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Kernel-smoothed conditional quantiles of randomly censored functional stationary ergodic data

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nonparametric models. For instance, Ferraty and Vieu (2004) established the strong consistency of kernel estimators of the regression function when the explanatory variable is functional and the response is scalar, and their study is extended to non standard regression problems such as time series prediction or curves discrimination by Ferraty *et al.* (2002) and Ferraty and Vieu (2003). The asymptotic normality result for the same estimator in the alpha-mixing case has been obtained by Masry (2005).
In addition to the regression function, other statistics such as quantile and mode regression could be with interest for both sides theory and practice. Quantile regression is a common way to describe the dependence structure between a response variable Y and some covariate X. Unlike the regression function (which is defined as the conditional mean) that relies only on the central tendency of the data, conditional quantile function allows the analyst to estimate the functional dependence between variables for all portions of the conditional distribution of the response variable. Moreover, quantiles are well-known by their robustness to heavy-tailed error distributions and outliers which allows to consider them as a useful alternative to the regression function.

Conditional quantiles for scalar response and a scalar/multivariate covariate have received considerable interest in the statistical literature. For completely observed data, several nonparametric approaches have been proposed, for instance, Gannoun *et al.* (2003) introduced a smoothed estimator based on double kernel and local constant kernel methods and Berlinet *et al.* (2001

where

$$W_{n,i}(x) = \frac{K\left(h_{n,K}^{-1} d(x, X_i)\right)}{\sum_{i=1}^{n} K\left(h_{n,K}^{-1} d(x, X_i)\right)}, \tag{6}$$

are the well-known Nadaraya-Watson weights. Here K is a real-valued kernel function, H a cumulative distribution function and $h_K := h_{n,K}$ (resp. $h_H := h_{n,H}$) a sequence of positive real numbers which decreases to zero as n tends to infinity. This estimator given by (5) has been introduced in Ferraty and Vieu (2006) in the general setting.

An appropriate estimator of the conditional distribution function $F(t \mid x)$ for censored data is then obtained by adapting (6) in order to put more emphasis on large values of the interest random variable T which are more censored than small one. Based on the same idea as in Carbonez *et al.* (1995) and Khardani *et al.* (2010), we consider the following weights

$$\widetilde{W}_{n,i}(x) = \frac{1}{h_K} K\left(h_K^{-1} d(x, X_i)\right) \frac{\delta_i}{\overline{G}(Y_i) \sum_{i=1}^{n} h_K^{-1} K\left(h_K^{-1} d(x, X_i)\right)}, \tag{7}$$

where $\overline{G}(\cdot) = 1 - G(\cdot)$. Now, we consider a *"pseudo-estimator"* of $F(t \mid x)$ given by:

$$\widetilde{F}_n(t \mid x) = \frac{\sum_{i=1}^{n} \delta_i \overline{G}^{-1}(Y_i) K\left(h_K^{-1} d(x, X_i)\right) H(h_H^{-1}(t - Y_i))}{\sum_{i=1}^{n} K\left(h_K^{-1} d(x, X_i)\right)} := \frac{\widetilde{F}_n(x, t)}{\ell_n(x)}, \tag{8}$$

where

$$\widetilde{F}_n(x, t) = \frac{1}{n\mathbb{E}(\Delta_1(x))} \sum_{i=1}^{n} \delta_i \overline{G}^{-1}(Y_i)\, H\left(h_H^{-1}(t - Y_i)\right) \Delta_i(x),$$

and

$$\ell_n(x) = \frac{1}{n\mathbb{E}(\Delta_1(x))} \sum_{i=1}^{n} \Delta_i(x),$$

where $\Delta_i(x) = K(d(x, X_i)/h_K)$. In practice G is unknown, we use the Kaplan and Meier (1958) estimator of G given by:

$$\overline{G}_n(t) = \begin{cases} \prod_{i=1}^{n} \left(1 - \frac{1 - \delta_{(i)}}{n - i + 1}\right)^{\mathbb{1}\{Y_{(i)} \le t\}} & \text{if } t < Y_{(n)}, \\ 0 & \text{Otherwise}, \end{cases}$$

where $Y_{(1)} < Y_{(2)} < \cdots < Y_{(n)}$ are the order statistics of $(Y_i)_{1 \le i \le n}$ and $\delta_{(i)}$ is the concomitant of $Y_{(i)}$. Therefore, the estimator of $F(t \mid x)$ is given by:

$$\widehat{F}_n(t \mid x) = \frac{\widehat{F}_n(x, t)}{\ell_n(x)}, \tag{9}$$

where

$$\widehat{F}_n(x, t) = \frac{1}{n\mathbb{E}(\Delta_1(x))} \sum_{i=1}^{n} \delta_i \overline{G}_n^{-1}(Y_i)\, H(h_H^{-1}(t - Y_i))\, \Delta_i(x).$$

Then a natural estimator of $q_\alpha(x)$ is given by:

$$\hat{q}_{\alpha,n}(x) = \inf\{y : \hat{F}_n(y \mid x) \geq \alpha\}, \tag{10}$$

which satisfies:

$$\hat{F}_n(\hat{q}_{\alpha,n}(x) \mid x) = \alpha. \tag{11}$$

3 Assumptions and main results

In order to state our results, we introduce some notations. Let $\mathcal{F}_i$ be the σ-field generated by $((X_1, T_1), \ldots, (X_i, T_i))$ and $\mathcal{G}_i$ the one generated by $((X_1, T_1), \ldots, (X_i, T_i), X_{i+1}).$ Let $B(x, u)$ be the ball centered at $x \in E$ with radius u. Let $D_i(x) := d(x, X_i)$ so that $D_i(x)$ is a nonnegative real-valued random variable. Working on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let $F_x(u) = \mathbb{P}(D_i(x) \leq u) := \mathbb{P}(X_i \in B(x, u))$ and $F_x^{\mathcal{F}_{i-1}}(u) = \mathbb{P}(D_i(x) \leq u \mid \mathcal{F}_{i-1}) = \mathbb{P}(X_i \in B(x, u) \mid \mathcal{F}_{i-1})$ be the distribution function and the conditional distribution function, given the σ-field $\mathcal{F}_{i-1}$, of $(D_i(x))_{i \geq 1}$ respectively. Denote by $o_{a.s.}(u)$ a real random function ℓ such that $\ell(u)/u$ converges to zero almost surely as $u \to 0.$ Similarly, define $O_{a.s.}(u)$ a real random function ℓ such that $\ell(u)/u$ is almost surely bounded. Furthermore, for any distribution function L, let $\tau_L = \sup\{t; \text{such that } L(t) < 1\}$ be the support's right endpoint. Let S be a compact set such that $q_\alpha(x) \in S \cup (-\infty, \tau]$, where $\tau < \tau_G \wedge \tau_F.$

3.1 Rate of strong consistency

Our results are stated under some assumptions we gather hereafter for easy reference.

(A1) K is a nonnegative bounded kernel of class C^1 over its support $[0, 1]$ such that $K(1) > 0$. The derivative K' exists on $[0, 1]$ and satisfy the condition $K'(t) < 0,$ for all $t \in [0, 1]$ and $\left|\int_0^1 (K^j)'(t)dt\right| < \infty$ for $j = 1, 2.$

(A2) For $x \in E$ and

(ii) $\int_{\mathbb{R}} |t| f(t \mid x)\,dt < \infty$, for all $x \in E$,

(ii) For any $x \in E$, there exist $V(x)$ a neighborhood of x, some constants $C_x > 0$, $\beta > 0$ and $\nu > 0$, such that for $j = 0, 1$, we have $\forall (t_1, t_2) \in S \times S$, $\forall (x_1, x_2) \in V(x) \times V(x)$,

$$\left| F^{(j)}(t_1 \mid x_1) - F^{(j)}(t_2 \mid x_2) \right| \le C_x \left(d(x_1, x_2)^{\beta} + |t_1 - t_2|^{\nu} \right).$$

(A4) For any $m \ge 1$ and $j = 0, 1$, $\mathrm{E}\left[\left(H^{(j)}(h_H^{-1}(t - T_i)) \right)^m \mid \mathcal{G}_{i-1} \right] = \mathrm{E}\left[\left(H^{(j)}(h_H^{-1}(t - T_i)) \right)^m \mid X_i \right]$

(A5) The distribution function H has a first derivative $H^{(1)}$ which is positive and bounded and satisfies $\int |u|^{\nu} H^{(1)}(u)\,du < \infty$.

(A6) For any $x' \in E$ and $m \ge 2$, $\sup_{t \in S} |g_m(x', t)| := \sup_{t \in S} |\mathrm{E}[H^m(h_H^{-1}(t - T_1)) \mid X_1 = x']| < \infty$ and $g_m(x', t)$ is continuous in $V(x)$ uniformly in t:

$$\sup_{t \in S} \sup_{x' \in B(x,h)} |g_m(x', t) - g_m(x, t)| = o(1).$$

(A7) $(C_n)_{n \ge 1}$ and $(T_n, X_n)_{n \ge 1}$ are independent.

Comments on hypothesis: Conditions (A1) involves the ergodic nature of the data and the small ball

3.2 Asymptotic normality

The aim of this section is to establish the asymptotic normality which induces a confidence interval of the conditional quantiles estimator. For that purpose we need to introduce further notations and assumptions. We assume, for $k = 1, 2$, that $E\left[|\delta_1 G^{-1}(Y_1) H(h_H^{-1}(t - Y_1))|^k\right] < \infty$ and that, for a fixed $x \in E$, the conditional variance, of $\delta_1 G^{-1}(Y_1) H(h_H^{-1}(t - Y_1))$ given $X_1 = x$, say,
$W_2(t | x) := E\left[\left(\delta_1 G^{-1}(Y_1) H(h_H^{-1}(t - Y_1)) - F(t | x)\right)^2 | X_1 = x\right]$ exists.

(A8) (i) The conditional variance of $\delta_i G^{-1}(Y_i) H(h_H^{-1}(t - Y_i))$ given the σ-field $\mathcal{G}_{i-1}$ depends only on X_i, i.e., for any $i \geq 1$, $E\left[\left(\delta_i G^{-1}(Y_i) H(h_H^{-1}(t - Y_i)) - F(t | X_i)\right)^2 | \mathcal{G}_{i-1}\right] = W_2(t | X_i)$ almost surely.

(ii) For some $\delta > 0$, $E[|\delta_1 G^{-1}(Y_1) H(h_H^{-1}(t - Y_1))|^{2+\delta}] < \infty$ and the function $\overline{W}_{2+\delta}(t | u) := E(|\delta_1 G^{-1}(Y_i) H(h_H^{-1}(t - Y_i)) - F(t | x)|^{2+\delta} | X_i = u)$, $u \in E$, is continuous in a neighborhood of x.

(A9) The distribution function of the censored random variable, G has bounded first derivative $G^{(1)}$.

Theorem 3.3 *Assume that assumptions (A1)-(A9) hold true and condition (12) is satisfied, then we have*

$$\sqrt{n\phi(h_K)}\left(\widehat{F}_n(t | x) - F(t | x)\right) \xrightarrow{D} N\left(0, \sigma^2(x; t)\right);$$

where $\xrightarrow{D}$ *denotes the convergence in distribution and*

$$\sigma^2(x; t) = \frac{M_2}{M_1^2} \frac{F(t | x)\left(G^{-1}(t) - F(t | x)\right)}{f_1(x)};$$

where $M_j = K^j(1) - \int_0^1 (K^j)'(u)\tau_0(u)du$.

Theorem 3.4 *Under the same assumptions and conditions of Theorem 3.3, we have*

$$\sqrt{n\phi(h_K)}\left(\widehat{q}_{n,\alpha}(x) - q_\alpha(x)\right) \xrightarrow{D} N\left(0, \sigma^2(x; q_\alpha(x))\right);$$

$$\sigma^2(x; q_\alpha(x)) = \frac{\sigma^2(x; q_\alpha(x))}{f^2(q_\alpha(x) | x))} = \frac{M_2}{M_1^2 f_1 \; q}$$

3.3 Application to predictive interval

Corollary 3.5 *Assume that conditions (A1)-(A9) hold true, K' and $(K^2)'$ are integrable functions and*

$$\frac{M_{1,n}}{\sqrt{M_{2,n}}}\hat{f}_n(\hat{q}_{\alpha,n}(x)\,|\,x)\sqrt{\frac{nF_{x,n}(h_K)}{G_n^{-1}(\hat{q}_{\alpha,n}(x))}}\left(\hat{q}_{n,\alpha}(x)-q_\alpha(x)\right)\xrightarrow{D}N(0,1),$$

where $\hat{f}_n(\cdot\,|\,x)$ is an estimator of the conditional density function $f(\cdot\,|\,x)$.

The corollary 3.5 can be now used to provide the $100(1-\alpha)\%$ confidence bands for $q_\alpha(x)$ which is given, for $x \in E$, by

$$\hat{q}_{n,\alpha}(x) \pm c_{\alpha/2}\frac{\sqrt{M_{2,n}}}{M_{1,n}\,\hat{f}_n(\hat{q}_{\alpha,n}(x)\,|\,x)}\sqrt{\frac{G_n^{-1}(\hat{q}_{\alpha,n}(x))}{nF_{x,n}(h_K)}}.$$

where $c_{\alpha/2}$ is the upper $\alpha/2$ x x

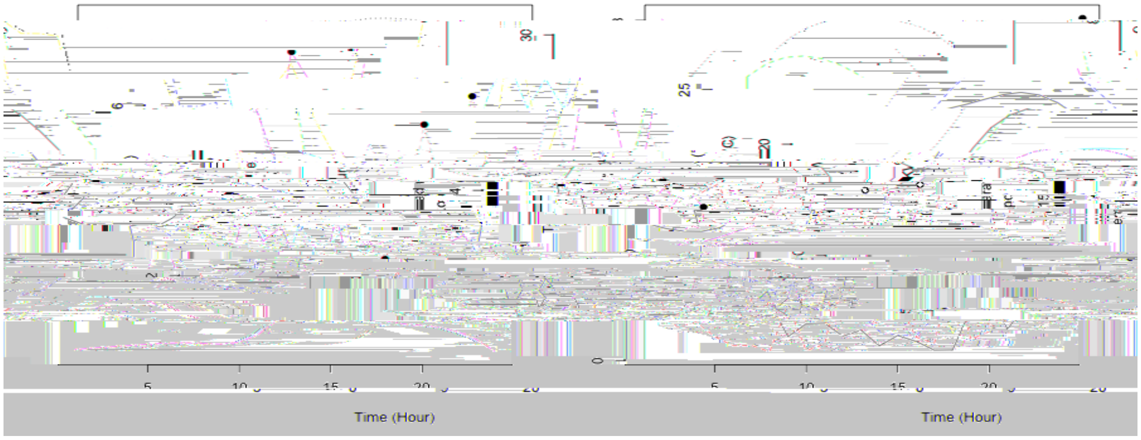

Figure 2: A sample of 10 daily temperature curves and the associated electricity demand curves. Observed daily peaks are in solid circle.

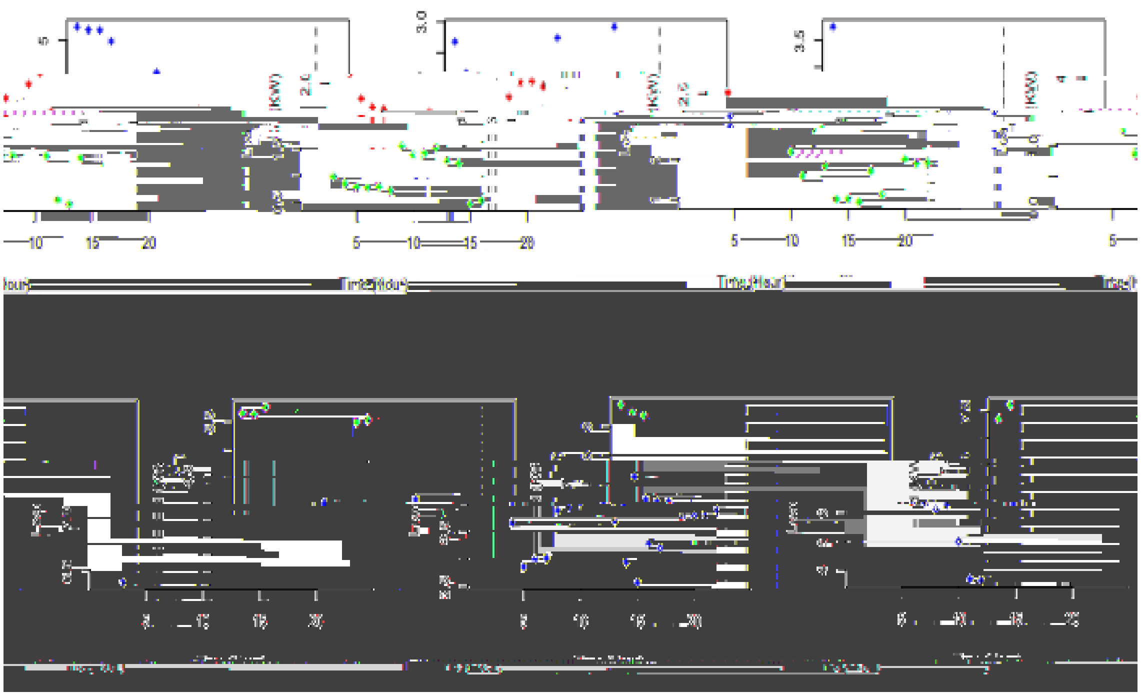

Figure 3: A sample of 6 censored daily load curves. Observed values of electricity consumption are plotted in star points, dashed line corresponds to the time of censorship for each day.

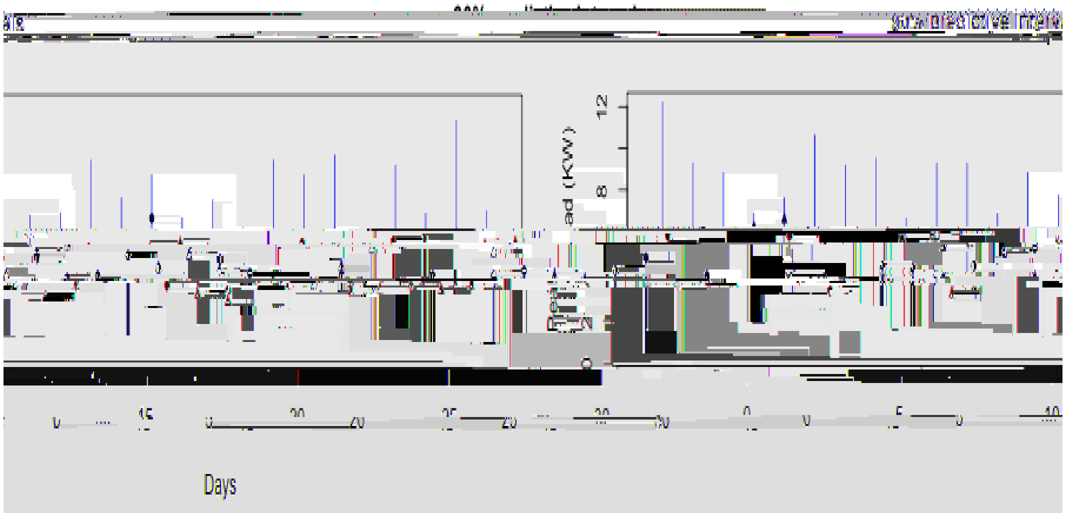

Figure 4: 90% predictive intervals of the peak demand for the last 30 days.

and $Y_d = P_d$ for completely observed days and $Y_d = C_d$ for censored ones. Here, we investigate, for each day $d = 971, \ldots, 1000$, the conditional quantile functions of Y_d given the predicted temperature curve X_d. The 5% and 95% quantiles consists of the 90% confidence intervals of the last 30 peak load in the testing sample, say $[q_{0.05}(X_d), q_{0.95}(X_d)]$ for $d = 971, \ldots, 1000.$ Note that these confidence intervals are derived directly from the conditional quantile functions given by (10). To estimate conditional quantiles we chose the quadratic kernel defined by $K(u) = 1.5(1 - u^2)\mathbb{1}_{[0,1]}$. Because the daily temperature curves are very smooth, we chosed as semi-metric $d(\cdot, \cdot)$ the L_2 distance between the sec ond derivative of the curves. Finally, we considered the optimal bandwidth $h := h_K = h_H$ chosen by the cross-validation method on the k-nearest neighbors (see Ferraty and Vieu (2006), p.102 for more details). Figure 4 provides our results for the peak load interval prediction for the testing sample. The true peaks are plotted in solid triangles. Solid circles represent the conditional median values. On can easily observe that the conditional median is a consistent predictor of the peak. In fact, let us define the Mean Absolute Prediction Error as

$$\text{MAPE} = \frac{1}{30} \sum_{d=1}^{30} \frac{|P_d - \hat{q}_{0.5}(X_d)|}{P_d},$$

where P_d is the true value of the peak for the day d and $\hat{q}_{0.5}(X_d)$ its predicted value based on the conditional median. We obtain here MAPE $= 0.24.$ Observe that we over-estimate the peak of the 16th day.

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5 Proofs of main results

In order to proof our results, we introduce some further notations. Let

$$\overline{\widetilde{F}}_n(x;t) = \frac{1}{n\mathbb{E}(\Delta_1(x))}\sum_{i=1}^{n} \mathbb{E}\left[\delta_i G^{-1}(Y_i)\, H(h_H^{-1}(t - Y_i))\, \Delta_i(x) \mid \mathcal{F}_{i-1}\right]$$

and

$$\bar{\ell}_n(x) = \frac{1}{n\mathbb{E}(\Delta_1(x))}\sum_{i=1}^{n} \mathbb{E}[\Delta_i(x) \mid \mathcal{F}_{i-1}].$$

Now, lets introduce the decomposition hereafter. For $x \in E$, set

$$\widehat{F}_n(t \mid x) - F(t \mid x) = \widehat{F}_n(t \mid x) - \widetilde{F}_n(t \mid x) + \widetilde{F}_n(t \mid x) - F(t \mid x). \tag{14}$$

To get the proof of Proposition 3.1, we establish the following Lemmas.

Definition 5.1 *A sequence of random variables $(Z_n)_{n\geq 1}$ is said to be a sequence of martingale differences with respect to the sequence of σ-fields $(\mathcal{F}_n)_{n\geq 1}$ whenever Z_n is $\mathcal{F}_n$ measurable and $\mathbb{E}(Z_n \mid \mathcal{F}_{n-1}) = 0$ almost surely.*

In this paper we need an exponential inequality for partial sums of unbounded martingale differences that we use to derive asymptotic results for the Nadaraya-Watson-type multivariate quantile regression function estimate built upon functional ergodic data. This inequality is given in the following lemma.

Lemma 5.2 *Let $(Z_n)_{n\geq 1}$ be a sequence of real martingale differences with respect to the sequence of σ-fields $(\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n))_{n\geq 1}$, where $\sigma(Z_1, \ldots, Z_n)$ is the σ-filed generated by the random variables $Z_1, \ldots, Z_n$. Set $S_n = \sum_{i=1}^{n} Z_i$. For any $p \geq 2$ and any $n \geq 1$, assume that there exist some nonnegative constants C and d_n such that*

$$\mathbb{E}(Z_n^p \mid \mathcal{F}_{n-1}) \leq C^{p-2} p!\, d_n^2 \quad \text{almost surely.} \tag{15}$$

Then, for any $\epsilon > 0$, we have

$$\mathbb{P}(|S_n| > \epsilon) \leq 2\exp\left\{-\frac{\epsilon^2}{2(D_n + C\epsilon)}\right\},$$

where $D_n = \sum_{i=1}^{n} d_i^2$.

As mentioned in Laib and Louani (2011) the proof of this lemma follows as a particular case of Theorem 8.2.2 due to de la Peña and Giné (1999).

We consider also the following technical lemma whose proof my be found in Laib and Louani (2010).

Lemma 5.3 *Assume that assumptions (A1) and (A2)(i), (A2)(ii) and (A2)(iv) hold true. For any real numbers $1 \le j \le 2+\delta$ and $1 \le k \le 2+\delta$ with $\delta > 0$, as $n \to \infty$, we have*

$$\text{(i)} \quad \frac{1}{\phi(h_K)} \mathbb{E}\left[\Delta_i^j(x) \mid \mathcal{F}_{i-1}\right] = M_j f_{i,1}(x) + O_{a.s.}\left(\frac{g_{i,x}(h_K)}{\phi(h_K)}\right);$$

$$\text{(ii)} \quad \frac{1}{\phi(h_K)} \mathbb{E}\left[\Delta_i^j(x)\right] = M_j f_1(x) + o(1),$$

$$\text{(iii)} \quad \frac{1}{\phi^k(h_K)} \left(\mathbb{E}(\Delta_1(x))\right)^k = M_1^k f_1^k(x) + o(1).$$

Lemma 5.4 *Assume that hypotheses (A1)-(A2) and the condition (12) are satisfied. Then, for any $x \in E$, we have*

$$\text{(i)} \quad \ell_n(x) - \bar{\ell}_n(x) = O_{a.s.}\left(\sqrt{\frac{\log n}{n\,\phi(h_K)}}\right),$$

$$\text{(ii)} \quad \lim_{n \to \infty} \ell_n(x) = \lim_{n \to \infty} \bar{\ell}_n(x) = 1 \quad a.s.$$

Proof. See the proof of Lemma 3 in Laib and Louani (2010). $\square$

Proof. *of Proposition 3.1*
Making use of the decomposition (14), the result follows as a direct consequence of Lemmas 5.5 and 5.6 below.

Lemma 5.5 *Under Assumptions (A1)-(A7) and the condition (12), we have*

$$\sup_{t \in S} \left| \widetilde{F}_n(t \mid x) - F(t \mid x) \right| = O_{a.s.}(h_K + h_H) + O_{a.s.}\left(\sqrt{\frac{\log n}{n\,\phi(h_K)}}\right).$$

Lemma 5.6 *Assume that hypothesis (A1)-(A7) and the condition (12) hold, we have*

$$\sup_{t \in S} \left| \widehat{F}_n(t \mid x) - \widetilde{F}_n(t \mid x) \right| = O_{a.s.}\left(\sqrt{\frac{\log \log n}{n}}\right).$$

$\square$

We provide, in the following lemma, the almost sure consistency, without rate, of $\widehat{\theta}_{n,\alpha}(x)$.

Lemma 5.7 *Under assumptions of Proposition 3.1, we have*

$$\lim_{n \to \infty} \left| \widehat{\theta}_{n,\alpha}(x) - q_\alpha(x) \right| = 0, \quad a.s.$$

Proof. of Lemma 5.7

Following the similar steps as in Ezzahrioui and Ould-Said (2008), the proof of this lemma is based in the following decomposition. As $F(\ \mid x)$ is a distribution function with a unique quantile of order , then for any $\ > 0$, let:

$$(\ ) = \min\{F(q\ (x) + \ \mid x) \quad F(q\ (x) \mid x); F(q\ (x) \mid x) \quad F(q\ (x) \quad \mid x)\};$$

then

$$\forall \ > 0; \forall t > 0; \quad |q\ (x) \quad t| \quad\ ) \ |F(q\ (x) \mid x) \quad F(t \mid x)| \quad (\ )$$

which is enough, while considering Proposition 3.1, to complete the proof of Theorem 3.2.

□

Proof. *of Theorem 3.3*
To proof our result we need to introduce the following decomposition

$$\widehat{F}_n(t \mid x) - F(t \mid x) := J_{1;n} + J_{2;n} + J_{3;n},$$

where $J_{1;n} := \widehat{F}_n(t \mid x) - \widetilde{F}_n(t \mid x)$, $J_{2;n} := \widetilde{F}_n(t \mid x) - \overline{\widetilde{F}}_n(t \mid x)$ and $J_{3;n} := \overline{\widetilde{F}}_n(t \mid x) - F(t \mid x)$. First, we establish that $J_{1;n}$ and $J_{3;n}$ are negligible, as $n \to \infty$, whereas $J_{2;n}$ is asymptotically normal. Observe that the term $J_{1;n} := \widehat{F}_n(t \mid x) - \widetilde{F}_n(t \mid x)$ has been studied in Lemma 5.6, then we have

$$J_{1;n} = O_{a.s.}\left(\sqrt{\frac{\log_2 n}{n}}\right). \tag{20}$$

On the other hand the term $J_{3;n} := \overline{\widetilde{F}}_n(t \mid x) - F(t \mid x)$ is equal to $B_n(x;t)$ which uniformly converges almost surely to zero (with rate $h_K + h_H$) by the Lemma 5.11 given in the Appendix. Then, we have

$$J_{3;n} = O_{a.s.}(h_K + h_H). \tag{21}$$

Now, let us consider the term $J_{2;n}$ which will provide us the asymptotic normality. For this end, we consider the following decomposition of the term $J_{2;n}$.

$$\begin{aligned} J_{2;n} &= \widetilde{F}_n(t \mid x) - \overline{\widetilde{F}}_n(t \mid x) \\ &:= \frac{Q_n(x;t) + R_n(x;t)}{\ell_n(x)}, \end{aligned} \tag{22}$$

where $Q_n(x;t) := [\widetilde{F}_n(x;t) - \overline{\widetilde{F}}_n(x;t)] - F(t \mid x)(\ell_n(x) - \bar{\ell}_n(x))$ and $R_n(x;t) := -B_n(x;t)(\ell_n(x) - \bar{\ell}_n(x))$, where $B_n(x;t) := \frac{\overline{\widetilde{F}}_n(x;t)}{\bar{\ell}_n(x)} - F(t \mid x)$. Using results of Lemma 5.11, we have, for any fixed $x \in E$, $B_n(x;t)$ and therefore $R_n(x;t)$ converge almost surely to zero when n goes to infinity. Thus, the asymptotic normality will be provided by the term $Q_n(x;t)$ which is treated by the Lemma 5.9 below.

Lemma 5.9 *Suppose that assumptions (A1)-(A3), (A6), (A8)-(A9]TJ/F16a65(th)-1(-dd((A)93(th50A1ce)-38ndi*

Define the "*pseudo-conditional bias*" of the conditional distribution function estimate of Y_i given $X = x$ as

$$B_n(x;t) = \frac{\bar{\widehat{F}}_n(x;t)}{\bar{\ell}_n(x)} - F(t \mid x).$$

Consider now the following quantites

$$R_n(x;t) = -B_n(x;t)(\widehat{\ell}_n(x) - \bar{\ell}_n(x)),$$

and

$$Q_n(x;t) = (\widehat{F}_n(x;t) - \bar{\widehat{F}}_n(x;t)) - F(t \mid x)(\widehat{\ell}_n(x) - \bar{\ell}_n(x)).$$

It is then clear that the following decomposition holds

$$\widehat{F}_n(t \mid x) - F(t \mid x) = B_n(x;t) + \frac{R_n(x;t) + Q_n(x;t)}{\widehat{\ell}_n(x)}. \tag{25}$$

Remark 5.10 *Using statement (29) and Lemma 5.4, one can easily get, for all $x \in E$,*

$$\sup_{t \in S} |Q_n(x;t)| = O_{a.s.}\left(\sqrt{\frac{\log n}{n\,\phi(h_K)}}\right).$$

Finally, the combination of results given in Lemma 5.11 and Remark 5.10 achieves the proof of Lemma 5.5. □

$_n(x;t$ 5.10 This is of stAf 4(1)129 $_n(x;t$

fact that $\mathbb{1}_{\{T_i \; C_i\}}\varphi(Y_i) = \mathbb{1}_{\{T_i \; C_i\}}\varphi(T_i)$, we get

$$\begin{aligned}
\overline{\widetilde{F}}_n(x;t) &= \frac{1}{n\mathbb{E}(\;_1(x))}\sum_{i=1}^{n}\mathbb{E}\;\;_i(x)\mathbb{E}\;\;_i G^{\;1}(Y_i)\,H(h_H^{\;1}(t\;\;Y_i))\,|\,\mathcal{G}_{i\;1};T_i\;\,|\,\mathcal{F}_{i\;1}\\
&= \frac{1}{n\mathbb{E}(\;_1(x))}\sum_{i=1}^{n}\mathbb{E}\;\;_i(x)\mathbb{E}\;\;_i G^{\;1}(Y_i)\,H(h_H^{\;1}(t\;\;Y_i))\,|\,X_i;T_i\;\,|\,\mathcal{F}_{i\;1}\\
&= \frac{1}{n\mathbb{E}(\;_1(x))}\sum_{i=1}^{n}\mathbb{E}\;G^{\;1}(T_i)\,H(h_H^{\;1}(t\;\;T_i))\;\;_i(x)\mathbb{E}\;\mathbb{1}_{\{T_i\;C_i\}}\,|\,X_i;T_i\;\,|\,\mathcal{F}_{i\;1}\\
&= \frac{1}{n\mathbb{E}(\;_1(x))}\sum_{i=1}^{n}\mathbb{E}\;\;_i(x)\,H(h_H^{\;1}(t\;\;T_i))\,|\,\mathcal{F}_{i\;1}
\end{aligned}$$

Then, by a double conditioning with respect to $\mathcal{G}_{i\;1}$, we have

$$\overline{\widetilde{F}}_n(x;t)\;\;\bar{\;}_n(x)F(t\,|\,x) = \frac{1}{n\mathbb{E}(\;_1(x))}\sum_{i=1}^{n}\mathbb{E}\;\;_i(x)[\mathbb{E}(H(h_H^{\;1}(t\;\;T_i))\,|\,X_i)\;\;F(t\,|\,x)]\,|\,\mathcal{F}_{i\;1}$$

Now, because of conditions (A3) and (A5), we get

$$\mathbb{E}(H(h_H^{\;1}(t\;\;T_i))\,|\,X_i)\;\;F(t\,|\,x)\;\;C_x\int$$

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Proof. *of Lemma 5.12*

Observe that

$$\widetilde{F}_n(x;t) - \overline{\widetilde{F}}_n(x;t) = \frac{1}{n\mathbb{E}(\Delta_1(x))}\sum_{i=1}^{n} L_{i;n}(x;t);$$

where $L_{i;n}(x;t) = \delta_i G^{-1}(Y_i)H(h_H^{-1}(t-Y_i))\Delta_i(x) - \mathbb{E}\left[\delta_i G^{-1}(Y_i)H(h_H^{-1}(t-Y_i))\Delta_i(x) \mid \mathcal{F}_{i-1}\right]$ is a martingale difference. Therefore, we can use Lemma 5.2 to obtain an exponential upper bound relative to the quantity $\widetilde{F}_n(x;t) - \overline{\widetilde{F}}_n(x;t)$. Let us now check the conditions under which one can obtain the mentioned exponential upper bound. In this respect, for any $p \in \mathbb{N} \setminus \{0\}$, observe that

$$L^p_{n;i}(x;t) = \sum_{k=0}^{p} C_p^k \left(\frac{\delta_i}{G(Y_i)} H(h_H^{-1}(t-Y_i))\Delta_i(x)\right)^k (-1)^{p-k}\left(\mathbb{E}\left[\frac{\delta_i}{G(Y_i)} H(h_H^{-1}(t-Y_i))\Delta_i(x) \mid \mathcal{F}_{i-1}\right]\right)^{p-k}$$

In view of condition (A4), $\left(\mathbb{E}\left[\delta_i G^{-1}(Y_i)\, H(h_H^{-1}(t-Y_i))\Delta_i(x) \mid \mathcal{F}_{i-1}\right]\right)^{p-k}$ is $\mathcal{F}_{i-1}$-measurable, it follows then that

$$\mathbb{E}(L^p_{i;n}(x;t) \mid \mathcal{F}_{i-1}) = \sum_{k=0}^{p} C_p^k \mathbb{E}\left[\left(\delta_i G^{-1}(Y_i)\, H(h_H^{-1}(t-Y_i))\Delta_i(x)\right)^k \mid \mathcal{F}_{i-1}\right](-1)^{p-k}$$
$$\times\left(\mathbb{E}\left[\delta_i G^{-1}(Y_i)\, H(h_H^{-1}(t-Y_i))\Delta_i(x) \mid \mathcal{F}_{i-1}\right]\right)^{p-k}.$$

Thus,

$$|\mathbb{E}(L^p_{i;n}(x;t) \mid \mathcal{F}_{i-1})| \le \sum_{k=0}^{p} C_p^k \mathbb{E}\left[\left|\delta_i G^{-1}(Y_i)\, H(h_H^{-1}(t-Y_i))\Delta_i(x)\right|^k \mid \mathcal{F}_{i-1}\right]$$
$$\times\left|\mathbb{E}\left[\delta_i G^{-1}(Y_i)\, H(h_H^{-1}(t-Y_i))\Delta_i(x) \mid \mathcal{F}_{i-1}\right]\right|^{p-k}.$$

Making use of Jensen inequality, one can write

$$\mathbb{E}\left[\left|\delta_i G^{-1}(Y_i)\, H(h_H^{-1}(t-Y_i))\Delta_i(x)\right|^k \mid \mathcal{F}_{i-1}\right]\left|\mathbb{E}\left[\delta_i G^{-1}(Y_i)\, H(h_H^{-1}(t-Y_i))\Delta_i(x) \mid \mathcal{F}_{i-1}\right]\right|^{p-k}$$
$$\le \mathbb{E}\left[\left|\delta_i G^{-1}(Y_i)\, H(h_H^{-1}(t-Y_i))\Delta_i(x)\right|^k \mid \mathcal{F}_{i-1}\right]\mathbb{E}\left[\left|\delta_i G^{-1}(Y_i)\, H(h_H^{-1}(t-Y_i))\Delta_i(x)\right|^{p-k} \mid \mathcal{F}_{i-1}\right]$$

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In view of assumption (A6), we have

$$\mathrm{E} \quad \frac{i}{G(Y_i)} H(h_H^{1}(t \quad Y_i)) \quad {}_i(x) \quad ^{m} j \, F_{i \quad 1} \qquad 1$$

where C_1 is a positive constant. Therefore, choosing η_0 large enough, we obtain

$$\sum_{n \geq 1} P\left(\widetilde{F}_n(x;t) - \overline{\widetilde{F}}_n(x;t) > \eta_0 \sqrt{\frac{\log n}{n\,\phi(h_K)}} \right) < \infty .$$

Finally, we achieve the proof by Borel-Cantelli Lemma. $\square$

Proof. *of Lemma 5.6*
From (8) and (9) we have

$$\widehat{F}_n(t \mid x) - \widetilde{F}_n(t \mid x) \leq \frac{1}{n\mathbb{E}[\Delta_1(x)]\,\ell_n(x)} \sum_{i=1}^{n} \delta_i \Delta_i(x)\, H(h_H^{-1}(t - Y_i)) \left| \frac{1}{G(Y_i)} - \frac{1}{G_n(Y_i)} \right|$$

$$\leq \frac{\sup_{t \in S} |G_n(t) - G(t)|}{}$$

770]TJ 0 -6SinceJTJ/F56 10.9091 Tf 4.1 20[([( )]TJ/F15 10.9091 Tf 12.28 2.758 Td [( )]TJ/F56 10.9091 T

$$\frac{}{G(t)}\int_{\mathbb{R}}$$

)8TJ/F56 10.9091 Tf 5.081 0 Td [()8TJ/F56 10.9091 Tf 9.626

Let us now examine the term K_{n1},

$$
\begin{aligned}
K_{n1} &= \mathrm{E}\ \frac{{}_i}{G^2(Y_i)} H^2\ \frac{t \quad Y_i}{h_H}\ \ j\, X_i \qquad \mathrm{E}\ \frac{{}_i}{G(Y_i)} H\ \frac{t \quad Y_i}{h_H}\ \ j\, X_i \quad {}^2 \\
&= I_1 + I_2.
\end{aligned}
$$

The first term of the last equality can be developed as follow,

$$
\begin{aligned}
I_1 &= \mathrm{E}\ \ H^2\ \frac{t \quad Y_i}{h_H}\ \ \frac{1}{G(Y_i)}\, j\, X_i \\
&= \int_{\mathbb{R}} H^2\ \frac{t \quad z}{h_H}\ \ \frac{1}{G(z)} f(z\ j\ X_i)dz \\
&= \int_{\mathbb{R}} H^2(v)\frac{1}{G(t \quad h_H v)} dF(t \quad h_H v\ j\ X_i).
\end{aligned}
$$

By the first order Taylor expansion of the function $G\ {}^1(\ )$ around zero one gets

$$
\begin{aligned}
I_1 &:= \int_{\mathbb{R}} H^2(v)\frac{1}{G(t)} dF(t \quad h_H v\ j\ X_i) + \frac{h_H}{G^2(t)}\int_{\mathbb{R}} vH(v)G^{(1)}(t^{?})dF(t \quad hv\ j\ X_i) + o(1) \\
&=: I_1^{0} + I_2^{0};
\end{aligned}
$$

where $t^{?}$ is between t and $t \quad h_H v.$

Under assumption (A9), we have $I_2^{0} \quad h_H^2 \frac{\sup_{u\,2\,\mathbb{R}} jG^{(1)}(u)j}{G^2(t)} \int_{\mathbb{R}} vf(t \quad h_H v\ j\ X_i)dv$. Then, using assumption (A3), we get $I_2^{0} = O(h_H^2).$

On the other hand, by integrating by part we have

$$I_1^{0} = \frac{1}{G(t)}\int_{\mathbb{R}} 2H^{0}(v)H(v)F\mathrm{t}\ h_H v v^{01} =$$

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