School of Mathematical
and Physical Sciences

Department of Mathematics and Statistics

Preprint MPS-2012-15

Acoustic trapped modes in a three-dimensional waveguide of slowly varying cross section

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9th July 2012

Abstract

In this paper we develop an asymptotic scheme to approximate the trapped mode solutions to the time harmonic wave equation in a three-dimensional waveguide with a smooth but otherwise arbitrarily shaped cross section and a single, slowly varying 'bulge', symmetric in the longitudinal direction.

Extending the work in [4], we first employ a WKBJ-type ansatz to identify the possible *quasi-mode* solutions which propagate only in the thicker region, and hence find a finite *cut-on* region of oscillatory behaviour and asymptotic decay elsewhere. The WKBJ expansions are used to identify a *turning point* between the cut-on and *cut-off* regions. We note that the expansions are nonuniform in an interior layer centred on this point, and we use the method of matched asymptotic expansions to connect the cut-on and cut-off regions within this layer.

The behaviour of the expansions within the interior layer then motivates the construction of a uniformly valid asymptotic expansion. Finally, we use this expansion and the symmetry of the waveguide around the longitudinal centre, $x = 0$, to extract trapped mode wavenumbers, which are compared with those found using a numerical scheme and seen to be extremely accurate, even to relatively large values of the small parameter.

Keywords: trapped modes; quasi-modes; slowly varying waveguide; perturbation methods; WKBJ; turning point

1 Introduction

By *trapped modes*, we are referring to acoustic modes wherein we find a localised region of finite oscillatory energy, and which decay elsewhere. The existence of trapped modes in various types of topographically slowly varying waveguides has been demonstrated previously, in the cases of elastic plates and rods (e.g., [8,10]), two-dimensional acoustic waveguides (e.g., [4,11]) and in weakly curved quantum waveguides (e.g., [5]). Additionally, there has been previous research into the calculation of *quasi-modes* — perturbed modes of a slowly varying waveguide — in both two- and three-dimensional acoustic waveguides with a slow curvature (e.g., [2,6]), and in two-dimensional elastic plates (e.g., [7]), amongst others.

In particular, in [4], an asymptotic scheme is developed to approximate the trapped mode solutions of the time-harmonic wave equation in a two-dimensional waveguide with a slowly varying symmetric bulge around the longitudinal centre $x = 0$. This slowness is incorporated into the scheme as a large longitudinal length scale, $L_x \sim 1/\epsilon$, where $\epsilon \ll 1$ is taken as the small parameter used to construct an asymptotic approximation to the solution. By expanding both the amplitude and phase in powers of ϵ, the cut-on and cut-off regions are determined by the relative magnitudes of the wavenumber, k, and the eigenvalues of the one-dimensional duct cross section at each x. A turning point between the two regions is identified, and the solution behaviour around this motivates the construction of a uniformly valid approximation, which is used to calculate the trapped mode eigensolutions.

In this paper, we extend this work to create an asymptotic approximation to the trapped mode solutions within a three-dimensional waveguide with a smooth and simply connected, but otherwise arbitrarily shaped cross section with a slow, symmetric variation centred on the plane $x = 0$. In Section 2, the slowly varying nature of the waveguide boundary is introduced in terms of the small parameter, $0 < \epsilon \ll 1$, and the geometry is non-dimensionalised and mapped onto a stretched coordinate system, such that this parameter is incorporated into the differential operator.

In Section 3, we first determine the possible oscillatory or evanescent quasi-mode solutions (see, e.g., [2–4, 6]) that may be present within the waveguide by expanding the solution using a WKBJ-type ansatz of the form $\phi = A\exp(P)$, wherein the amplitude and phase terms are expanded as $A = A_0 + \epsilon A_1 + \dots$ and $P = \epsilon^{-1}P_{-1} + P_1 + \dots$, respectively. A hierarchy of equations in the A_i and P_i terms is obtained, which we solve iteratively to obtain a leading order quasi-mode approximation to ϕ. It will be seen that the behaviour of the solution depends upon the relative sizes of the wavenumber, k, and the transverse eigenvalues of the waveguide cross section, which we denote $\lambda(x)$.

In Section 4, the existence of a turning point is discussed, around which the solution is either oscillatory or evanescent. Motivated by the quasi-mode solutions found in Section 3, we find this to be a point x_*, for which $\lambda(x_*) = k$. Taking established results in domain perturbation theory (e.g., [9, 12]), we are able to assert that a positive turning point will be found in a region for which the cross-sectional area is decreasing, such that we expect a solution to be oscillatory only in the bulge, up until the turning point, and decaying toward infinity.

An interior layer is then introduced around the turning point, wherein an interior solution is found and asymptotically matched with the oscillatory and evanescent solutions on either side of the region. We find that the interior solution takes the form of an Airy function in the x-direction, which motivates the uniformly valid expansion introduced in Section 6. Using this expansion, we are able to extract trapped mode wavenumbers by considering the symmetry of the waveguide about the $x = 0$ plane. As such, we find that to leading order, the wavenumbers of the trapped modes which are either symmetric or antisymmetric in the x-direction are those for which $\phi_x(0; y; z) = 0$ or $\phi(0; y; z) = 0$, respectively. We find there to be a sequence of trapped mode wavenumbers in each case, of the form $k_{n,p}$, $n, p \in \mathbb{N}$, for the Laplacian eigenvalues $\lambda_n(x)$ of the waveguide cross section at x.

Lastly, in Section 8, both the asymptotic scheme and a spectral collocation method are applied to calculate the trapped mode wavenumbers of two example problems, to demonstrate the accuracy of the former. The first test case considered is that of an azimuthally constant 'cylindrical' waveguide with a slow variation. Due to this azimuthal invariance, the problem in this particular geometry may be reduced to a sequence of two-dimensional problems, so that wavenumbers may be quickly extracted via the spectral method, and we see an excellent agreement between these and those obtained using the asymptotic scheme. Further benefits of the asymptotic method are shown in the second example problem, wherein no such reduction is possible and a three-dimensional spectral collocation or similar numerical method proves to be prohibitively expensive. In such a case, we see that the asymptotic scheme can still be used to efficiently approximate the trapped mode wavenumbers to a high degree of accuracy.

2 Formulation

We are interested in time harmonic solutions to the wave equation, and hence omit the common factor $\exp(-i\omega t)$ throughout. The problem is then equivalent to that of finding trapped mode solutions to the Helmholtz equation within a three-dimensional waveguide of arbitrary cross section and smooth boundary. Using standard cartesian coordinates with x oriented along the waveguide, we take the duct to be infinite in the x-direction and symmetric about the plane $x = 0$. We will take the cross section at x to be a simply connected domain $D(x) \in \mathbb{R}^2$, with smooth boundary $\partial D(x)$. We seek eigensolutions of

$$\phi_{xx} + \phi_{yy} + \phi_{zz} + k^2\phi = 0 \qquad \begin{cases} (y, z) \in D(x), \\ -\infty < x < \infty, \end{cases}$$

$$\phi \to 0 \quad \text{as } x \to \pm\infty. \tag{2.1}$$

We will consider individually both the sound soft and sound hard waveguide, i.e., that with either a Dirichlet or Neumann condition on the waveguide boundary, respectively, given by

$$\phi(x, y, z) = 0 \quad \text{for } (y, z) \in \partial D(x), \tag{2.2a}$$

$$\nabla\phi(x, y, z) \cdot n(x, y, z) = 0 \quad \text{for } (y, z) \in \partial D(x), \tag{2.2b}$$

with $n(x;y;z)$ denoting the outward normal vector to $D(x)$ at $(x;y;z)$. If we define a non-dimensionalisation constant L_{yz}, which characterises the length scale of the waveguide in the y- and z-directions, by

$$L_{yz} = \lim_{x \to \infty} \max_{\mathbf{p_1};\mathbf{p_2} \in \partial D} ||\mathbf{p_1} - \mathbf{p_2}||$$

3.1 Sound soft boundary

Note that to satisfy (2.5a), it is su cient to choose

$$A$$

so that, as before,

$$\tilde{\ }(\ ) = \frac{\tilde{A}}{|f^0(\ )|^{1/2}};$$

for constant $\tilde{A}$. Hence, reintroducing the subscript notation, we have the sound hard quasi-mode solutions

$$\tilde{\ }_n = \frac{\tilde{A}_n}{|\tilde{f}_n^0|^{1/2}} \tilde{E}_n \exp\left\{ \ ^{1} \tilde{f}_n \right\} + O(\ ) \tag{3.16}$$

where

$$\tilde{f}_n(\ ) = \begin{cases} i \displaystyle\int \sqrt{k^2 - \tilde{\ }_n^2(\ )}\, d & \text{for } |k| \quad \tilde{\ }_n(\ ) \\ \displaystyle\int \sqrt{\tilde{\ }_n^2(\ ) - k^2}\, d & \text{for } |k| \quad \tilde{\ }_n(\ ); \end{cases} \tag{3.17}$$

for constants $\tilde{A}_n$.

These solutions thus imply that the presence of oscillations in the -direction depend upon the relative magnitudes of the wavenumber, k, and the eigenvalues, $_n(\ )$, of the Laplacian over the domain $D(\ )$. In the case presented here, in which we have a waveguide symmetric about the plane $= 0$, we require, for a trapped mode solution, a region of oscillatory energy up to a particular point, $0 \quad | \ | \quad | \ |$, and decay for $| \ | \quad | \ |$. In fact, we may restrict our attention to just the $\ 0$ half of the waveguide, since it is symmetric, and we see from (3.17) that we hence require a turning point > 0 satisfying $_n(\ ) = k$ and $^0_n(\ ) > 0$ for such a solution to exist.

4 Turning point

We briefly discuss a result in domain perturbation theory for a necessary condition for a suitable turning point to exist in the Dirichlet waveguide. Given an arbitrary, piecewise smooth domain $(x) \quad \mathbb{R}^2$ and a domain $(x) \quad \mathbb{R}^2$, with a small (i.e., $O(\ )$ for $0 < \quad 1$), but not necessarily localised, perturbation from (x), it follows from the work of [9], [12] that the perturbed Dirichlet eigenvalues can be expressed as a uniform asymptotic expansion in . In particular, if we denote by $_j$ a particular Dirichlet eigenvalue on (x), and by $_j$ the associated perturbed eigenvalue on (x), then we can expand

$$_j = \ _j + \ _{j;1} + \ ^2 \ _{j;2} + \cdots :$$

We then find that for the leading order variation, $\mathrm{sgn}(\ _{j;1}) = \ \mathrm{sgn}(f)$, wherein f is the normal distance between the boundaries ∂ and ∂ . We may say, then, that a decrease in the size of the domain coincides with an increase in the magnitude of the eigenvalues of the Dirichlet Laplacian on it. In terms of our sound soft waveguide, a necessary condition for the existence of a turning point > 0, with the properties described above, is for the size of the waveguide cross section to be decreasing at .

In the example waveguide geometry that we will return to in Section 8.1, for instance, we consider a rotationally symmetric waveguidec]TJ -14.944 -11F11 9.9626 Tf 165.8 9.9626 Tf 7.19lEr3(p)-27281(Section)-281(8.1,)

5 Asymptot[illegible]ching

Given the quasi-mode[illegible] (3.7), with (3.8), and the existence of a turning point, > 0, we now discuss the beha[illegible]he solution near this turning point, and show that the asymptotic expansions exhibit no[illegible]y within a small layer ar[illegible]do so, we consider the related re ection-transmissio[illegible] formed by the half-wa[illegible]) in $2\ [0; 1)$ and the constant duct $D(\ )\quad D(0)$ for[illegible]0].

5.1 Region of [illegible]ormity

We consider for clarit[illegible] sound-soft[illegible], but note that the analysis shown also holds for the sound-hard du[illegible] that the[illegible]mum is at $= 0$ and the waveguide tapers with increasing , we can w[illegible]eneral[illegible] solutions in the cut-on and cut-o regions as

$$_o = \frac{I E \exp\left\{ i \int^1 [\text{illegible}] \right\} [\text{illegible}] \left\{ i \int^1 \sqrt{k^2 - {}^2(\ )}\, d \right\}}{[\text{illegible}]} + O(\ ) \quad \text{for} \quad k \quad (\ ); \tag{5.1a}$$

$$_d = \frac{[\text{illegible}] \int^1 \sqrt{{}^2(\ ) [\text{illegible}]}}{[\text{illegible}] (\ {}^2(\ ) \quad k^2)^{1=4}} [\text{illegible}] (\ ) \quad \text{for} \quad k \quad (\ ); \tag{5.1b}$$

respectively, where th[illegible]y solution is comprised[illegible]nt and re ected component. We expect there to be a l[illegible]uniformity between these[illegible]within which these expansions break down around a t[illegible]nt where $(\ ) = k$, and [illegible]behaviour of the quasi-mode solutions as 4 2(k, [illegible]) the behaviour ofk , anda Td [(4 [(2 73 0 Td [( )]TJ/F13 6.9738 Tf 4.359 -1.494 Td [( )]TJ/F8 9.9626 Tf

The $O(1)$ terms vanish and we are left with

$$^{2/3} \; X'' E \Big|_{=} \; X \frac{\partial}{\partial} (\; ^{2}(\;) E) \Big|_{=} + X \; ^{2}(\;) E \Big|_{=} + O(\; ^{4/3}) = 0;$$

$$\Longrightarrow \; ^{2/3} \; X'' E \Big|_{=} \; 2 \;\; 3 \;\; 2 \; ^{2/3} \; 3 \; X E \Big|_{=} \;\; 3 \;\; ;$$

= $^{2}(\;)$ 00 E $^{2}($ E $=$ ^{3}-78( )]TJ 18Td [( )]TJ/F8 9.9626 Tf 9.686 -14.048 Td [(+)]TJ/F11 F7 6.973 3 3 provided that

$$T = \frac{\ ^{1/6}\ \ ^{1/6}}{2\sqrt{\ }}F = e^{i\ /4}I. \tag{5.6}$$

We therefore have the exterior expansions

6.1 Sound soft boundary

7 Trapped modes

Given the leading order uniform expansions (6.9) and (6.12), valid for $\in [0; \infty)$, we notice that by considering a reflection of this semi-infinite duct through the (;)-plane, we may extract leading order trapped mode solutions by considering those solutions which are either antisymmetric or symmetric about = 0. We hence choose appropriate wavenumbers, k, for which either (0; ;) = 0 or (0; ;) = 0, respectively. Taking just the Dirichlet expansion, for example, this is equivalent to requiring either

Antisymmetric: Ai

all positive , $h^0(\ )$ 0, and hence ${}^0_{mn}(\ ) > 0$. Thus, as discussed in Section 4, we may reasonably anticipate the existence of one or more turning points per wavenumber, and hence associated trapped mode solutions.

We notice that the separability and behaviour of (8.2) in the -direction permits the rewriting of (8.1) as

$$^2 r^2$$

and

problem described by (8.1), with the waveguide pro le given instead by

$$h(\; ; \;) = \frac{2 + 2\,(h_1 \quad 1)\,\mathrm{sech} \quad \exp\left\{ \quad 16 \sin \quad \frac{}{2} \quad \frac{}{4} \right\}^2}{\sqrt{\cos^2(\;) + 4 \sin^2(\;)}} ; \tag{8.4}$$

and a bulge height $h_1 = 1{:}5$, as shown in Figures 2(a) and 2(b). We map this geometry onto a

(a) Waveguide with wall pro le given by (8.4). (b) Waveguide cross section at = 0.

Figure 2: Elliptical waveguide with an axially symmetric bulge around = 0.

circular cross section cylinder of unit radius in order to apply a spectral collocation method, using the transformation $= r/h(\; ; \;)$. Thus (8.1) becomes

$$^2 \; ^2h^2 \quad + 2 \quad + \quad + \quad ^2 + \; ^2 \quad + \quad + \quad ^2 + 2 \quad + \quad + \quad + \; ^2h^2k^2 = 0;$$

where, similarly to the previous case,

$$= \frac{(\; + 1)\,h}{h}; \qquad = \frac{(\; + 1) \;\; 2h^2 \quad hh}{h^2};$$

$$= \frac{(\; + 1)\,h}{h}; \qquad = \frac{(\; + 1) \;\; 2h^2 \quad hh}{h^2}$$

and such that $2\,[0;1)$, $2\,[0;2\;]$ and $2\,(\; 1;1)$. It is of note here that, following the methods of [13], we map onto the interval $[0;1)$, as opposed to the \`full' Chebyshev interval $[\;1;1)$, and later discard the negative Chebyshev interpolation points. This ensures a much more even spacing of Chebyshev points in the -direction and thus far greater accuracy for a marginal computational cost. We note, however, that even with such a modi cation, a full spectral collocation method with M, N and P interpolation points in the -, r- and -directions, respectively, involves calculating the eigenvalues of an $(M \quad N \quad P)$-by-$(M \quad N \quad P)$ matrix. For the purposes of this veri cation, we apply this method with an interpolation grid of (18 18 20) points, such that the problem is discretised into a 6480-by-6480 matrix.

$k_{np}^{(a;s)}$	Asymptotic	Spectral	Rel. error
k_{11}^{s}	1.785610713	1.781332958	0.2401%
k_{11}^{a}	1.803862094	1.798011732	0.3254%
k_{12}^{s}	1.81886783	1.814169297	0.2590%
k_{12}^{a}	1.833185541	1.828951766	0.2315%
k_{13}^{s}	1.845360806	1.842142284	0.1747%
k_{13}^{a}	1.856266363	1.853653319	0.1410%
k_{14}^{s}	1.865307241	1.863482336	0.0979%
k_{14}^{a}	1.872942432	1.87174977	0.0637%

Table 3: Relative error between the asymptotic and spectral collocation calculations of the rst eight wavenumbers for the duct described in pro le by (8.4), with small parameter $= 0:1$ and $h_1 = 1:5$.

Figure 3: Cross sections of the rst four (in the azimuthal / radial directions) antisymmetric trapped mode solutions at $= 0:$

A Appendix

Given a smooth surface $S = (x; y; z)$, parameterised by

$$x = x; \qquad y = \ (x; \ ); \qquad z = \ (x; \ );$$

it is clear that the rate of change of the cross section with x is given by $v = (\ _x; \ _x)$. We also have that the surface normal is given by the cross product

$$n = \frac{@S}{}$$