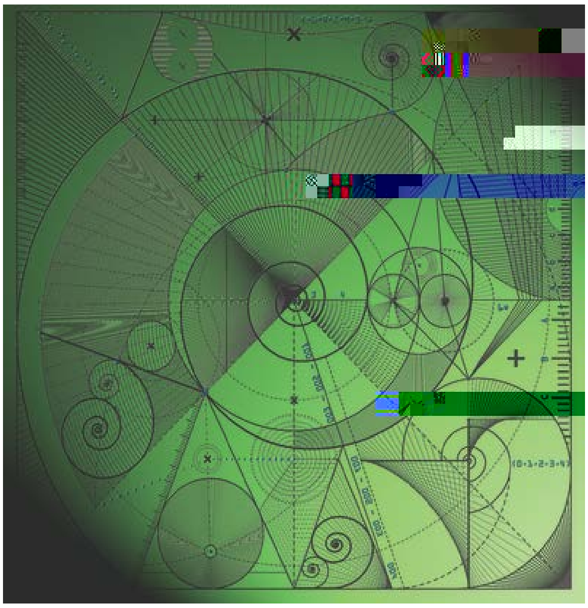


problems. They are the simplest canonical problems that exhibit multiple di raction, yet have applications in acoustics (see, e.g., [39, 23]), electromagnetics (see, e.g., [14, 42]) and water waves (the \breakwater" problem, see e.g. [2], [26, chapter 4.7]). In this paper, we propose a numerical method (supported by a complete analysis) that we believe to be the rst method of any kind (numerical or analytical) for this problem that is provably e ective at all frequencies. Precisely, we prove that increasing the number of degrees

Figure 1: Total field u, solving Problem P, for $\mathbf{d} = (1/\sqrt{2};\ \ 1/\sqrt{2})$ with $k = 5$ (upper) and $k = 20$ (lower).

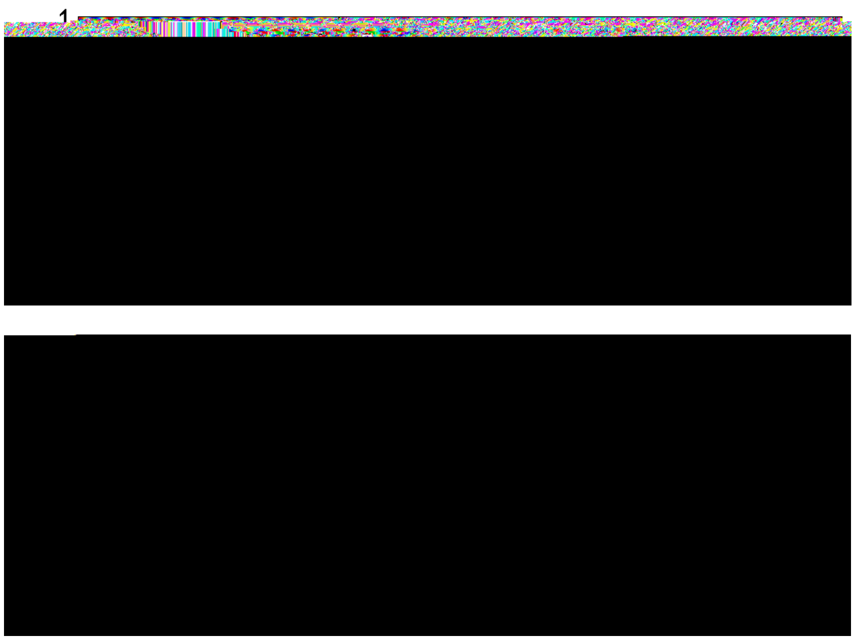


Figure 2: Total field u^0, solving Problem P^0, for $\mathbf{d} = (1/\sqrt{2};\ \ 1/\sqrt{2})$ with $k = 5$ (upper) and $k = 20$ (lower).

An outline of the paper is as follows: we begin in $x2$ by reviewing details of the specific incident angles (see, e.g., [2, 31]), but these approaches still require a solution to those specific problems. Thus, in general, both Problems P and P' must be solved either numerically, or else asymptotically in the high ($k \to \infty$) or low ($k \to 0$) frequency limit.

Asymptotic and numerical approaches are usually viewed as being rather comple-

In future work, it might be of interest to use elements of our approximation space (defined in §5) with the weighted integral operators proposed by [4, 25], to see what

Then $C^\infty_{comp}(\Omega) := \{U|_\Omega : U \in C_0^\infty(\mathbb{R}^n)\}$ is a dense subset of $H^s(\Omega)$. Second, let $\widetilde{H}^s(\Omega)$ denote the closure of $C_0^\infty(\Omega) := \{U \in C_0^\infty(\mathbb{R}^n) : \mathrm{supp}(U) \subset \Omega\}$ in the space $H^s(\mathbb{R}^n)$, equipped with the norm $\|\cdot\|_{\widetilde{H}^s_k(\Omega)} := \|\cdot\|_{H^s_k(\mathbb{R}^n)}$. When Ω is sufficiently regular (e.g. when Ω is C^0, cf. [27, Thm 3.29]) we have that we de ne normal derivative operators $@_n : C^1_{comp}(U \ ) ! : C$

Theorem 3.2. Suppose that u' is a solution of Problem P$'$. Then the representation formula

$$u'(x) = \begin{cases} u^i(x) + u^r(x) - \mathcal{S}_k\{\partial u'/\partial n\}(x), & x \in U^+, \\ \mathcal{S}_k\{\partial u'/\partial n\}(x), & x \in U^-, \end{cases} \tag{14}$$

holds, where $\{\partial u'/\partial n\}(x) := \partial_n^+(\chi u') + \partial_n^-(\chi u') \in \tilde{H}^{-1/2}(\Gamma)$, and χ is an arbitrary element of $C^1_{0,1}(\mathbb{R}^2)$. Furthermore, $\{\partial u'/\partial n\}(x) \in \tilde{H}^{-1/2}(\Gamma)$ satisfies the integral equation (13). Conversely, suppose that $\varphi \in \tilde{H}^{-1/2}(\Gamma)$ satisfies (13). Then u', defined by $u' := u^i + u^r - \mathcal{S}_k\varphi$ in U^+ and $u' := \mathcal{S}_k\varphi$ in U^-, satisfies Problem P$'$, and $\{\partial u'/\partial n\} = \varphi$.

The following continuity and coercivity properties of the operator S_k have been proved recently in [7, 8]:

Lemma 3.3 ([7, Theorem 5.2]). Let $s \in \mathbb{R}$. Then $S_k : \tilde{H}^s(\Gamma) \to H^{s+1}(\Gamma)$ is bounded, and for $kL \geq c_0 > 0$ there exists a constant $C_0 > 0$, depending only on c_0 (specifically, $C_0 = C \log(2 + c_0^{-1})$, where C is independent of c_0), such that

$$\|S_k\|_{\tilde{H}^s_k(\Gamma) \to H^{s+1}_k(\Gamma)} \leq C_0(1 + \sqrt{kL})\,\| \cdot \|_{\tilde{H}^{s}_k(\Gamma)};$$

Hk k e exists as8.95036(s)]TJ -0.613]TJ/F37 7 ~ 5.119 1.793 Td [()]TJ/F61 1c]TJ/F34 7.97019.n21TJ/F61F

if k`

Problem P^0

Proposition 4.7. The solution u of Problem P satisfies the pointwise bound

$$|u(x)| \leq C \left(1 + \frac{1}{\sqrt{kL}}\right)\left(1 + \frac{1}{\sqrt{kd}}\right) \log\left(2 + \frac{1}{kd}\right) \log^{1/2}(2 + kL)(1 + \sqrt{kL}), \qquad x \in D; \tag{22}$$

where $d = \mathrm{dist}(x, \;)$, and $C > 0$ is independent of x, k and .

The second stage in the proof of Lemma 4.5 involves the derivation of a uniform bound on $|u(x)|$ valid on a neighbourhood of . We begin by using a separation of variables argument to bound $|u(x)|$ close to in terms of the L^2 norm of the scattered field in a neighbourhood of .

Lemma 4.8. Let be of the form (1), and let u be the corresponding solution of Problem P, with $u^s = u - u^i$. Let $\varepsilon := \min\{\ell_{\min}/2, \; = (3k$

$K_n := \sqrt{\int_{kR/2}^{kR} \frac{z\,dz}{|J_{n/2}(z)|^2}}$ and $A_{R/2,R}$ is the annulus defined by $A_{R/2,R} := \{(r,\theta) : R/2 < r < R,\ 0 \le \theta \le 2\pi\}$. To bound $|K_n|$, we note that (cf., e.g., [10, (3.12)])

$$\cos z \le \frac{J_\nu(z)\,\Gamma(1+\nu)}{(z/2)^\nu} \le 1, \quad 0 \le z \le \pi/2, \quad \nu > -1/2. \tag{25}$$

where $\Gamma(\cdot)$, in (25)–(27), denotes the Gamma function. Hence if $0 < kR \le \pi/3$ (so that $1/2 \le \cos z \le 1$ for $kR/2 \le z \le kR$) then

$$|K_n| \le 2^{1+n/2}\,\Gamma(1+n/2)\sqrt{\int_{kR/2}^{kR} z^{1-n}\,dz} \le \frac{2^{1+n}\,\Gamma(1+n/2)}{\sqrt{n}}(kR)^{1-n/2}. \tag{26}$$

Thus

$$|a_n| \le \frac{2^{4+n}\,\Gamma(1+n/2)(kR)^{-n/2}}{3\sqrt{\pi}R\sqrt{n}}\|u\|_{L^2(A_{R/2,R})}; \tag{27}$$

and, using (25) again,

$$|a_n J_{n/2}(kr)| \le \frac{16(2r/R)^{n/2}}{3\sqrt{\pi}R\sqrt{n}}\|u\|_{L^2(A_{R/2,R})}.$$

Then, for $x \in B_{R/2}(e_j)$,

$$|u(x)| = |u(r,\theta)| \le \frac{16}{3\sqrt{\pi}R}\sum_{n=1}^{\infty}(2r/R)^{n/2}\|u\|_{L^2(A_{R/2,R})} = \frac{16}{3\sqrt{\pi}R}\,\frac{(2r/R)^{1/2}}{1-(2r/R)^{1/2}}\,\|u\|_{L^2(A_{R/2,R})}.$$

In particular, for $x \in B_{R/4}(e_j)$ we have

$$|u(x)| \le \frac{16}{3\sqrt{\pi}(\sqrt{2}-1)R}\|u\|_{L^2(A_{R/2,R})}.$$

Recalling that $u = u^i + u^s$, and noting that $\|u^i\|_{L^2(A_{R/2,R})} \le \sqrt{3\pi}R/2$, this implies that

$$|u(x)| \le \frac{8}{\sqrt{3}(\sqrt{2}-1)} + \frac{16}{3\sqrt{\pi}(\sqrt{2}-1)R}\|u^s\|_{L^2(A_{R/2,R})}; \qquad x \in B_{R/4}(e_j). \tag{28}$$

To satisfy both $R \le \pi/(3k)$ and $R < \ell_{\min}$, it suffices to set, e.g., $R = R_j := \min\{\ell_{\min}/2, \pi/(3k)\}$. A similar estimate to (28) can be obtained in a neighbourhood of the right endpoint e_j'.

Now let x_j denote an interior point of Γ_j and let (r,θ) be polar coordinates centered at x_j, so that Γ_j is a subset of the lines $\theta = 0$ and $\theta = \pi$. By a similar analysis to that presented above, but with the separation of variables carried out only in a half-disk $0 \le \theta \le \pi$ or $\pi \le \theta \le 2\pi$ and $n/2$ replaced by n etc., we can show that, if $0 < R \le \pi/(3k)$ and $R < \min\{|x - e_j|, |x - e_j'|\}$, then

$$|u(x)| \le \frac{4\sqrt{2}}{\sqrt{3}} + \frac{16}{3\sqrt{\pi}R}\|u^s\|_{L^2(\tilde{A}_{R/2,R})}; \qquad x \in B_{R/4}(x_j); \tag{29}$$

where $\tilde{A}_{R/2,R} := \{(r,\theta) : R/2 < r < R,\ 0 \le \theta \le \pi\}$ is a semi-annulus centered at x_j.

To combine these results we note that if $\min\{|x - e_j|, |x - e_j'|\} > R_j/4$ then we can take $R = R_j/4$ in (29). Then the union of the balls $B_{R_j/16}(x_j)$ over all such x_j, together with the balls $B_{R_j/4}(e_j)$ and $B_{R_j/4}(e_j')$, certainly covers a $(R_j/32)$-neighbourhood of Γ_j. Hence we can conclude that

$$|u(x)| \le \left(\frac{8}{\sqrt{3}(\sqrt{2}-1)} + \frac{16}{\sqrt{3\pi}(\sqrt{2}-1)R_j}\right)\|u^s\|_{L^2((\Gamma_j)_{R_j})}, \qquad x \in (\Gamma_j)_{R_j/32}, \tag{30}$$

from which the result follows, since $1/R_j \le 2k(1 + (k\ell_{\min})^{-1})$. □

To use Lemma 4.8 we require an estimate of $\|u^s\|_{L^2((\Gamma)_\varepsilon)}$, which is provided by the following result:

Lemma 4.9. Let $\varepsilon > 0$. Then there exists a constant $C > 0$, independent of ε, k and Ω, such that

$$\|S_k\phi\|_{L^2((\Gamma)_\varepsilon)} \le C\sqrt{k\varepsilon}(1 + k\varepsilon)k^{-1}\log(2 + (kL)^{-1})(1 + (kL)^{1/2})\|\phi\|_{\tilde{H}_k^{-1/2}(\Gamma)}, \qquad \phi \in \tilde{H}^{-1/2}(\Gamma). \tag{31}$$

Proof. Arguing as in the proof of [7, Lemma 5.1 and Thm 5.2], one can show that for any $\varepsilon > 0$ (see also [8] for slightly sharper bounds)

$$\|S_k\phi(\cdot, x_2)\|_{L^2(\tilde{\Gamma}_\varepsilon)} \le C\log(2 + (kA)^{-1})(1 + (kA)^{1/2} + (k|x_2|)^{1/2}\log(2 + kA))\|\phi\|_{\tilde{H}_k^{-1}(\Gamma)},$$

where $A = L + \varepsilon$, $\tilde{\Gamma}_\varepsilon := \{x \in \mathbb{R} : \operatorname{dist}(x, \tilde{\Gamma}) < \varepsilon\}$, $x_2 \in \mathbb{R}$, and $C > 0$ is independent of k, Ω and ε. From this one can show that

$$\|S_k\phi(\cdot, x_2)\|_{L^2((\tilde{\Gamma})_\varepsilon)} \le C(1 + k\varepsilon)\log(2 + (kL)^{-1})(1 + (kL)^{1/2})\|\phi\|_{\tilde{H}_k^{-1}(\Gamma)}, \qquad |x_2| \le \varepsilon,$$

where again C is independent of k, Ω and ε. The estimate (31) then follows from integrating over $x_2 \in (-\varepsilon, \varepsilon)$ and noting that $\|\phi\|_{\tilde{H}_k^{-1}(\Gamma)} \le k^{-1/2}\|\phi\|_{\tilde{H}_k^{-1/2}(\Gamma)}$. □

Combining Lemmas 4.8 and 4.9 gives:

Proposition 4.10. Under the assumptions of Lemma 4.8, we have

$$|u(x)| \le C\left(1 + (k\ell_{\min})^{-1}\right)\log(2 + (kL)^{-1})(1 + kL), \qquad x \in (\Gamma)_{\varepsilon/32}, \tag{32}$$

where $C > 0$ is independent of x, k and Ω.

Proof. Noting that $u^s = -S_k[\partial u/\partial n]$, and that $[\partial u/\partial n] = S_k^{-1}u^i|_\Gamma$, the result follows from Lemmas 4.6(i), 4.8 and 4.9, the stability estimate (17), and the fact that $k\varepsilon = \pi/3$. □

The third and final stage in the proof of Lemma 4.5 involves combining Propositions 4.7 and 4.10 to obtain a bound which holds uniformly throughout D. Specifically, we combine (32), which holds in the region $d < \varepsilon/32$, with (22), applied in the region $d \ge \varepsilon/32$. Noting that in the latter case we have $(kd)^{-1} \le 32/(k\varepsilon) \le C(1 + (k\ell_{\min})^{-1})$, we can obtain the following estimate in which the constant C is independent of both k and Ω:

$$|u(x)| \le C\left(1 + \frac{1}{k\ell_{\min}}\right)\log\left(1 + \frac{1}{k\ell_{\min}}\right)(1 + kL), \qquad x \in D.$$

The statement of Lemma 4.5 then follows immediately.

5 hp approximation space and best approximation results

Our numerical method for solving the integral equation (13) uses a hybrid numerical-asymptotic approximation space based on Theorem 4.1. Rather than approximating itself using piecewise polynomials (as in conventional methods), we use the decomposition (20), with the factors v_j^+ and v_j replaced by piecewise polynomials. The advantage of our approach is that, as is quanti ed by Theorem 4.1, the functions v_j are non-oscillatory (cf. Remark 4.2), and can therefore be approximated much more e ciently than the full (oscillatory) solution . Explicitly, the function we seek to approximate is

$$' \ (s) := \frac{1}{k} (\ (x(s)) \qquad (x(s))) ; \quad s \ 2 \ \tilde{} \quad (0; L); \tag{33}$$

which represents the di erence between and its GO approximation (recall Remark 4.4), scaled by 1=k so that '

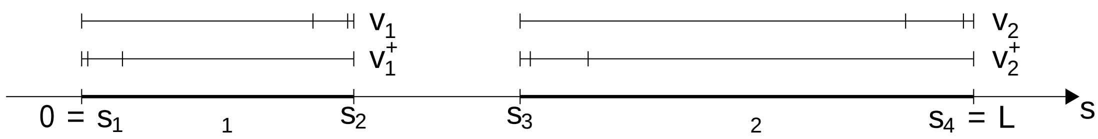


Figure 3: Illustration of the overlapping geometrically graded meshes used to approximate the amplitudes v_j in (34), in the case where comprises two components, $_1$ and $_2$.

for some $\in [0;1]$ and some integer $p \geq 0$ (the highest polynomial degree on the mesh). The choice $= 0$ corresponds to a constant degree across the mesh (this was the only choice considered in [21]), while for $\in (0;1]$ the degree decreases linearly in the direction of refinement.

For each $j = 1;\ldots;n_i$ let $n_j \geq 1$ and $p_j \in (N_0)^{n_j}$ denote respectively the number of layers and the degree vector associated with the approximation of the factor v_j in (34). The total number of degrees of freedom in $V_{N;k}$ is then

$$N := \dim(V_{N;k}) = \sum_{j=1}^{n_i} \left(\sum_{m=1}^{n_j^+} (p_j^+)_m + 1 + \sum_{m=1}^{n_j} (p_j)_m + 1 \right) : \qquad (37)$$

The regularity results provided by Theorem 4.1 allow us to prove that, under certain assumptions, the best approximation error in approximating ' by an element of $V_{N;k}$ decays exponentially as p, the maximum degree of the approximating polynomials, increases. Our best approximation results in the space $\tilde{H}^{1/2}(\tilde{\ })$ are stated in the following theorem, which is the main result of this section. For simplicity of presentation we assume that the mesh parameters are the same in each of the meshes used to approximate the different components v_j (similar estimates hold in the more general case).

Theorem 5.1. Let $k\ell_{min} \geq c_0 > 0$. Suppose that $n_j = n$ and $p_j = p$ for each $j = 1;\ldots;n_i$, where n and p are defined by (36) with $n \geq cp$ for some constant $c > 0$. Then, for any $0 < < 1/2$, there exists a constant $C_3 > 0$, depending only on , , n

The following result is essentially stated in [6, equation (A.7)], but we need to restate it here as we are working with a k-dependent norm and want k-explicit estimates.

Lemma 5.3. For $1 \le q \le 2$ and $s < 1/2 - 1/q$, $L^q(\mathbb{R})$ can be continuously embedded in $H^s(\mathbb{R})$, with

$$\|\phi\|_{H^s_k(\mathbb{R})} \le c(s,k,\beta)\,\|\phi\|_{L^q(\mathbb{R})}, \quad \phi \in L^q(\mathbb{R}); \tag{39}$$

where β is as defined in Lemma 5.2 and

$$c(s,k,\beta) = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} (k^2+\xi^2)^{s/\beta}\, d\xi\right)^{\beta/2}.$$

Proof. By the density of $C_0^\infty(\mathbb{R})$ in $L^q(\mathbb{R})$ it suffices to prove (39) for $\phi \in C_0^\infty(\mathbb{R})$. Let $\phi \in C_0^\infty(\mathbb{R})$, let $1 < q \le 2$ (the case $q = 1$ requires an obvious trivial modification of the proof), let r be such that $1/q + 1/r = 1$, and let $\beta = 2/q - 1$ as in Lemma 5.2. Provided that $s < 1/2 - 1/q$, we have $s/\beta < -1/2$, so that the function $(k^2+\xi^2)$ is in $L^{1/\beta}(\mathbb{R})$, and Hölder's inequality gives

$$\begin{aligned}
\|\phi\|^2_{H^s_k(\mathbb{R})} = \int_{\mathbb{R}} (k^2+\xi^2)^s |\hat\phi(\xi)|^2\, d\xi &\le \left(\int_{\mathbb{R}} (k^2+\xi^2)^{s/\beta}\, d\xi\right)^{\beta} \left(\int_{\mathbb{R}} (|\hat\phi(\xi)|^2)^{r/2}\, d\xi\right)^{2/r} \\
&= c(s,k,\beta)^2 (2\pi)\, \|\hat\phi\|^2_{L^r(\mathbb{R})} \\
&\le c(s,k,\beta)^2 \|\phi\|^2_{L^q(\mathbb{R})};
\end{aligned}$$

the final inequality following from an application of Lemma 5.2. □

Corollary 5.4. For $1 < q \le 2$, $L^q(\mathbb{R})$ can be continuously embedded in $H^{-1/2}(\mathbb{R})$ with

$\|\phi\|_{H\,(\mathbb{R})} \|_{1}$ 2 H4.553-4.98.746 cm5

Theorem 5.5. Suppose that a function $g(z)$ is analytic in $\mathrm{Re}[z] > 0$ and satisfies the bound

$$|g(z)| \leq \hat{C}|z|^{-1/2}, \quad \mathrm{Re}[z] > 0,$$

for some $\hat{C} > 0$. Given $l > 0$, $\sigma \in [0,1]$, and integers $n \geq 1$ and $p \geq 0$, let the degree vector $\mathbf{p}$ be defined by (36), and suppose that $n \geq cp$ for some constant $c > 0$. Then for any $0 < \epsilon < 1/2$ there exists a constant $C_\epsilon > 0$, depending only on ϵ and σ (with $C_\epsilon \to \infty$ as $\epsilon \to 0$ or $\epsilon \to 1/2$), and a constant $\tau > 0$, depending only on ϵ, σ, and c (with $\tau \to 0$ as $\sigma \to 0$), such that

$$\inf_{v \in P_{\mathbf{p},n}(0,l)} \|g - v\|_{\tilde{H}_k^{-1/2}(0,l)} \leq C_\epsilon \hat{C} k^{-1/2} (kl)^{\epsilon} e^{-p\tau}. \tag{42}$$

Proof. Our aim is to use Corollary 5.4 to derive a best approximation error estimate in the $\tilde{H}_k^{-1/2}$ norm in terms of estimates in L^q norms, $1 < q < 2$. For the sharpest results (in terms of k-dependence) one might want to take $q = 2$ in Corollary 5.4. However, this is not possible because g cannot be assumed to be square integrable at $s = 0$; this is why we assume that $1 < q < 2$.

We begin by defining a candidate approximant $V \in P_{\mathbf{p},n}(0,l)$, which we take to be zero on $(0, x_1)$, and on (x_{i-1}, x_i), $i = 2, \ldots, n$, to be equal to the L^∞ best approximation to $g|_{(x_{i-1}, x_i)}$ in

Now, since

$$(p)_i \quad 1 \quad + \frac{(i \quad 1)}{n} \quad p; \quad i = 2$$

6 Galerkin method

Having designed an approximation space $V_{N;k}$ which can efficiently approximate φ, we now select an element of $V_{N;k}$ using the Galerkin method. That is, we seek $\varphi_N \in V_{N;k} \subset \tilde{H}^{-1/2}(\Gamma)$ such that (recall (13) and (34))

$$\langle S_k \varphi_N ; v\rangle = \frac{1}{k}\langle f - S_k \Psi ; v\rangle ; \quad \text{for all } v \in V_{N;k}: \tag{47}$$

We note that since $\varphi_N ; v \in V_{N;k} \subset L^2(\Gamma)$ the duality pairings in (47) can be evaluated simply as inner products in $L^2(\Gamma)$ (see the discussion after (8) and the implementation details in §7). Existence and uniqueness of the Galerkin solution φ_N is guaranteed by the Lax-Milgram Lemma and Lemmas 3.3 and 3.4. Furthermore, by Cea's lemma we have the quasi-optimality estimate

$$\|\varphi - \varphi_N\|_{\tilde{H}_k^{-1/2}(\Gamma)} \leq \frac{C_0(1+\sqrt{kL})}{2\sqrt{2}} \inf_{v \in V_{N;k}} \|\varphi - v\|_{\tilde{H}_k^{-1/2}(\Gamma)} ; \tag{48}$$

where C_0 is the constant from Lemma 3.3. Combined with Theorem 5.1, this gives:

Theorem 6.1. Under the assumptions of Theorem 5.1, we have

$$\|\varphi - \varphi_N\|_{\tilde{H}_k^{-1/2}(\Gamma)} \quad 1$$

An object of interest in applications is the far field pattern of the scattered field. An asymptotic expansion of the representation (12) reveals that (cf. [12])

$$u^s(x) \sim \frac{e^{i\pi/4}}{2\sqrt{2\pi}} \frac{e^{ikr}}{\sqrt{kr}} F(\hat{x}); \quad \text{as } r := |x| \to \infty;$$

where $\hat{x} := x/|x| \in S^1$, the unit circle, and

$$F(\hat{x}) := \int e^{-ik\hat{x}\cdot y} \frac{\partial u}{\partial n}(y)\, ds(y); \qquad \hat{x} \in S^1. \tag{52}$$

An approximation

Hence $L_1 = 2\pi$, $L_2 = 2\pi/5$, $L_3 = 7\pi/10$, $L_4 = 2\pi$ and $L_5 = 39\pi/10$, so that the smallest component has length $2\pi/5$, the longest has length $39\pi/10$, and the sum of the length of all of the components is $\sum_{i=1}^{n_i} L_j = 9\pi = 9k\lambda/2$ (where $\lambda = 2\pi/k$ is the wavelength). We present results below for values of k ranging from $k = 10$ (in which case the smallest segment is two wavelengths long) up to $k = 10240$ (in which case the longest segment is nearly 20000 wavelengths long). The plots in Figures 1 and 2 show the total fields for the "non-grazing" incident direction $\mathbf{d} = (1/\sqrt{2}, -1/\sqrt{2})$; in our examples below we also consider the "grazing" incident direction $\mathbf{d} = (1, 0)$.

To describe our implementation of the HNA approximation space of §5, we write $\varphi_N \in V_{N,k}$ as

$$\varphi_N(\cdot) = \sum_{\ell=1}^{N} v_\ell \psi_\ell(\cdot); \tag{55}$$

where N is given by (37), v_ℓ, $\ell = 1, \ldots, N$, are the unknown coefficients to be determined, and ψ_ℓ, $\ell = 1, \ldots, N$, are the HNA basis functions, which we now define. Each basis function ψ_ℓ is supported on an interval $(a, b) \subset (s_{2j-1}, s_{2j})$ for some $j \in \{1, \ldots, n_i\}$, and takes the form

$$\psi_\ell(s) = \sqrt{\frac{2q+1}{b-a}}\, P_q\left(2\left(\frac{s-a}{b-a}\right) - 1\right) e^{\sigma iks}, \quad s \in (a, b);$$

where P_q, $q \leq p$, denotes the Legendre polynomial of order q, and either $\sigma = 1$ and $a \in \mathcal{F}_1$

instead, we compute $\|\phi\|$, defined by

$$\|\phi\| := \sqrt{|\langle S_k\phi, \phi\rangle|}, \qquad \phi \in \tilde{H}^{-1/2}(\Gamma);$$

which defines an equivalent norm on $\tilde{H}^{-1/2}(\Gamma)$ and is easier to compute (see, e.g., the discussion in [33, pp. A:29–A:30]). Specifically, it follows from (15) and (16) that

$$\frac{1}{\sqrt{2}\sqrt{2}}\|\phi\|_{\tilde{H}_k^{-1/2}(\Gamma)} \le \|\phi\| \le \sqrt{C_0(1+\sqrt{kL})}\,\|\phi\|_{\tilde{H}_k^{-1/2}(\Gamma)}, \qquad \phi \in \tilde{H}^{-1/2}(\Gamma);$$

and hence combining the right inequality with Theorem 6.1 we expect

$$\|\varphi - \varphi_N\| \le C_5\sqrt{C_0(1+\sqrt{kL})}\,k^{-1}(1+(kL)^{3/2+\epsilon})\,e^{-p}. \tag{57}$$

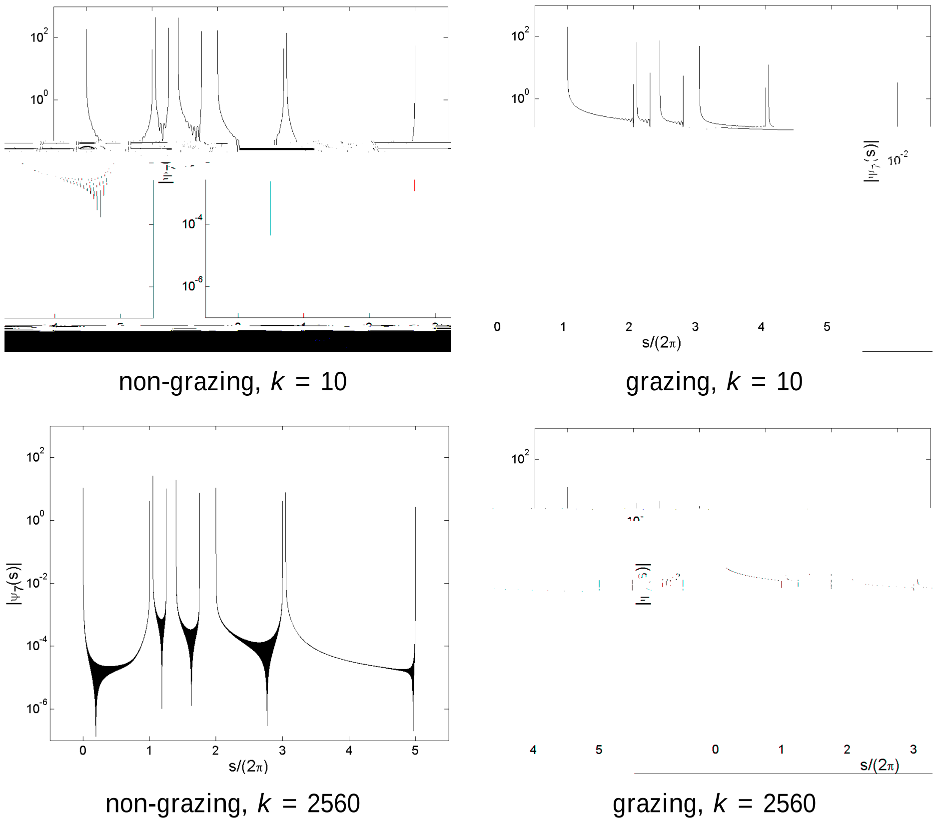
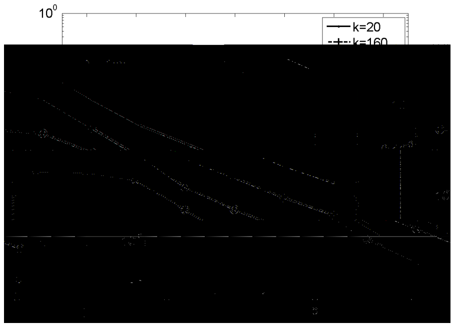


Figure 4: Boundary solution for grazing and non-grazing incidence, with k = 10 and k = 2560.

In Table 1 we also show the condition number (COND) of the N-dimensional linear system (56), and we investigate the dependence of the condition number on both k and p further in Figure 6. For fixed k, the condition number grows exponentially with respect to p (note the logarithmic scale on the vertical axis). This rapid growth in the condition number as p increases is not surprising: for weakly singular BIEs of the first kind, the condition number for standard hp Galerkin BEM, with a geometrically graded mesh (as used here), is known to grow exponentially with respect to the number of unknowns (see, e.g., [18]). For fixed p, the condition number decreases slowly as k increases (and hence as the average number of degrees of freedom per wavelength decreases), and we note that the condition numbers we encountered in our experiments were not so large as to cause problems for our direct solver. Furthermore, as remarked in §1, our best approximation results (though not our full analysis) hold regardless of the BIE used, so using our approximation space within a better conditioned BIE such as the second kind formulations proposed in [4, 25] might lead to reduced condition numbers.

Finally, in the last column of Table 1 we show the relative computing time (rel cpt) required for setting up and solving the linear system (we solve the system directly), measured with respect to the time required for k = 10. We emphasize the fact that the computing time is independent of k, reflecting that all of the integrals are evaluated using Filon quadrature in a k-independent way.



e_p

Figure 6: Condition number of the N-dimensional linear system (56).

non-grazing, $\max_{t \in [0;28]} |u_7(t) \quad u_p(t)|$

grazing, $\max_{t \in [0;28]} |u_7(t) \quad u_p(t)|$

non-grazing, max

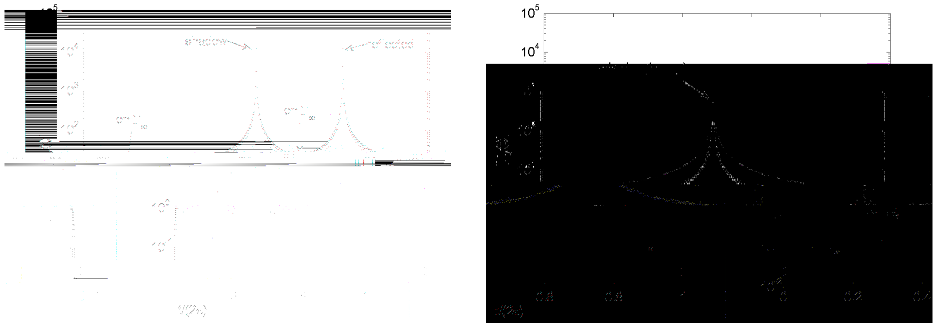
d	k	$N \big/ \sum_{j=1}^{n_i} L_j/\lambda$	e_p		r_p	COND	rel cpt
$(1/\sqrt{2}; 1/\sqrt{2})$	10	10.00	9.25×10^{-4}	-1.03	2.18×10^{-3}	1.50×10^{9}	1.00
	20	5.00	4.51×10^{-4}	-0.77	2.01×10^{-3}	1.03×10^{9}	0.98
	40	2.50	2.64×10^{-4}	-0.87	2.49×10^{-3}	7.12×10^{8}	0.98
	80	1.25	1.45×10^{-4}	-0.69	2.88×10^{-3}	4.97×10^{8}	0.99
	160	0.63	8.99×10^{-5}	-0.80	3.72×10^{-3}	3.50×10^{8}	1.00
	320	0.31	5.16×10^{-5}	-0.74	4.32×10^{-3}	2.47×10^{8}	0.99
	640	0.16	3.08×10^{-5}	-0.74	5.08×10^{-3}	1.75×10^{8}	1.00
	1280	0.08	1.85×10^{-5}	-0.67	6.48×10^{-3}	1.23×10^{8}	1.00
	2560	0.04	1.16×10^{-5}	-0.91	7.64×10^{-3}	8.72×10^{7}	1.00
	5120	0.02	6.18×10^{-6}	-0.83	9.14×10^{-3}	6.17×10^{7}	1.01
	10240	0.01	3.47×10^{-6}		1.01×10^{-2}	4.36×10^{7}	1.01
$(1; 0)$	10	10.00	3.39×10^{-4}	-0.38	4.52×10^{-4}	1.50×10^{9}	1.00
	20	5.00	2.60×10^{-4}	-0.61	5.84×10^{-4}	1.03×10^{9}	1.01
	40	2.50	1.70×10^{-4}	-0.60	6.43×10^{-4}	7.12×10^{8}	0.99
	80	1.25	1.12×10^{-4}	-0.71	7.13×10^{-4}	4.97×10^{8}	0.98
	160	0.63	6.84×10^{-5}	-0.69	7.31×10^{-4}	3.50×10^{8}	0.99
	320	0.31	4.23×10^{-5}	-0.68	7.59×10^{-4}	2.47×10^{8}	0.99
	640	0.16	2.64×10^{-5}	-0.72	7.97×10^{-4}	1.75×10^{8}	1.00
	1280	0.08	1.60×10^{-5}	-0.62	8.13×10^{-4}	1.23×10^{8}	1.00
	2560	0.04	1.04×10^{-5}	-0.73	8.92×10^{-4}	8.72×10^{7}	1.00
	5120	0.02	6.27×10^{-6}	-0.73	9.02×10^{-4}	6.17×10^{7}	1.01
	10240	0.01	3.78×10^{-6}		9.14×10^{-4}	4.36×10^{7}	1.00

Table 1: Errors e_p and relative errors r_p, for non-grazing ($d = (1/\sqrt{2}; 1/\sqrt{2})$) and grazing ($d = (1; 0)$) incidence, with $p = 5$ (and hence $N = 450$).

far field pattern computed with our finest discretization) for each of the two incident directions, for $k = 1280$, are shown in Figure 8. For non-grazing incidence, the peaks corresponding to the geometric shadow (i.e. the forward-scattering direction) and the specular reflection are indicated (compare Figure 8 with Figure 1). We also show the points at which $\hat{x}(t) \in \Gamma_1$. For grazing incidence, the shadow peak is much lower for than for non-grazing incidence; in the grazing case, there is no reflected peak.

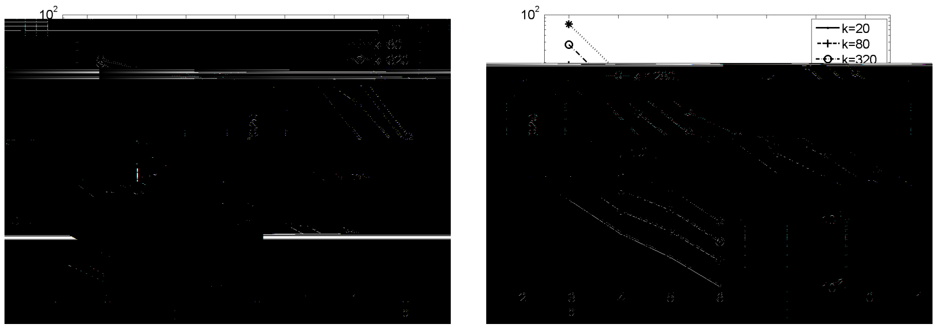
In Figure 9 we plot approximations to $\|F_\infty - F_p\|_{L^1(S^1)}$ and $\|F_\infty - F_p\|_{L^1(S^1)}/\|F_\infty\|_{L^1(S^1)}$ for $k = 20$, 80, 320 and 1280, for each of the two incident directions. To approximate the L^1 norm, we compute F_∞ and F_p at 50,000 evenly spaced points on the unit circle. The exponential decay as p increases, as predicted by Theorem 6.3, can be clearly seen (again, note the logarithmic scale on the vertical axes).

For fixed p, the errors $\|F_\infty - F_p\|_{L^1(S^1)}$ increase slowly as k increases. To investigate this behaviour more carefully, in Table 2 we show results for the two angles of incidence for $p = 5$ (and hence N



non-grazing grazing

Figure 8: Far field patterns, $|F_7(t)| \quad |F(t)|$, $k = 1280$.



non-grazing, $\|F_7 \quad F_p\|_{L^1(S^1)}$ grazing, $\|F_7 \quad F_p\|_{L^1(S^1)}$

non-grazing, $\dfrac{\|F_7 \quad F_p\|_{L^1(S^1)}}{\|F_7\|_{L^1(S^1)}}$ grazing, $\dfrac{\|F_7 \quad F_p\|_{L^1(S^1)}}{\|F_7\|_{L^1(S^1)}}$

Figure 9: Errors in the far field pattern (note the different scales on the upper and lower plots).

that $f_p(k)$ k as $k \to \infty$. The values of $\in (0.02, 0.75)$ for non-grazing incidence, and 0.5 for grazing incidence, are considerably lower than might be anticipated from the estimate (54), suggesting that our estimates are not sharp in terms of their k-dependence. In particular, the results are again consistent with the conjecture that $M(u) = O(1)$ (as discussed just before Lemma 4). In the last column of Table 2, we show how $k\|F_7\|_{L^\infty(S^1)}$ grows with k. For non-grazing incidence, $k\|F_7\|_{L^\infty(S^1)}$ grows approximately linearly with k, and so the relative error $\|F_7 - F_p\|_{L^\infty(S^1)}/\|F_7\|_{L^\infty(S^1)}$ decreases as k increases. For grazing incidence, $k\|F_7\|_{L^\infty(S^1)}$

optimality estimate (48).

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