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by

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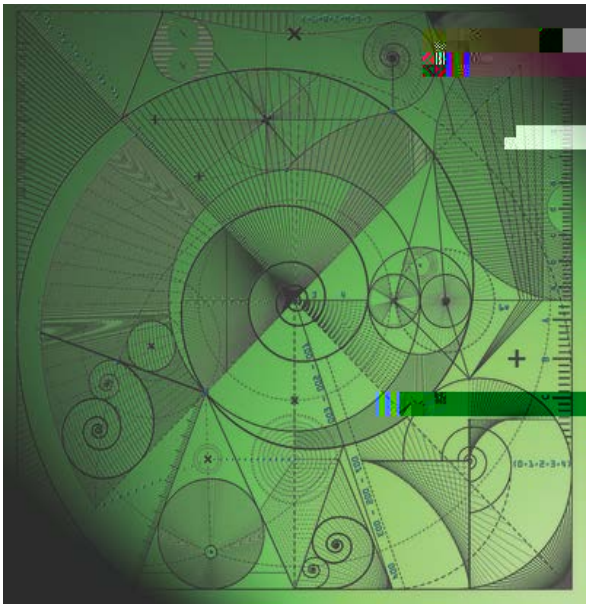

Wavenumber-explicit continuity and coercivity estimates in acoustic scattering by planar screens

S. N. Chandler-Wilde and D. P. Hewett

Abstract. We study the classical rst-kind boundary integral equation reformulations of time-harmonic acoustic scattering by planar sound-soft (Dirichlet) and sound-hard (Neumann) screens. We prove continuity and coercivity of the relevant boundary integral operators (the acoustic single-layer and hypersingular operators respectively) in appropriate fractional Sobolev spaces, with wavenumber-explicit bounds on the continuity and coercivity constants. Our analysis is based on spectral representations for the boundary integral operators, and builds on results of Ha-Duong (Jpn J Ind Appl Math 7:489{513 (1990) and Integr Equat Oper Th 15:427{453 (1992)).

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1. Introduction

This paper concerns the mathematical analysis of a class of time-harmonic acoustic scattering problems modelled by the Helmholtz equation

$$u + k^2 u = 0; \tag{1}$$

where u is a complex scalar function and $k > 0$ is the wavenumber We study the reformulation of such scattering problems in terms of boundary integral equations (BIEs), proving continuity and coercivity estimates for the associated boundary integral operators (BIOs) which are explicit in their k-dependence.

Our focus is on scattering by a thin planar screen occupying some bounded and relatively open set $_1 := \{x = (x_1; :::; x_n) \in R^n : x_n = 0\}$ (we assume throughout that $n = 2$ or 3), with (1) assumed to hold in the

domain $D := \mathbb{R}^n \setminus \overline{\ }$. We shall assume throughout that is a Lipschitz relatively open subset of $_1$ (in the sense of [26], viewing and $_1$ as subsets of $\mathbb{R}^{n\ 1}$). But in fact our continuity and coercivity estimates can be used to prove analogous estimates for the corresponding BIOs on arbitrary relatively open or relatively closed $_1$ (in suitable Sobolev spaces), as is discussed in [7, 6, 9].

We consider both the Dirichlet (sound-soft) and Neumann (sound-hard) boundary value problems (BVPs), which we now state. The function space notation in the following de nitions, and the precise sense in which the boundary conditions are to be understood, will be explained in *x*2.

De nition 1.1 (Problem D). Given $g_D \in H^{1/2}(\ )$, nd $u \in C^2(D) \cap W^1_{loc}(D)$ such that

$$u + k^2 u = 0; \quad \text{in } D; \tag{2}$$

$$u = g_D; \quad \text{on } ; \tag{3}$$

and u satis es the Sommerfeld radiation condition at in nity.

De nition 1.2 (Problem N). Given $g_N \in H^{\ 1/2}(\ )$, nd $u \in C^2(D) \cap W^1_{loc}(D)$ such that

$$u + k^2 u = 0; \quad \text{in } D; \tag{4}$$

$$\frac{\partial u}{\partial n} = g_N; \quad \text{on } ; \tag{5}$$

and u satis es the Sommerfeld radiation condition at in nity.

Example 1.3. Consider the problem of scattering by of an incident plane wave

$$u^i(x) := e^{ikx\ d}; \qquad x \in \mathbb{R}^n; \tag{6}$$

where $d \in \mathbb{R}^n$ is a unit direction vector. A 'sound-soft' and a 'sound-hard' screen are modelled respectively by problem D (with $g_D = \ u^i|\ $) and problem N (with $g_N = \ \partial u^i/\partial n|\ $). In both cases u represents the scattered eld, the total eld being given by $u^i + u$.

Such scattering problems have been well-studied, both theoretically [30, 29, 32, 17, 18, 23] and in applications [13, 12]. For $_1$ su ciently smooth, it is well known (see, e.g., [30, 29, 32, 23]) that problems D and N are uniquely solvable for all $g_D \in H^{1/2}(\ )$ and $g_N \in H^{\ 1/2}(\ )$. (Many of the references cited above assume $_1$ is C^1 smooth, or do not explicitly specify the regularity of , but in fact unique solvability extends to the more general Lipschitz case we consider in this paper, as clari ed e.g. in [7].) Their solutions can be represented respectively in terms of the single and double layer potentials (for notation and de nitions see *x*2)

$$S_k : H^{\ 1/2}(\ ) \to C^2(D) \cap W^1_{loc}(D); \qquad D_k : H^{\ 1/2}(\ ) \to C^2(D) \cap W^1_{loc}(D);$$

where $\Phi(x;y)$ denotes the fundamental solution of (1),

$$\Phi(x;y) := \begin{cases} \dfrac{e^{ik|x-y|}}{4\pi|x-y|}; & n=3; \\ \dfrac{i}{4}H_0^{(1)}(k|x-y|); & n=2; \end{cases} \qquad x,y \in \mathbb{R}^n: \tag{8}$$

The densities of the potentials are the unique solutions of certain first-kind BIEs involving the single-layer and hypersingular BIOs (again, for definitions see *x*2)

$$S_k : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) = (\tilde{H}^{-1/2}(\Gamma))' ;$$
$$T_k : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) = (\tilde{H}^{1/2}(\Gamma))' ;$$

which for $\Gamma \in \mathcal{D}(\Gamma)$ and $x \in \Gamma$ have the following integral representations:

$$S_k\phi(x) = \int_\Gamma \Phi(x;y)\,\phi(y)\,ds(y); \qquad T_k\phi(x) = \frac{\partial}{\partial n(x)}\int_\Gamma \frac{\partial\Phi(x;y)}{\partial n(y)}\,\phi(y)\,ds(y): \tag{9}$$

These standard statements are summarised in the following two theorems. Here $[u]$ and $[\partial u/\partial n]$ represent the jump across Γ of u and of its normal derivative respectively.

Theorem 1.4. Problem D has a unique solution u satisfying

$$u(x) = S_k[\partial u/\partial n](x); \qquad x \in D; \tag{10}$$

where $[\partial u/\partial n] \in \tilde{H}^{-1/2}(\Gamma)$ is the unique solution of the BIE

$$S_k[\partial u/\partial n] = g_D: \tag{11}$$

Theorem 1.5. Problem N has a unique solution satisfying

$$u(x) = D_k[u](x); \qquad x \in D; \tag{12}$$

where $[u] \in \tilde{H}^{1/2}(\Gamma)$ is the unique solution of the BIE

$$T_k[u] = g_N: \tag{13}$$

1.1. Main results and outline of the paper

In this paper we present new k-explicit continuity and coercivity estimates for the operators S_k and T_k appearing in (11) and (13). Our main results are contained in the following four theorems.

Theorem 1.6. For any $s \in \mathbb{R}$, the single-layer operator S_k defines a bounded linear operator $S_k : \tilde{H}^{-s}(\Gamma) \to H^{s+1}(\Gamma)$, and there exists a constant $C > 0$, independent of k and Γ, such that, for all $\phi \in \tilde{H}^{-s}(\Gamma)$, and with $L := \operatorname{diam}\Gamma$,

$$\|S_k\phi\|_{H^{s+1}_k(\Gamma)} \le \begin{cases} C(1+(kL)^{1/2})\,\|\phi\|_{\tilde{H}^{-s}_k(\Gamma)}; & n=3; \\ C\log(2+(kL)^{-1})(1+(kL)^{1/2})\,\|\phi\|_{\tilde{H}^{-s}_k(\Gamma)}; & n=2; \end{cases} \qquad k > 0: \tag{14}$$

Theorem 1.7. The sesquilinear form on H

very mildly as $k \to \infty$. This aim is provably achieved in certain cases, mainly 2D so far; see, e.g., [22, 8] and the recent review [5]. For 2D screen and aperture problems we recently proposed in [21] an HNA BEM which provably achieves a fixed accuracy of approximation with N growing at worst like $\log^2 k$ as $k \to \infty$, our numerical analysis using the wavenumber-explicit estimates of the current paper. Numerical experiments demonstrating the effectiveness of HNA approximation spaces for a 3D screen problem have been presented in [5, §7.6].

Clearly the results in this paper are a contribution to this endeavour. In particular, Theorems 1.6 and 1.8 provide upper bounds on $\|S_k\|_{\tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)}$ and $\|T_k\|_{\tilde{H}^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)}$ (with $\tilde{H}^{-1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ equipped with the wavenumber-dependent norms specified in §2). Further, as noted generically above, through providing lower bounds on the coercivity constant , Theorems 1.7 and 1.9 bound the inverses ximat6b5 Td [( )]TJ/F7 6.9.o586 2

$H^s(\mathbb{R}^n)$ de ned by

$$\|u\|^2_{H^s_k(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (k^2 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \, . \tag{26}$$

We emphasize that $\|\cdot\|_{H^s(\mathbb{R}^n)} := \|\cdot\|_{H^s_1(\mathbb{R}^n)}$ is the standard norm on $H^s(\mathbb{R}^n)$, and that, for $k > 0$, $\|\cdot\|_{H^s_k(\mathbb{R}^n)}$ is another, equivalent, norm on $H^s(\mathbb{R}^n)$. Explicitly,

$$\min\{1, k^s\}\|u\|_{H^s(\mathbb{R}^n)} \le \|u\|_{H^s_k(\mathbb{R}^n)} \le \max\{1, k^s\}\|u\|_{H^s(\mathbb{R}^n)}, \text{ for } u \in H^s(\mathbb{R}^n). \tag{27}$$

It is standard that $D(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$. It is also standard (see, e.g., [26]) that $H^{-s}(\mathbb{R}^n)$ is a natural isometric realisation of $(H^s(\mathbb{R}^n))^*$, the dual space of bounded antilinear functionals on $H^s(\mathbb{R}^n)$, in the sense that the mapping $u \mapsto u^*$ from $H^{-s}(\mathbb{R}^n)$ to $(H^s(\mathbb{R}^n))^*$, de ned by

$$u^*(v) := \langle u, v\rangle_{H^{-s}(\mathbb{R}^n),\, H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)} \, d\xi \, ; \quad v \in H^s(\mathbb{R}^n); \tag{28}$$

is a unitary isomorphism. The duality pairing $\langle \cdot, \cdot\rangle_{H^{-s}(\mathbb{R}^n),\, H^s(\mathbb{R}^n)}$ de ned in (28) represents a natural extension of the $L^2(\mathbb{R}^n)$ inner product in the sense that if $u_j, v_j \in L^2(\mathbb{R}^n)$ for each j and $u_j \to u$ and $v_j \to v$ as $j \to \infty$, with respect to the norms on $H^{-s}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ respectively, then $\langle u, v\rangle_{H^{-s}(\mathbb{R}^n),\, H^s(\mathbb{R}^n)} = \lim_{j\to\infty} (u_j, v_j)_{L^2(\mathbb{R}^n)}$.

We de ne two Sobolev spaces on Ω when Ω is a non-empty open subset of $\mathbb{R}^n$. First, let $D(\Omega) := C_0^\infty(\Omega) = \{u \in D(\mathbb{R}^n) : \operatorname{supp} u \subset \Omega\}$, and let $D'(\Omega)$ denote the associated space of distributions (continuous antilinear functionals on $D(\Omega)$). We set

$$H^s(\Omega) := \{u \in D'(\Omega) : u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^n)\};$$

where $U|_\Omega$ denotes the restriction of the distribution U to Ω (cf. [26, p. 66]), with norm

$$\|u\|_{H^s_k(\Omega)} := \inf_{U \in H^s(\mathbb{R}^n);\, U|_\Omega = u} \|U\|_{H^s_k(\mathbb{R}^n)}.$$

Then $D(\overline{\Omega}) := \{u \in C^\infty(\Omega) : u = U|_\Omega \text{ for some } U \in D(\mathbb{R}^n)\}$ is dense in $H^s(\Omega)$. Second, let

$$\tilde{H}^s(\Omega) := \overline{D(\Omega)}^{H^s(\mathbb{R}^n)}$$

denote the closure of $D(\Omega)$ in the space $H^s(\mathbb{R}^n)$, equipped with the norm $\|\cdot\|_{\tilde{H}^s_k(\Omega)} := \|\cdot\|_{H^s_k(\mathbb{R}^n)}$. When Ω is su ciently regular (e.g. when Ω is C^0, cf. [26, Thm 3.29]) we have that $\tilde{H}^s(\Omega) = H^s_{\overline{\Omega}} := \{u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subset \overline{\Omega}\}$. (But for non-regular Ω $\tilde{H}^s(\Omega)$ may be a proper subset of $H^s_{\overline{\Omega}}$, see [9].)

For $s \in \mathbb{R}$ and Ω any open, non-empty subset of $\mathbb{R}^n$ it holds that

H^{-s} s , with respe16 Td [(it)-333(holds)-334(that)]TJ/F11 9.9626 Tf

in the sense that the natural embeddings $\iota : H^s(\Omega) \to (\tilde{H}^{-s}(\Omega))'$ and $\iota : \tilde{H}^{-s}(\Omega) \to (H^s(\Omega))'$,

$$(\iota u)(v) := \langle u, v\rangle_{H^s(\Omega)\times\tilde{H}^{-s}(\Omega)} := \langle U, v\rangle_{H^s(\mathbb{R}^n)\times H^{-s}(\mathbb{R}^n)},$$
$$(\iota v)(u) := \langle v, u\rangle_{\tilde{H}^{-s}(\Omega)\times H^s(\Omega)} := \langle v, U\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)},$$

where $U \in H^s(\mathbb{R}^n)$ is any extension of $u \in H^s(\Omega)$ with $U|_\Omega = u$, are unitary isomorphisms. We remark that the representations (29) for the dual spaces are well known when Ω is su ciently regular. However, it does not appear to be widely appreciated, at least in the numerical analysis for PDEs community, that (29) holds without constraint on the geometry of Ω, proof of this given recently in [7, Thm 2.1].

Sobolev spaces can also be de ned, for $s \geq 0$, as subspaces of $L^2(\mathbb{R}^n)$ satisfying constraints on weak derivatives. In particular, given a non-empty open subset Ω of $\mathbb{R}^n$, let

$$W^1(\Omega) := \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\},$$

where ∇u is the weak gradient. Note that $W^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$ with

$$\|u\|^2_{H^1_k(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla u(x)|^2 + k^2|u(x)|^2 \, dx.$$

Further [26, Theorem 3.30], $W^1(\Omega) = H^1(\Omega)$ whenever Ω is a Lipschitz open set. It is convenient to de ne

$$W^1_{loc}(\Omega) := \{u \in L^2_{loc}(\Omega) : \nabla u \in L^2_{loc}(\Omega)\},$$

where $L^2_{loc}(\Omega)$ denotes the set of locally integrable functions u on Ω for which $\int_G |u(x)|^2 dx < \infty$ for every bounded measurable $G \subset \Omega$.

To de ne Sobolev spaces on the screen $\Gamma_1 := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$ we make the natural associations of Γ_1 with $\mathbb{R}^{n-1}$ and of Ω with $\tilde{\Omega} := \{x \in \mathbb{R}^{n-1} : (x, 0) \in \Omega\}$ and set $H^s(\Gamma_1) := H^s(\mathbb{R}^{n-1})$, $H^s(\Omega) := H^s(\tilde{\Omega})$ and $\tilde{H}^{-s}(\Omega) := \tilde{H}^{-s}(\tilde{\Omega})$ (with $C^1(\Gamma_1)$, $D(\Gamma_1)$, $D(\bar{\Omega})$ and $D(\Omega)$ de ned analogously).

2.2. Traces, jumps and boundary conditions

Letting $U^+ := \{x \in \mathbb{R}^n : x_n > 0\}$ and $U^- := \mathbb{R}^n \setminus \overline{U^+}$ denote the upper and lower half-spaces, respectively, we de ne trace operators $\gamma : D(\Omega) \to H$

for all $\phi \in D(\mathbb{R}^n)$. We then define

$$[u] := \gamma^+(\chi u) - \gamma^-(\chi u) \in \tilde{H}^{1/2}(\Gamma_\infty)$$

$$[\partial u/\partial n] := \partial_n^+(\chi u) - \partial_n^-(\chi u) \in \tilde{H}^{-1/2}(\Gamma_\infty),$$

where χ is any element of $D_{1,\Gamma}(\mathbb{R}^n) := \{\phi \in D(\mathbb{R}^n) : \phi = 1$ in some neighbourhood of $\overline{\Gamma}\}$.

The boundary conditions (3) and (5) can now be stated more precisely: by (3) and (5) we mean that

$$\gamma^\pm(\chi u)|_\Gamma = g_D, \quad \text{and} \quad \partial_n^\pm(\chi u)|_\Gamma = g_N, \quad \text{for every } \chi \in D_{1,\Gamma}(\mathbb{R}^n).$$

2.3. Layer potentials and boundary integral operators

We can now give precise definitions for the single and double layer potentials

$$\mathcal{S}_k : \tilde{H}^{-1/2}(\Gamma) \to C^2(D) \cap W^1_{\mathrm{loc}}(D), \qquad \mathcal{D}_k : \tilde{H}^{1/2}(\Gamma) \to C^2(D) \cap W^1_{\mathrm{loc}}(D),$$

namely

$$\mathcal{S}_k\phi(x) := \left\langle \gamma^\pm(\chi\Phi_k(x,\cdot)), \overline{\phi} \right\rangle_{H^{1/2}(\Gamma_\infty)\times \tilde{H}^{-1/2}(\Gamma_\infty)}, \quad x \in D, \quad \phi \in \tilde{H}^{-1/2}(\Gamma),$$

$$\mathcal{D}_k\psi(x) := \left\langle \psi, \overline{\partial_n^\pm(\chi\Phi_k(x,\cdot))} \right\rangle_{\tilde{H}^{1/2}(\Gamma_\infty)\times H^{-1/2}(\Gamma_\infty)}, \quad x \in D, \quad \psi \in \tilde{H}^{1/2}(\Gamma),$$

where χ is any element of $D_{1,\Gamma}(\mathbb{R}^n)$ with $x \notin \operatorname{supp}\chi$. The single-layer and hypersingular boundary integral operators

$$S_k : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \qquad T_k : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma),$$

are then defined by

$$S_k\phi := (\gamma^\pm \mathcal{S}_k\phi)|_\Gamma :=$$

To evaluate $\hat{\Phi}_c$ we note that for a function $f(x) = F(r)$, where $r = |x|$ for $x \in \mathbb{R}^d$, $d = 1, 2$, the Fourier transform of f is given by (cf. [15, §B.5]) [4]

$$\hat{f}(\xi) = \begin{cases} \displaystyle\int_0^\infty F(r) J_0(|\xi| r)\, r\, dr, & d = 2, \\ \displaystyle\sqrt{\frac{2}{\pi}} \int_0^\infty F(r) \cos(\xi r)\, dr, & d = 1. \end{cases} \tag{38}$$

This result, combined with the identities [14, (6.677), (6.737)] and [1, (10.16.1), (10.39.2)], gives

$$\hat{\Phi}_c(\xi; x_n) = \frac{i\, e^{i|x_n| Z(\xi)}}{2(2\pi)^{(n-1)/2} Z(\xi)},$$

where $Z(\xi)$ is defined as in (32). The representation (33) is then obtained by Fourier inversion.

The representation (31) for $\mathcal{D}_k$ can be then obtained from (30) by noting that

$$\frac{\partial \Phi(x, y)}{\partial n(y)} = \frac{\partial \Phi(x, y)}{\partial y_n} = -\frac{\partial \Phi(x, y)}{\partial x_n}, \qquad x \in D, \; y \in \Gamma,$$

and the representations for S_k and T_k follow from taking the appropriate traces of (30) and (31).

Finally, (35) and (36) follow from viewing $S_k^\infty \phi$ and $T_k^\infty \phi$ as elements of $C^\infty(\mathbb{R}^{n-1}) \cap S^*(\mathbb{R}^{n-1})$ and recalling the definition of the Fourier transform of a distribution, e.g., for S_k,

$$\begin{aligned}(S_k \phi, \psi)_{L^2(\Gamma)} &= \int_{\mathbb{R}^{n-1}} S_k^\infty \phi(\tilde{x}) \overline{\psi(\tilde{x})}\, d\tilde{x} = \int_{\mathbb{R}^{n-1}} \widehat{S_k^\infty \phi}(\xi) \overline{\hat{\psi}(\xi)}\, d\xi \\ &= \frac{i}{2} \int_{\mathbb{R}^{n-1}} \frac{1}{Z(\xi)} \hat{\phi}(\xi) \overline{\hat{\psi}(\xi)}\, d\xi.\end{aligned}$$

4. k-explicit analysis of S_k

Our k-explicit analysis of the single-layer operator S_k makes use of the following lemma.

Lemma 4.1. Given $L > 0$ let

$$\Phi_L(\tilde{x}; x_n) := \begin{cases} \Phi_c(\tilde{x}; x_n), & |\tilde{x}| \le L, \\ 0, & |\tilde{x}| > L, \end{cases} \tag{39}$$

where Φ_c is defined as in the proof of Theorem 3.1. Then there exists a constant $C > 0$, independent of k, L, Γ and x_n, such that, for all $k > 0$,

[4] Strictly speaking, [15, §B.5] only provides (38) for $f \in L^1(\mathbb{R}^d)$. But for the functions $f = \Phi_c(\cdot; x_n)$ one can check using the dominated convergence theorem that (38) holds.

$\xi \in \mathbb{R}^{n-1}$, and $x_n \in \mathbb{R}$,

$$|\hat{c}_L(\xi; x_n)| \sqrt{k^2 + |\xi|^2} \le \begin{cases} C(1 + (kL)^{1/2}), & n = 3, \\ C\left(\log(2 + (kL)^{-1}) + (kL)^{1/2}\right), & n = 2. \end{cases} \qquad (40)$$

Proof. It is convenient to introduce the notation $\rho := |\xi|$, and by $C > 0$ we denote an arbitrary constant, independent of k, L, ρ, and x_n, which may change from occurrence to occurrence. To prove (40) we proceed by estimating $|\hat{c}_L(\xi; x_n)|$ directly, using the formula (38). We treat the cases $n = 3$ and $n = 2$ separately. We will make use of the following well-known properties of the Bessel functions (cf. [1, Sections 10.6, 10.14, 10.17]), where $\mathcal{B}_n$ represents either J_n or $H_n^{(1)}$:

$$|J_n(z)| \le 1, \qquad n \in \mathbb{N},\ z > 0; \qquad (41)$$

$$|H_0^{(1)}(z)| \le C(1 + |\log z|), \qquad 0 < z \le 1 \qquad (42)$$

$$|H_1^{(1)}(z)| \le C z^{-1}, \qquad 0 < z \le 1 \qquad (43)$$

$$\left|H_1^{(1)}(z) + \frac{2i}{\pi z}\right| \le C z^{-1/2}, \qquad z > 0; \qquad (44)$$

$$|\mathcal{B}_n(z)| \le C z^{-1/2}, \qquad n = 0, 1,\ z \ge 1; \qquad (45)$$

$$\mathcal{B}_0'(z) = -\mathcal{B}_1(z), \qquad z > 0; \qquad (46)$$

$$\frac{d}{dz}(z\mathcal{B}_1(z)) = z\mathcal{B}_0(z), \qquad z > 0. \qquad (47)$$

(i) In the case $n = 3$, $|\hat{c}_L(\xi; x_3)| \le |I(L)|/(4\pi)$, where

$$I(L) := \int_0^L \frac{e^{ik\sqrt{r^2 + x_3^2}}}{\sqrt{r^2 + x_3^2}} J_0(\rho r)\, r\, dr, \qquad \text{for } L > 0.$$

Using (41), we see that $|I(L)| \le 1/\rho$, if $L \le 1/\rho$. If $L > 1/\rho$ then, integrating by parts using the relation (47),

$$I(L) - I(1/\rho) = \frac{1}{\rho}\left[\frac{r e^{ik\sqrt{r^2 + x_3^2}}}{\sqrt{r^2 + x_3^2}} J_1(\rho r)\right]_{1/\rho}^{L} - \frac{1}{\rho}\int_{1/\rho}^{L} r^2 e^{ik\sqrt{r^2 + x_3^2}}\left(\frac{ik}{r^2 + x_3^2} - \frac{1}{(r^2 + x_3^2)^{3/2}}\right) J_1(\rho r)\, dr,$$

so that, substituting $t = \rho r$ and using (41),

$$|I(L)| \le |I(1/\rho)| + |I(L) - I(1/\rho)| \le \frac{3}{\rho} + \frac{1}{\rho}\int_1^{\rho L}\left(\frac{k}{\rho} + t^{-1}\right)|J_1(t)|\, dt. \qquad (48)$$

Using the bound (45) in (48), it follows that

$$|I(L)| \le \frac{3}{\rho} + \frac{C}{\rho}\left(\frac{kL^{1/2}}{\rho^{1/2}} + 1\right), \qquad L > 0,$$

so that

$$|c_1(\ ; x_3)| \ \sqrt{k^2 + \ +}$$

so that

$$|I_0(L)| \quad \frac{2}{} \quad 3 + \int_1^L \frac{1}{t^2}\,dt \quad < \frac{8}{}:$$

Using these bounds on $I_0(L)$ and the bound (44) on $F_1(z)$, we see that

$$|I(L)$$

since $jk^2 \quad j \quad j^2 j \quad 2kj \quad j + j \quad j^2 \quad 3k \quad ^2$, for $j \quad j$. Also, for the same choice of and all t 0,

$$k \quad k^2_{\tilde{H}^{t}_{k}(\ )} \quad \frac{1}{k^{2t}} \int_{R^{n-1}} j\,b(\ )j^2\, d \ = \ \frac{1}{k^{2t}} \int_{R^{n-1}} j\,b(\ )j^2\, d \quad 2k \quad ^{2t} I(\ ); \tag{54}$$

for su ciently large. Further, for su ciently large and k > 1,

$$I(\ ) \quad \int_{j \ j>} (2+2j \ j+j \ j^2)^t j\,b(\ )j^2\, d \quad \int_{j \ j>} 1 + \frac{}{k} + đ \ ^2 \ ^t j\,b(\ )j^2\, d \ ;$$

so that, since $k^2 + j \ + kđj^2 \quad k^2 + (\ + k)^2 \quad 5k^2 \ ^2$ for $j \ j$,

$$k \quad k^2_{\tilde{H}^{t}_{k}(\ )} = \int_{R^{n-1}} k^2 + \ + kđ \ ^2 \ ^t j\,b(\ )j^2\, d \quad 6k^{2t} \ ^{2t} I(\ ): \tag{55}$$

Combining (53), (54) and (55) we see that, for every $s \in R$, if 1 is su ciently large, there exists $C > 0$, depending on and s but independent of k, such that

$$j(S_k \ ; \ )_{L^2(\ )} j \quad C k^{1=2} k \quad k_{\tilde{H}_k^{(s+1)}(\ )} \ k \quad k_{\tilde{H}^s_k(\ )} :$$

But, on the other hand,

$$j(S_k \ ; \ )_{L^2(\ )} j = j h S_k \ ; \quad i_{H^{s+1}(\ ) \ \tilde{H}^{(s+1)}(\ )} j \quad kS_k \quad k_{H^{s+1}_k(\ )} \ k \quad k_{\tilde{H}_k^{(s+1)}(\ )} ;$$

so that, for this particular choice of ,

$$kS_k \quad k_{H^{s+1}_k(\ )} \quad C k^{1=2} k \quad k_{\tilde{H}^s_k(\ )} ;$$

which demonstrates the sharpness of (14) in the limit $k \; ! \; 1$.

Remark 4.3. Theorem 1.6 bounds $S_k : \tilde{H}^s(\ ) \; ! \; H^{s+1}(\ )$. We can also bound S_k as a mapping $S_k : \tilde{H}^s(\ ) \; ! \; H^s(\ )$. Since $k \quad k_{\tilde{H}^{s-1}(\ )} \quad k^{-1} k \quad k_{\tilde{H}^s(\ )}$ for $2 \tilde{H}^s(\ )$, it follows from Theorem 1.6 that, for $kL \quad 1$, $k^{-1} k \quad : 2(t)$

Remark 4.4. We can also show that the bound in Theorem 1.7 is sharp in its dependence on k as $k \to \infty$. Let $0 \neq \ \in D(\ )$ be independent of k. Then, by (35), and since $\hat{\ }(\ )$ is rapidly decreasing as $k \to \infty$,

$$|a_D(\ ;\ )| \quad 1$$

Defining

$$J := k\,\|\phi\|^2_{H_k^{1/2}(\Gamma)} = \int_{\mathbb{R}^{n-1}} (k^2+|\xi|^2)^{1/2}|\hat{\phi}(\xi)|^2\,d\xi;$$

the problem of proving (17) reduces to that of proving

$$I \geq Ck\,J, \qquad k \geq k_0, \tag{63}$$

for some $C > 0$ depending only on k_0.

The difficulty in proving (63) is that the factor $|Z(\xi)|$ in I vanishes when $|\xi| = k$. To deal with this, we write the integrals I and J as

$$I = I_1 + I_2 + I_3 + I_4, \qquad J = J_1 + J_2 + J_3 + J_4,$$

corresponding to the decomposition

$$\int_{\mathbb{R}^{n-1}} = \int_{0<|\xi|<k-\varepsilon} + \int_{k-\varepsilon<|\xi|<k} + \int_{k<|\xi|<k+\varepsilon} + \int_{|\xi|>k+\varepsilon},$$

where $0<\varepsilon \leq k$ is to be specified later. We then proceed to estimate the integrals $J_1,\ldots,J_4$ separately. Throughout the remainder of the proof $c > 0$ denotes an absolute constant whose value may change from occurrence to occurrence.

We first observe that, for $0 < |\xi| < k-\varepsilon$,

$$\frac{k^2+|\xi|^2}{k^2-|\xi|^2} \leq \frac{k^2+(k-\varepsilon)^2}{k^2-(k-\varepsilon)^2} \leq \frac{2k^2}{\varepsilon(2k-\varepsilon)} \leq \frac{2k}{\varepsilon},$$

so that

$$\begin{aligned} J_1 &:= \int_{0<|\xi|<k-\varepsilon} (k^2+|\xi|^2)^{1/2}|\hat{\phi}(\xi)|^2\,d\xi \\ &= \int_{0<|\xi|<k-\varepsilon} \frac{(k^2+|\xi|^2)^{1/2}}{(k^2-|\xi|^2)^{1/2}}|Z(\xi)|\,|\hat{\phi}(\xi)|^2\,d\xi \\ &\leq c\sqrt{\frac{k}{\varepsilon}}\,I_1. \end{aligned}$$

Similarly, for $|\xi| > k+\varepsilon$,

$$\frac{k^2+|\xi|^2}{|\xi|^2-k^2} \leq \frac{k^2+(k+\varepsilon)^2}{(k+\varepsilon)^2-k^2} \leq \frac{5k}{2\varepsilon}$$

$4 \sin^2 (t=2) \quad t^2$, so that, for $_1$;

Arguing similarly, again assuming that $0<\varepsilon<k/3$, but using (64) and (66) with the plus rather than the minus sign, gives

$$J_3 := \int_{k<|\xi|<k+\varepsilon} (k^2+|\xi|^2)^{1/2}|\hat\phi(\xi)|^2\, d\xi \le c\sqrt{\frac{k}{\varepsilon}}\, I_4 + c\varepsilon^3 k^{n-2} J.$$

Combining the above estimates we see that, for $0<\varepsilon<k/3$,

$$J = J_1+J_2+J_3+J_4 \le c\sqrt{\frac{k}{\varepsilon}}\,(I_1+I_4) + c\varepsilon^3 k^{n-2} J;$$

which implies that

$$J\left(1-c\varepsilon^3k^{n-2}\right) \le c\sqrt{\frac{k}{\varepsilon}}\, I. \qquad (67)$$

Now, given $k_0>0$, choose $\tilde c>0$ such that $\tilde c<1/(2c)$ and $\tilde c k^{-(n-2)/3}\le k/3$, for $k\ge k_0$. Then, for $k\ge k_0$, setting $\varepsilon=\tilde c k^{-(n-2)/3}$ in (67), it follows from (67) that

$$J \le c\tilde c^{-1/2} k^{\beta} I;$$

where $\beta = 1/2+(n-2)/6$. Thus (63) holds, which completes the proof.

Remark 5.1. We can show that the bounds established in Theorem 1.8 are sharp in their dependence on k as $k\to\infty$. Let $0\neq\phi\in D(\Gamma)$ be independent of k. Then, by (62), and since $a_N(\phi,\phi)=(T_k\phi,\phi)_{L^2(\Gamma)}$ and $\hat\phi(\xi)$ is rapidly decreasing,

$$|(T_k\phi,\phi)_{L^2(\Gamma)}| \ge \frac{1}{2\sqrt2}\int_{\mathbb{R}^{n-1}}\sqrt{|k^2-|\xi|^2|}\,|\hat\phi(\xi)|^2\, d\xi \sim \frac{k}{2\sqrt2}\int_{\mathbb{R}^{n-1}}|\hat\phi(\xi)|^2\, d\xi, \qquad (68)$$

as $k\to\infty$. Also, for every $s\in\mathbb{R}$, $|(T_k\phi,\phi)_{L^2(\Gamma)}|\le\|T_k\phi\|_{H_k^{s-1}(\Gamma)}\|\phi\|_{\tilde H_k^{1-s}(\Gamma)}$ and

$$\|\phi\|^2_{\tilde H_k^{s}(\Gamma)} = \int_{\mathbb{R}^{n-1}}(k^2+|\xi|^2)^s|\hat\phi(\xi)|^2\, d\xi \sim k^{2s}\int_{\mathbb{R}^{n-1}}|\hat\phi(\xi)|^2\, d\xi, \qquad (69)$$

as $k\to\infty$. Combining (68) and (69) we see that, for every $s\in\mathbb{R}$ and $C<1/(2\sqrt2)$, it holds for all sufficiently large k that $|(T_k\phi,\phi)_{L^2(\Gamma)}|\ge C\|\phi\|_{\tilde H_k^{1-s}(\Gamma)}\|\phi\|_{\tilde H_k^{s}(\Gamma)}$, so that

$$\|T_k\phi\|_{H_k^{s-1}(\Gamma)} \ge C\|\phi\|_{\tilde H_k^{s}(\Gamma)};$$

for all k sufficiently large, which demonstrates the sharpness of (16) in the limit $k\to\infty$.

Remark 5.2. We can also show that the bound established in Theorem 1.9 is sharp in its dependence on k as $k\to\infty$, in the case $n=2$. As in Remark 4.2, let $g(x) := e^{\ldots}$ 34 - g 0 G -/F36 9.9626 Tf -54.638 4(b)] its dependencesharp

$\|T_k$

independent of k, so that $\hat{\phi}(\xi) = \hat{\psi}(\eta)$, where $\eta = \xi - k\mathbf{d}$. Since $|k^2 - |\xi|^2| \leq 2k|\eta| + |\eta|^2$, and since $\hat{\psi}(\eta)$ is rapidly decreasing,

$$|a_N(\phi;\phi)| \leq \frac{1}{2}\int_{\mathbb{R}^{n-1}} \sqrt{2k|\eta| + |\eta|^2}\, |\hat{\psi}(\eta)|^2 \, d\eta \sim \frac{k^{1/2}}{\sqrt{2}} \int_{\mathbb{R}^{n-1}} |\eta|^{1/2} |\hat{\psi}(\eta)|^2 \, d\eta, \tag{70}$$

as $k \to \infty$. Further,

$$\|\phi\|^2_{\tilde{H}_k^{-1/2}(\Gamma)} = \int_{\mathbb{R}^{n-1}} \left(2k^2 + 2k\,\eta\cdot\mathbf{d} + |\eta|^2\right)^{-1/2} |\hat{\psi}(\eta)|^2 \, d\eta \sim \frac{1}{\sqrt{2}k} \int_{\mathbb{R}^{n-1}} |\hat{\psi}(\eta)|^2 \, d\eta, \tag{71}$$

as $k \to \infty$. Combining (70) and (71) we see that, for some constant $C > 0$ independent of k,

$$|a_N(\phi;\phi)| \leq C k^{1/2} \|\phi\|^2_{\tilde{H}_k^{-1/2}(\Gamma)},$$

for all sufficiently large k. This demonstrates the sharpness of (17) in the limit $k \to \infty$, for the case $n = 2$. In the case $n = 3$ it may be that (17) holds with the value of β increased from $-2/3$ to $-1/2$, i.e., to its value for $n = 2$.

6. Norm estimates in $H^{-1/2}(\Gamma)$

In this section we derive k-explicit estimates of the norms of certain functions in $H^{-1/2}(\Gamma)$, which are of relevance to the numerical solution of the Dirichlet boundary value problem D, when it is solved via the integral equation formulation (11). For an application of the results presented here see [21].

The motivation for the estimates we prove in Lemma 6.1 below comes

Given an estimate of $\|v\|_{\tilde{H}_k^{1/2}(\ )}$, an estimate of $|I|$ follows from

$$|I| = |\langle w, \nabla\rangle_{H^{1/2}(\ )\ \tilde{H}^{-1/2}(\ )}| \quad \|w\|_{H_k^{1/2}(\ )}\|v\|_{\tilde{H}_k^{1/2}(\ )};$$

provided we can bound $\|w\|_{H_k^{1/2}(\ )}$. We now do this for the choices of w noted above.

Lemma 6.1. Let $k > 0$, let be an arbitrary nonempty relatively open subset of $_1$, and let $L := \operatorname{diam}$.

(i) Let $d \in \mathbb{R}^n$ with $|d| \quad 1$. Then, for $s \quad 0$, there exists $C_s > 0$, dependent only on s, such that

$$\|e^{ikd\cdot(\ )}\|_{H_k^s(\ )} \quad C_s L^{(n \quad 1 \quad 2s)/2}(1 + kL)^s: \tag{73}$$

(ii) Let $x \in D := \mathbb{R}^n \ \overline{n}$. Then there exists $C > 0$, independent of k, , and x, such that

k

and, using that $|F(z)|$ is monotonic for $z > 0$,

$$\|u - r_L\|^2_{L^2(\mathbb{R}^{n-1})} \le Ck^{2n-4}d^{n-3}|F(kd/2)|^2 + L^{-2}\left(\int_{S_{5L}\setminus S_d} |F(kr(y))|^2\,dy\right)$$
$$\le Ck^{2n-4}d^{n-3}|F(kd/2)|^2 + Ck^{n-3}L^{-2}\log(5L/d). \qquad (82)$$

Combining (78) and (80)–(82), we see that, for $d < 4L$,

$$\|u\|_{H^1_k(\Gamma)} \le Ck^{(n-3)/2}(k + L^{-1})\log^{1/2}(5L/d) + Cd^{(1-n)/2} + Ck^{n-2}d^{(n-3)/2}|F(kd/2)|. \qquad (83)$$

Now, in the case $n = 3$, for which $|F(kd/2)| \le C/(kd)$, it follows from (79) and (83), and noting that $\log^{1/2}(5L/d) \le CP_3(L/d) \le CL/d$ for $4L > d$, that

$$\|u\|_{H^1_k(\Gamma)} \le \frac{C}{d}\left(1 + kd\,P_3(L/d)\right). \qquad (84)$$

For $n = 2$, $F(kd/2) \le C\log(2+(kd)^{-1})(1+kd)^{-1/2} \le C(kd)^{-1/2}$, by (42) and (45). Hence, and by (79) and (83) and as $\log^{1/2}(5L/d) \le CP_2(L/d) \le CL/d$ for $4L > d$,

$$\|u\|_{H^1_k(\Gamma)} \le \frac{C}{d^{1/2}}\left((kL)^{-1/2} + \log(2+(kd)^{-1}) + (kd)^{1/2}P_2(L/d)\right). \qquad (85)$$

Part (ii) of the lemma then follows from (76), (84), and (85).

As an application we use Lemma 6.1 to prove a k-explicit pointwise bound on the solution of the sound-soft screen scattering problem considered in Example 1.3.

Corollary 6.2. The solution u of problem D, with $g_D = -u^i|_\Gamma$, satisfies the pointwise bound

$$|u(x)| \le \begin{cases} C\sqrt{kL}\sqrt{1+kL}\left(\dfrac{1}{kd} + P_3(L/d)\right), & n = 3;\\[2ex] C\sqrt{1+kL}\left(\dfrac{1}{\sqrt{kd}}\left(\dfrac{1}{\sqrt{kL}} + \log\left(2 + \dfrac{1}{kd}\right)\right) + P_2(L/d)\right), & n = 2;\end{cases}$$

where $x \in D$, $d := \mathrm{dist}(x, \Gamma)$, $L := \mathrm{diam}\,\Gamma$, and $C > 0$ is independent of

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