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Approximation by harmonic polynomials in star-shaped domains and exponential convergence of Trefftz *HP*-DGFEM

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APPROXIMATION BY HARMONIC POLYNOMIALS IN STAR-SHAPED DOMAINS AND EXPONENTIAL CONVERGENCE OF TREFFTZ HP-DGFEM*

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Abstract. We study the approximation of harmonic functions by means of harmonic polynomials in two-dimensional, bounded, star-shaped domains. Assuming that the functions possess analytic extensions to a -neighbourhood of the domain, we prove exponential convergence of the approximation error with respect to the degree of the approximating harmonic polynomial. All the constants appearing in the bounds are explicit and depend only on the shape-regularity of the domain and on .

We apply the obtained estimates to show exponential convergence with rate $(\exp(-b\sqrt{N}))$, N being the number of degrees of freedom and b 0, of a p-dGFEM discretisation of the Laplace equation based on piecewise harmonic polynomials. This result is an improvement over the classical rate $(\exp(-b\sqrt[3]{N}))$, and is due to the use of *harmonic polynomial spaces*, as opposed to complete polynomial spaces.

1. Introduction. We fix a domain that meets the following requirements, see Figure 1.1.

ASSUMPTION 1.1. *The domain* D $\mathbb{C}$ *is open and satisfies* gleaned in Section 3 by means of fairly intricate estimates. A result similar to Theorem 1.2 was stated in [25, Theorem 2.2.10]; the novelty of the present contribution lies in the *explicit expressions* for the constants $\mathbf{C}$ and $\mathbf{b}$ in terms of the parameters $\boldsymbol{\delta}$, $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_0$ only.

7(u)0.433137((s)-0.355331(e)0.443552.)-0.235254]TJ)0.3.412(o)10.122(o)o

such that, for any $\mathbf{w}$ $\partial \mathbf{D}$,

a) there exists a cone[3] with vertex $\mathbf{w}$, opening angle $\Lambda\boldsymbol{\pi}$ and height $\mathbf{H}_0$ contained in $\overline{\mathbf{D}}$,

b) there exists an infinite cone with vertex $\mathbf{w}$ and opening angle $\boldsymbol{\lambda\pi}$ contained in $\mathbb{C} \setminus \mathbf{D}$.

The proof is postponed to Lemma A.1 in Appendix A. The uniform cone conditions imply that $\mathbf{D}$ is Lipschitz (see, e.g., [16, Theorem 1.2.2.2]).

REMARK 1.3. *If $\mathbf{D}$ is convex, we could choose $\boldsymbol{\rho}_0 = \boldsymbol{\rho}$. However, in order to avoid the discussion of special cases, we will always assume $\boldsymbol{\rho}_0 < \boldsymbol{\rho}$, obviously with no loss of generality.*

We also notice that, in the convex case, the exterior cone condition holds with $\boldsymbol{\lambda} = 1$ (the cone is a half plane through $\mathbf{w}$ that does not intersect $\mathbf{D}$), while for the interior cone condition one ala4.4nta4.4one ala4.4nta4tia4-0.239f 5d69893(h).2401(i)-0.240155(o)0.431299(r)-336.398(c)0.44 1336.396(c)0.44

3. Distance estimates for level lines of ϕ_D. We need precise quantitative information of how far the level lines L_h move away from ∂D as h increases. It is provided by the following key result.

THEOREM 3.1. *Let L_h be the h-level line of the conformal mapping of D. Define* $0 < \xi \quad 1$ *as*

$$\xi := \begin{cases} \frac{2}{\pi} \arcsin \quad \rho \\ \end{cases}$$

the minimum is $\mathbf{h}^2$

Then $\psi'(0) \quad \frac{(1-\rho)}{\rho}$ and the proof is complete. □

The inverse of Ψ is given by $\Psi^{-1}(re^{i\theta}) = \frac{1}{\psi(\theta)}re^{i\theta}$ or, in Cartesian coordinates (after the identification of $\mathbb{C}$ with $\mathbb{R}^2$),

$$\Psi^{-1}(r\cos\theta, r\sin\theta) = \quad \frac{r}{\psi(\theta)}\cos\theta, \frac{r}{\psi(\theta)}\sin\theta \quad =: (F_1, F_2). \tag{4.1}$$

Of course, Ψ^{-1} is Lipschitz continuous as well, and an estimate for its Lipschitz constant is given in the next Lemma.

LEMMA 4.2. *The function* $\Psi^{-1} : \mathbb{C} \quad \mathbb{C}$

where

$$C_D = 4\sqrt{\pi}\ \sqrt{2} L_\psi L_{-} \,,$$

with L_ψ *and* L_{-} *as in Lemma 4.1 and Lemma 4.2, respectively.*

Proof. Fix $w_0 \quad L_h$, and assume, with no loss of generality, that w_0 is on the positive real axis. Define $d := w_0 - \psi(0)$ and notice that $d(w_0, \partial D) \quad d \quad 1$.

Setting $w(\theta) := \Psi(e^{i\theta}) = \psi(\theta)e^{i\theta} \quad \partial D$, using Lemma 4.2, Lemma 4.3 and $\psi(\theta) < 1$, we obtain, for all $\theta \quad [-\pi, \pi]$,

$$
\begin{aligned}
w(\theta) - w_0 {}^2 \quad & L_{-}^{-2} \quad \Psi^{-1}(w(\theta)) - \Psi^{-1}(w_0) {}^2 = L_{-}^{-2} \quad e^{i\theta} - w_0/\psi(0) {}^2 \\
& L_{-}^{-2} \; C_B^2 \;\; \theta^2 + \quad \frac{w_0}{\psi(0)} - 1 \quad {}^2 \quad = L_{-}^{-2} \; C_B^2 \;\; \theta^2 + \quad \frac{w_0 - \psi(0)}{\psi(0)} \quad {}^2 \\
> \; & L_{-}^{-2} \; C_B^2 \;\; \theta^2 + (w_0 - \psi(0))^2 \quad = \frac{4}{\pi^2} L_{-}^{-2} \; (\theta^2 + d^2) =: L_D^2 (\theta^2 + d^2).
\end{aligned}
\tag{4.2}
$$

Then,

$$
\begin{aligned}
\int_{\partial D} \frac{1}{w - w_0} \, dw \; &= \int_{-\pi}^{\pi} \frac{1}{w(\theta) - w_0} \; w'(\theta) \; d\theta \overset{\text{Lem. 4.1}}{\quad} L_\psi \int_{-\pi}^{\pi} \frac{1}{w(\theta) - w_0} \, d\theta \\
&\overset{(4.2)}{\quad} L_\psi L_D^{-1} \int_{-\pi}^{\pi} \frac{1}{\sqrt{\theta^2 + d^2}} \, d\theta \quad \sqrt{2} \quad 2 L_\psi L_D^{-1} \int_0^{\pi} \frac{1}{\theta + d} \, d\theta \\
&\quad \sqrt{2} \quad 2 L_\psi L_D^{-1} \big(\log(\pi + d) - \log
\end{aligned}
$$

Define the sequence of complex polynomials $\omega_p \;\; {}_{p\in\mathbb{N}}$ with

$$\omega_p(w) := \prod_{k=0}^{p-1} \big(w - \phi(e^{2\pi ik/p})\big),$$

where ϕ is the exterior conformal mapp(p)0.433749(52264 Tf 263269k2.49((52264 Tf 263269k2.49((51.376(m)0.1935d((p0.917

Since ϕ is a curve parametrisation $\phi : \partial B_{1+3h} \quad L_{3h}$,

$$\operatorname{length}(L_{3h}) \quad 2\pi(1+3h) \sup_{|z|=1+3h} \phi'(z) \ ;$$

this, together with the lower bound of $d(L_h, L_{3h})$ and the upper bound of $\phi'(z)$ given in Lemma 2.1, and the bounds in Lemma 4.6, gives

$$\begin{aligned} \| f - q_p \|_{L^\infty(\operatorname{Int} L_h)} \quad & \frac{8(1+3h)^5 \phi'(\quad)^2}{6h^3\rho^2} (3h^2)^{-C_D} \quad \frac{1+h}{1+3h}^{\ p} \| f \|_{L^\infty(\operatorname{Int} L \quad h)} \\ & \frac{4\phi'(\quad)^2}{3^{1+C_D}\rho^2} h^{-3-2C_D} \quad \frac{1+h}{1+3h}^{\ p} (1+3h)^5 \| f \|_{L^\infty(\operatorname{Int} L \quad h)} \\ & \frac{20(1-\rho)^2}{3\rho^2} h^{-3-2C_D} \quad \frac{1}{1+h}^{\ p} \| f \|_{L^\infty(\operatorname{Int} L \quad h)} , \end{aligned}$$

where in the last step we have used $3^{1+C_D} > 3$, $\phi'(\quad) < 1-\rho$, $\frac{1+h}{1+3h} \quad \frac{1}{1+h}$, and $(1+3h)^5 < 5$, since $h \quad \rho/4 \cdot \quad h < 1/8$. The use of Lemma 4.6 (and thus of Lemma 4.4) is legitimate due to the hypothesis imposed on h and δ. The result of the theorem follows from the bound of C_D derived in Remark 4.5. □

Obviously, Theorem 1.2 from the Introduction is an immediate consequence of Theorem 4.7: given $0 < h < h^*$, just define $C := C_{\mathrm{appr}}(h^*(\delta))^{-\in\in} \quad \mathcal{T}$

$$\|u\|_{W^{1,\infty}(S)} := \|u\|_{L^\infty(S)} + \mathrm{diam}(D_\delta)\,\|\nabla u\|_{L^\infty(S)}.$$

THEOREM 4.10. *Fix $0 < \delta \le 1/2$, and let h satisfy* (4.4)*. For any real, harmonic function u in the inflated domain D_δ defined in* (4.3)*, there is a sequence of harmonic polynomials $\{Q_p\}_{p\geq 1}$ of degree at most p such that*

$$\|u - Q_p\|_{L^\infty(D)} \le C_{\mathrm{appr}} h^{-\alpha}(1+h)^{-p}\|u\|_{W^{1,\infty}(\mathrm{Int}\, L_h)},$$

$$|u - Q_p|_{W^{j,\infty}(D)} \le C_{\mathrm{appr}} \left(\frac{2j}{C_I h^2}\right)^j h^{-\alpha}(1+h)^{-p}\|u\|_{W^{1,\infty}(\mathrm{Int}\, L_h)},$$

$\|u - Q$ [illegible]

and the previous inequalities. ∎

From Theorem 1.2, with the same considerations as in the proof of Theorem 4.10, we obtain the following result.

COROLLARY 4.11. *Fix $0 < \delta \le 1/2$ and $j \in \mathbb{N}_0$. There exist $C > 0$ and $b > 0$,* depending only on ρ, ρ_0, δ and j, *such that, for any real-valued, harmonic function u which is bounded along with its first-order derivatives in the inflated domain D_δ defined in (4.3), there is a sequence of harmonic polynomials $\{Q_p\}_p$ of degree at most p such that*

$$\|u - Q_p\|_{W^{j,\infty}(D)} \le C e^{-bp} \|u\|_{W^{1,\infty}(D_\delta)},$$
$$\|u - Q_p\|_{H^{j}(D)} \le C e^{-bp} \|u\|_{W^{1,\infty}(D_\delta)}.$$

REMARK 4.12. *The constants C and b in Theorem 1.2 and Corollary 4.11 depend on δ only through $h^*(\delta)$ defined in (4.4).*

The boundedness of f, u and ∇u in Theorem 1.2 and Corollary 4.11 is assumed only in order to write estimates with L^∞-norms in the whole D_δ on the right-had side. Actually, the estimates hold true also with $\|f\|_{L^\infty(\mathrm{Int}\, L_h)}$ and $\|u\|_{W^{1,\infty}(\mathrm{Int}\, L_h)}$ respectively, on the right-hand side, for any $0 < h < h^*$, with no need of assuming boundedness of f, u and ∇u in D_δ.

REMARK 4.13. *The interpolating polynomials q_p (and Q_p) in Theorem 1.2, Theorem 4.7 and Corollary 4.9 (Theorem 4.10 and Corollary 4.11, respectively) interpolate exactly the function f (u, respectively) in at least $p + 1$ points lying on the boundary of D. The exact location of the points depend on the conformal map ϕ_D. This fact follows from the definition of q_p given in the proof of Theorem 4.7 and the relations $u = \mathrm{Re}\, f$ and $Q_p = \mathrm{Re}\, q_p$.*

5. Application: exponential convergence of Trefftz hp-dGFEM. In this section, we outline how to apply the estimates of Corollary 4.11 and prove exponential convergence of a *Trefftz hp-dGFEM* for the mixed Laplace boundary value problem (BVP), i.e. a FEM with discontinuous, piecewise harmonic, polynomial basis functions on a geometrically graded mesh. We establish exponential convergence with rate $O(\exp(-b\sqrt{N}))$, for some $b > 0$, in terms of the overall number N of degrees of freedom. This result is an improvement over the classical rate $O(\exp(-b\sqrt[3]{N}))$ shown for inhomogeneous problems in [2, 4]; this improvement is due to the use of harmonic polynomials, instead of complete polynomials, in the trial spaces.

Since we rely on the hp-dGFEM theory from [37], we restrict ourselves to the case of (straight) polygonal domains and meshes comprising (straight) triangles or parallelograms. The extension to curvilinear domains and mesh elements would require to develop, for such elements, several tools as polynomial hp-inverse estimates, scaling estimates of Sobolev seminorms, and approximation estimates for linear and bilinear polynomials near corners. This goes beyond the scope of this paper.

5.1. The Laplace BVP. Without further explanation, we use the notation for the weighted Sobolev spaces ($H^{m,l}_{\underline{\beta}}(\Omega)$) and the countably normed spaces ($\mathcal{B}^l_{\underline{\beta}}(\Omega)$ and $\mathcal{C}^l_{\underline{\beta}}(\Omega)$) from [2, §2], along with the analyticity and analytic continuation results given in [2–5].

Let $\Omega \subset \mathbb{R}^2$ be a bounded, Lipschitz polygon with corners c_ν, $1 \le \nu \le n_a$, whose boundary is partitioned into a Dirichlet and a Neumann boundary $\Gamma^{[0]}$ and $\Gamma^{[1]}$, respectively, such that the interiors of $\Gamma^{[0]}$ and $\Gamma^{[1]}$ do not overlap and $\overline{\Gamma^{[0]}} \cup \overline{\Gamma^{[1]}} = \partial\Omega$. Moreover, we assume that $\Gamma^{[0]}$ has positive 1-dimensional measure. Consider the following (well-posed) boundary value problem: given $g^{[i]}$, $i = 0, 1$, find $u \in H^1(\Omega)$

$$\gamma_0 \bar{u} = g^{[0]} \quad \text{on } \Gamma^{[0]}, \quad \gamma_1 \bar{u} = g^{[1]} \quad \text{on } \Gamma^{[1]}. \tag{5.1b}$$

Here, γ_0 and γ_1 denote trac[illegible]rmal derivative operators, respectively.

There exists a weight vect[illegible]$(0,1)^{n_a}$ such that, if $g^{[i]} \in \mathcal{B}^{-i}_{\beta}(\Gamma^{[i]})$, $i = 0, 1$, problem (5.1) admits a unique solutio[illegible]ch belongs to $\mathcal{C}^2_{\beta}(\Omega)$, [2, Theorem 3.5]. Moreover, as in [2, page 841], it can be prove[illegible]ere exist two constants $C_u > 0$ and $d_u \geq 1$ such that

$$(D^\alpha u)(x_0) \leq C_u \frac{d_u}{\Phi(x_0)} [illegible] \; x_0 \in \Omega, \; \alpha \in \mathbb{N}_0^2, \; |\alpha| = k \geq 1, \tag{5.2}$$

where $\Phi(x_0) := \prod_{\nu=1}^{n_a} \min\{1, |x_0 - c_\nu|\}$ [illegible] admits a real analytic continuation to the set

$$\mathcal{N}(u) := \bigcup_{x_0 \in \bar{\Omega} \setminus \bigcup_{\nu}^{n_a} c_\nu} \{ x [illegible] - x_0| < \frac{\Phi(x_0)}{2 d_u} \} \subset \mathbb{R}^2. \tag{5.3}$$

5.2. Trefftz hp-dGFEM. We now for[illegible]e hp-dGFEM discretisation of the BVP (5.1) on *geometric mesh families* $M_\sigma = \{\mathcal{T}$[illegible]n Ω, with increasing number ℓ of layers and geometric grading factor $0 < \sigma < 1$.

5.2.1. Geometric meshes. Given ℓ [illegible] mesh $\mathcal{T}^\ell_\sigma$ is a partition of the domain Ω into open triangles or parallelograms Ω^ℓ_{ij} [illegible]at $\overline{\Omega} = \bigcup_{i,j} \overline{\Omega^\ell_{ij}}$ and $\Omega^\ell_{ij} \cap \Omega^\ell_{i'j'} = \emptyset$ if $(i,j) \neq (i',j')$). The elements are groupe[illegible]ers, denoted by $\mathcal{L}^\ell_{\sigma,i}$, $1 \leq i \leq \ell$, such that

$$\mathcal{T}^\ell_\sigma = [illegible]$$

(GM3) *The size of an element* $\Omega^{\ell}_{\mathbf{ij}}$ *depends geometrically on its layer index* $\mathbf{i}$*:* $0 < \kappa_{3-} \quad \kappa_{3+} <$, *independent of* σ, ℓ, $\mathbf{i}$ *and* $\mathbf{j}$, *such that for all* $\mathcal{T}^{\ell}_{\sigma} \quad \mathbf{M}_{\sigma}$ *and* $\Omega^{\ell}_{\mathbf{ij}} \quad \mathcal{T}^{\ell}_{\sigma}$,

$$\kappa_{3-}\,\sigma^{\mathbf{i}} \quad h^{\ell}_{\mathbf{ij}} \quad \kappa_{3+}\,\sigma^{\mathbf{i}}.$$

(GM4) *For* $\ell \quad 2$, $\mathcal{T}^{\ell}_{\sigma}$ *is obtained from* $\mathcal{T}^{\ell-1}_{\sigma}$ *by only refining the elements in the layer* $\mathcal{L}^{\ell-1}_{\sigma,\ell-1}$ *adjacent to the domain corners, forming two new layers* $\mathcal{L}^{\ell}_{\sigma,\ell-1}$ *and* $\mathcal{L}^{\ell}_{\sigma,\ell}$*. Equivalently, the elements of* $\mathcal{L}^{\ell}_{\sigma,\mathbf{i}}$ *are uniquely defined for all* $\ell \quad \mathbf{i}+1$*:*

$$\mathcal{L}^{\ell}_{\sigma,\mathbf{i}} = \mathcal{L}^{\ell'}_{\sigma,\mathbf{i}} \quad \mathbf{i} \quad 1, 2, \ldots, \min(\ell,\ell') - 1 \;; \qquad \mathcal{L}^{\ell}_{\sigma,\ell} = \sum_{\mathbf{i}=\ell}^{\ell'} \mathcal{L}^{\ell'}_{\sigma,\mathbf{i}} \quad \ell' > \ell \quad 1. \tag{5.4}$$

Note that (GM2) and (GM3) imply that the diameter of an element $\Omega^{\ell}_{\mathbf{ij}}$ is proportional to its distance from the domain corners:

$$\frac{\kappa_{3-}}{\kappa_{2+}}\, r^{\ell}_{\mathbf{ij}} \quad h^{\ell}_{\mathbf{ij}} \quad \frac{\kappa_{3+}}{\kappa_{2-}}\, r^{\ell}_{\mathbf{ij}} \qquad 1 \quad \mathbf{i} < \ell,\ 1 \quad \mathbf{j} \quad \mathbf{J}(\ell). \tag{5.5}$$

Using (GM1) and (GM3), we can control the area $\Omega^{\ell}_{\mathbf{ij}}$ of each element: for all $\Omega^{\ell}_{\mathbf{ij}} \quad \mathcal{T}^{\ell}_{\sigma}$, $\ell \quad \mathbb{N}$,

$$(h^{\ell}_{\mathbf{ij}})^2 \quad \Omega^{\ell}_{\mathbf{ij}} \quad \mathbf{B}_{\rho^{\ell}_{\mathbf{ij}}}(x^{\ell}_{\mathbf{ij}} \;\; \sigma$$

PROPOSITION 5.3. *[37, Theorem 2.3.7, Corollary 2.4.2] Let* $\underline{\boldsymbol{\beta}} \quad (0,1)^{n_a}$ *be such that the analytical solution* $\mathbf{u}$ *to* (5.1) *belongs to* $\mathcal{C}^2_{\underline{\boldsymbol{\beta}}}(\Omega)$*. If either* $\boldsymbol{\theta} = 1$ *and* $\boldsymbol{\alpha}$ *is positive, or* $\boldsymbol{\theta} = -1$ *and* $\boldsymbol{\alpha}$ *is sufficiently large, then the* $\mathbf{hp}$*-dGFEM* (5.9) *admits a unique solution.*

Moreover, let $\boldsymbol{\pi}_{\mathcal{T}} : \mathbf{H}^{2,2}_{\underline{\boldsymbol{\beta}}}(\Omega) \quad \mathbf{V}_{\mathbf{p}}(\mathcal{T}^{\square}_{\boldsymbol{\sigma}})$ *be an arbitrary operator such that, for every element* $\mathbf{K} \quad \mathcal{T}^{\square}_{\boldsymbol{\sigma}}$*, there exist at least two zeros of* $\boldsymbol{\eta} := \mathbf{u} - \boldsymbol{\pi}_{\mathcal{T}}\mathbf{u}$ *in* $\overline{\mathbf{K}}$*. For* $\boldsymbol{\theta} = \pm 1$ *(with sufficiently large* $\boldsymbol{\alpha}$*, if* $\boldsymbol{\theta} = -1$*), it holds*

$$\left\| \mathbf{u} - \mathbf{u}^{\boldsymbol{\theta}}_{\mathbf{h}} \right\|^2_{\mathbf{dG}} \qquad \mathbf{C}\,\mathbf{p}^2 \sum_{\mathbf{K} \in \mathcal{T}^{\square}_{\boldsymbol{\sigma}}} \boldsymbol{\eta}\ {}^2_{\mathbf{H}}\ (\mathbf{K}) + \sum_{\mathbf{K} \in \mathcal{T}^{\square}_{\boldsymbol{\sigma}} \setminus \mathcal{K}^{\square}_{\boldsymbol{\sigma}}} \mathbf{h}^2_{\mathbf{K}}\ \boldsymbol{\eta}\ {}^2_{\mathbf{H}}\ (\mathbf{K}) + \sum_{\mathbf{K} \in \mathcal{K}^{\square}_{\boldsymbol{\sigma}}} \mathbf{h}^{2(1-\boldsymbol{\beta}\ \underline{\mathbf{K}}\ )}_{\mathbf{K}}\ \boldsymbol{\eta}\ {}^2_{\mathbf{H}_{\underline{\boldsymbol{\beta}}}'}\ (\mathbf{K}) \tag{5.10}$$

where $\mathbf{C} > 0$ *is independent of* $\boldsymbol{\sigma}$*,* $\square$ *and* $\mathbf{p}$*. Here,* $\mathcal{K}^{\square}_{\boldsymbol{\sigma}} := \mathcal{L}^{\square}_{\boldsymbol{\sigma},\square} \quad \mathcal{T}^{\square}_{\boldsymbol{\sigma}}$ *designates the set of elements abutting at domain corners and, for any* $\mathbf{K} \quad \mathcal{K}^{\square}_{\boldsymbol{\sigma}}$*,* $\boldsymbol{\beta}_{[\mathbf{K}]} := \sup\ \boldsymbol{\beta}_{\boldsymbol{\nu}} : \mathbf{c}_{\boldsymbol{\nu}} \quad \partial\mathbf{K}$ *.*

5.3. Exponential convergence of $\mathbf{hp}$-dGFEM. We apply the approximation estimates proved in Section 4.2 to establish exponential convergence of the $\mathbf{hp}$-dGFEM scheme. We begin with the following lemma, which puts in relation the domain of analyticity of $\mathbf{u}$ and the geometric mesh $\mathbf{M}_{\boldsymbol{\sigma}}$.

LEMMA 5.4. *Let* $\mathbf{M}_{\boldsymbol{\sigma}}$ *be a family of geometric meshes* $\mathcal{T}^{\square}_{\boldsymbol{\sigma}}$ *on* Ω *satisfying Assumption 5.1, and let* $\mathbf{u}$ *be the solution of the BVP* (5.1) *on* Ω*. Then, there exists* $\boldsymbol{\delta}_* > 0$ *E* (

figs1 k505.55(.4332.569(l)-0299(i)-02g)0.433137(e)0o87L91(m)0.19-0.240155(h)0.47]E /R1.0 0 0 10 0 0 cm BT /R2a0 0 018444e-0.2s

i.e., Ω_{ij}

Remark 4.13 guarantees that the interpolation is exact in at least $p+1$ points on the boundary of Ω^{ℓ}_{ij}. From the usual scaling of Sobolev seminorms $|\cdot|_{H^k(\Omega^{\ell}_{ij})} \le C(h^{\ell}_{ij})^{1-k}|\cdot|_{H^k(\widehat{\Omega}^{\ell}_{ij})}$, we obtain

$$\sum_{1\le i\le \ell-1,\,1\le j\le \widehat{J}(i)} \left(|\eta|^2_{H(\Omega^{\ell}_{ij})} + (h^{\ell}_{ij})^2 |\eta|^2_{H(\Omega^{\ell}_{ij})} \right) \le C\ell e^{-b\ell},$$

with C and b depending only on u, σ, Ω and M_{σ}. Here we used the fact that the number of elements in $\mathcal{T}^{\ell}_{\sigma}$ is $O(\ell)$, as proved in Lemma 5.2.

The assertion is then obtained by combining the last bound with the one previously obtained for the elements incident to the corners, using $\ell = Q(\sqrt{N})$, and noting that $\pi_{\mathcal{T}}(u)$ interpolates u at least in two points per element, thus Proposition 5.3 applies, and the hp-dGFEM error is bounded by the approximation error. ∎

REMARK 5.6. *In standard FEM convergence analysis, approximation estimates are derived only for* few reference elements, *which are then mapped to the "physical" mesh elements. For Trefftz schemes this is usually not possible: spaces made of harmonic functions (or harmonic polynomials) are not invariant under general a* has opening θ

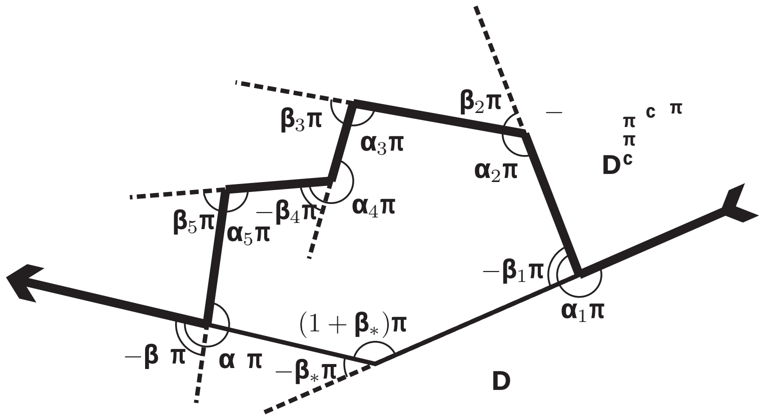
since $\theta_w \quad \theta$, we have the second (exterior) cone condition. □

REMARK A.2. *If $\mathcal{D}$ is a polygon with interior angles $\alpha_k\pi\ {}_{k=1}^{N}$ and satisfies the hypothesis of Lemma A.1, then*

$$\frac{2}{\pi}\arcsin\frac{\rho_1}{\rho_2} \quad \alpha_k \quad 2-\frac{2}{\pi}\arcsin\frac{\rho_1}{\rho_2} \qquad k=1,\dots,N.$$

Appendix B. Proof of the upper bound (3.2) **for non convex domains.** We consider first the case of polygonal domains (with straight sides) in Section B.1, then we extend the result to more general curvilinear domains in Section B.2. We recall that we are assuming $0 < h \quad 1$.

B.1. Polygonal domains. Denote by $\alpha_k^{C}\pi\ {}_{k=1}^{n_C}$ and $\alpha_k^{NC}\pi\ {}_{k=1}^{n_{NC}}$ the convex and non convex internal angles, respectively, of D, by $w_k^{C}\ {}_{k=1}^{n_C}$ and $w_k^{NC}\ {}_{k=1}^{n_{NC}}$


$\beta_2\pi$
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$\alpha_3\pi$
$\alpha_2\pi$
$\beta_5\pi$
$\alpha_5\pi$
$-\beta_4\pi$
$\alpha_4\pi$
$-\beta_1\pi$
$\alpha_1\pi$
$(1+\beta_*)\pi$
$-\beta\ \pi$
$\alpha\ \pi$
$-\beta_*\pi$
D

consequently, as can be inferred from Figure B.2,

$$\left|1 - \bar{z}^{C}_{n_{ar},}\right| \quad \left|1 - \bar{z}^{NC}_{2}\right| . \tag{B.2}$$

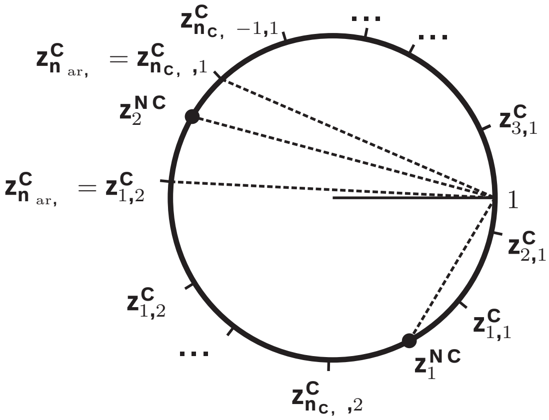

FIG. B.2. *The location of the pre-vertices z_k's in case ii) with two non consecutive non convex corners. The four dashed segments have lengths* $\max\left\{\left| \; - \bar{z}_1^{NC}\right|; \; \right| \; -$

In order to conclude, we only need to prove (B.3).
Consider the counterclockwise oriented part of ∂D formed by the consecutive (oriented) sides s_i, $i = 1, \dots, m := n_{C,1} + 3$, abutting w_1^{NC}, $w_{j,1}^{C}$, $j = 1, \dots, n_{C,1}$, and w_2^{NC}. Let ℓ_i be the oriented line containing s_i, $i = 1, \dots, m$.
Since D is star-shaped with respect to B_ρ , then B_ρ lies in the intersection of the half planes lying on the left of the ℓ_i's.
Let K be the infinite cone obtained by intersecting the right half planes generated by ℓ_1 and ℓ_m. Its opening is $(1 + \beta^*)\pi < \pi$, with $\beta^* < 0$ (*cf.* Figures B.1 and B.3).
Define $D' := D \setminus \overline{K}$; D'

of consecutive convex angles. With a similar notation as before, we can write

$$T \le \int_S \prod_{i=1}^{n} \prod_{j=1}^{n_{NC,i}} \left| y - z^{NC}_{j,i} \right|^{\beta^{NC}_{j,i}} \prod_{j=1}^{n_{C,i}} \left| y - z^{C}_{j,i} \right|^{\beta^{C}_{j,i}} \, \mathrm{d}y.$$

Setting, for $i = 1, \dots, n$,

$$n_{\mathrm{far},i} = \operatorname*{arg\,max}_{j=1,\dots n_{C,i}} \left| 1 - z^{C}_{j,i} \right|, \qquad n_{\mathrm{near},i} = \operatorname*{arg\,min}_{j=1,\dots n_{NC,i}} \left| 1 - z^{NC}_{j,i} \right|,$$

we can bound T as

$$T \le \int_S \prod_{i=1}^{n} \left| y - z^{NC}_{\mathrm{near},i} \right|^{\sum_j \beta^{NC}_{j,i}} \left| y - z^{C}_{\mathrm{far},i} \right|^{\sum_j \beta^{C}_{j,i}} \, \mathrm{d}y =: \int_S P(y) \, \mathrm{d}y.$$

We order the blocks in such a way that

$$\begin{aligned} \left| 1 - z^{NC}_{\mathrm{near},i} \right| &\le \left| 1 - z^{NC}_{\mathrm{near},i+1} \right| \qquad i = 1, \dots, n-1, \\ \left| 1 - z^{C}_{\mathrm{far},i} \right| &\le \left| 1 - z^{C}_{\mathrm{far},i+1} \right| \qquad i = 1, \dots, n-1; \end{aligned}$$

consequently (see Figure B.4),

$$\left| 1 - z^{C}_{\mathrm{far},i} \right| \le \left| 1 - z^{NC}_{\mathrm{near},i+1} \right| \qquad i = 1, \dots, n-1. \tag{B.6}$$

Thus, we have

$$P(y) \le \left| y - z^{NC}_{\mathrm{near},1} \right|^{\sum_j \beta^{NC}_{j,1}} \prod_{i=1}^{n-1} \left| y - z^{NC}_{\mathrm{near},i+1} \right|^{\sum_j \beta^{C}_{j,i} + \sum_j \beta^{NC}_{j,i+1}} (2+h)^{\sum_j \beta^{C}_{j,n}}.$$

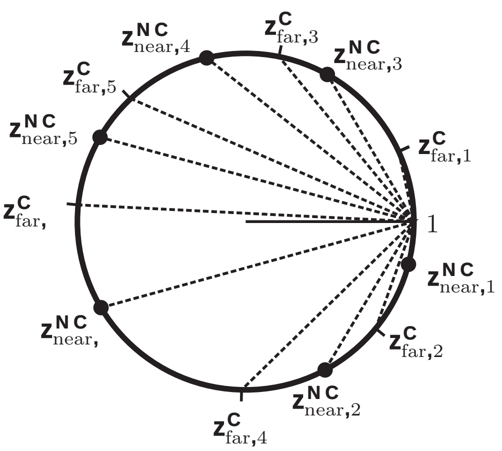

FIG. B.4. *The pre-vertices $z^{\cdot}_k$ satisfy the ordering relation (B.6). Notice that $z^{NC}_{\mathrm{near},1}$ and $z^{C}_{\mathrm{far},n}$ (in the picture $n =$) do not enter the relation. Therefore it is not relevant which one between $z^{C}_{\mathrm{far},1}$ and $z^{NC}_{\mathrm{near},1}$ is closest to . The number of pre-vertices lying in the upper and in the lower half of the complex plane does not affect the ordering of the distances.*

We consider the term with index $n-1$ in the product and look at its exponent ($\sum_j \beta^{C}_{j,n-1} + \sum_j \beta^{NC}_{j,n}$);

a) if it is 0

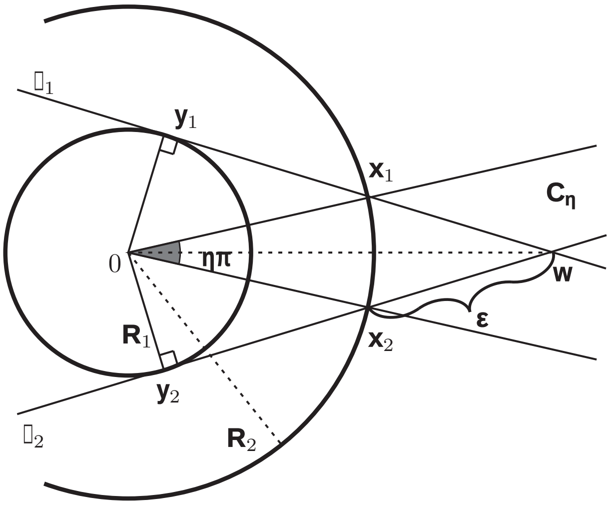

FIG. B.5. *The geometric configuration in Lemma B.1.*

Proof. We consider the limit case $\eta = \frac{2}{\pi} \arcsin \frac{R}{R} < 1$. Then, $R_2 \sin \frac{\eta\pi}{2} = R_1$ and, as depicted in Figure B.6, the lines ℓ_1 and ℓ_2 are parallel to each other. Therefore, whenever η is smaller than this threshold value, ℓ_1 and ℓ_2 will intersect on the central half line of

124883c1.4R9609737619

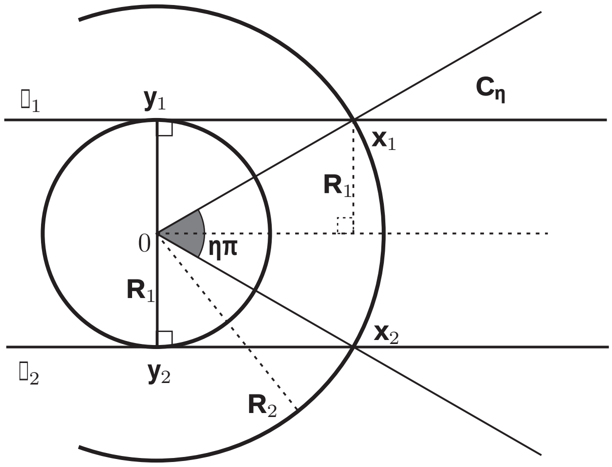

FIG. B.6. *The limit case* $= \frac{2}{}$

REFERENCES

[1] D. N. ARNOLD, F. BREZZI, B. COCKBURN, AND L. D. MARINI, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., 39 (2002), pp. 1749–1779.

[2] I. BABUŠKA AND B. Q. GUO, *The -p version of the finite element method for domains with curved boundaries*, SIAM J. Numer. Anal., 25 (1988), pp. 837–861.

[3] ———, *Regularity of the solution of elliptic problems with piecewise analytic data. I. Boundary value problems for linear elliptic equation of second order*, SIAM J. Math. Anal., 19 (1988), pp. 172–203.

[4] ———, *The -p version of the finite element method for problems with nonhomogeneous essential boundary condition*, Comput. Methods Appl. Mech. Engrg., 74 (1989), pp. 1–28.

[5] ———, *Regularity of the solution of elliptic problems with piecewise analytic data. II. The trace spaces and application to the boundary value problems with nonhomogeneous boundary conditions*, SIAM J. Math. Anal., 20 (1989), pp. 763–781.

[6] G. A. BAKER, *Finite element methods for elliptic equations using nonconforming elements*, Math. Comp., 31 (1977), pp. 45–59.

[7] C. E. BAUMANN AND J. T. ODEN, *A discontinuous p finite element method for convection-diffusion problems*,,cosMefn

Available at http://e-collection.library.ethz.ch/view/eth:4515.
[29] A. Moiola, R. Hiptmair, and I. Perugia, *Vekua theory for the Helmholtz operator*, Z. Angew. Math. Phys., 62 (2011), pp. 779–807.
[30] P. Monk and D.Q. Wang, *A least squares method for the Helmholtz equation*, Comput. Methods Appl. Mech. Eng., 175 (1999), pp. 121–136.
[31] R. Nevanlinna and V. Paatero, *Introduction to complex analysis*, Translated from the German by T. Kövari and G. S. Goodman, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
[32] B. Rivière, M. F. Wheeler, and V. Girault, *Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems. I*, Comput. Geosci., 3 (1999), pp. 337–360 (2000).
[33] C. Schwab, *p- and p-Finite Element Methods. Theory and Applications in Solid and Fluid Mechanics*, Numerical Mathematics and Scientific Computation, Clarendon Press, Oxford, 1998.
[34] I. N. Vekua, *New methods for solving elliptic equations*, North Holland, 1967.
[35] J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Fifth edition. American Mathematical Society Colloquium Publications, Vol. XX, American Mathematical Society, Providence, R.I., 1969.
[36] R. Webster, *Convexity*, Oxford Science Publications, Oxford University Press, New York, 1994.
[37] T. P. Wihler, *Discontinuous Galerkin FEM for Elliptic Problems in Polygonal Domains*, PhD thesis, Swiss Federal Institute of Technology Zurich, 2002. Available at