

Department of Mathematics and Statistics

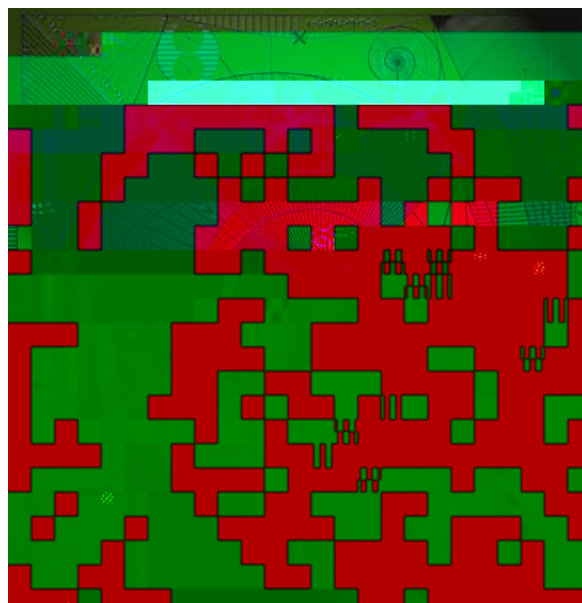
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Resolution of sharp fronts in the presence of model error in variational data assimilation

by

M.A. Freitag, N.K. Nichols and C.J. Budd



solution, fronts are resolved more accurately than with the standard ℓ_2 -norm regularisation of 4DVar.

The aim of this paper is to examine the potential benefits of using ℓ_1 -norm regularisation in variational data assimilation. It presents a preliminary study showing that the method has potential to give improvement over existing approaches. Further investigation remains to be done in order to evaluate the technique in an operational setting.

Section 2 gives an introduction to 4DVar and shows its relation to Tikhonov regularisation. In Section 3 we introduce the new algorithm and in Section 4 we explain how we solve the ℓ_1 -norm regularisation problem and the mixed TV ℓ_1 - ℓ_2 -norm regularisation problem. In Section 5 we state the model equations. Section 6 presents numerical examples, where the new ℓ_1 -norm regularisation is compared to standard 4DVar. In our examples we introduce several kinds of model error. Under these conditions it can be seen that ℓ_1 -norm regularisation outperforms 4DVar when sharp fronts are present (see Sections 6). We conclude with a section on future work.

2 4DVar and its relation to Tikhonov regularisation

In nonlinear 4DVar we aim to minimise the objective function

$$\mathbf{J}(x) = \frac{1}{2}(x - x^b)^T B^{-1} (x - x^b) + \frac{1}{2} \sum_{i=1}^N$$

are vectors whereas $\mathbf{A}_i, i = 0 \dots N$ are square matrices of the dimension of the system state.

The approximate Hessian $\widetilde{\mathbf{H}}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$ are

Firstly, in this paper we consider the effects of -

subject to

$$u = \text{ and } u \geq 0 \quad (25)$$

where

$$\begin{aligned} u &= \begin{bmatrix} - \\ - \end{bmatrix} \quad \lambda = \begin{bmatrix} 2(\phi^T \phi + \mu^2 I) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -2\phi^T \\ 1 \\ 1 \end{bmatrix} \quad = \begin{bmatrix} C_B^{I^2} & -I & I \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & I \end{bmatrix} \quad = - \quad x^b \end{aligned}$$

and the block matrices I and 0 as well as the vectors 1 of all ones in the matrices λ , λ , and λ are of appropriate size. The objective function in (24) is convex as λ is symmetric positive semi-definite. In order to solve the quadratic programming problem (24) with constraints (25) we use the MATLAB in-built function `quadprog.m`.

In the following section we consider a square wave advected using the linear advection equation as an example. We use a ‘true’ model (from which we take the observations) and another model, which is different from the truth and hence introduces a model error. The different models we use are introduced in the next section. In all examples we observe that the new edge-preserving mixed TV - γ_2 -norm regularisation indeed gives better results than the standard γ_2 -norm approach and the simple γ_2 -norm regularisation.

In all the examples we keep the regularisation parameter μ fixed, as we are only investigating the influence of the norm in the regularisation term, but not the size of the regularisation parameter μ .

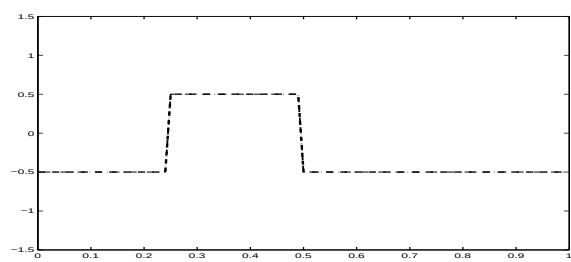
5 Models

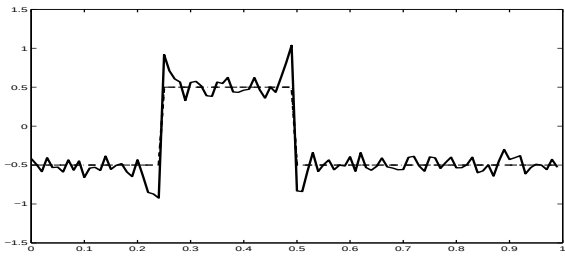
In this section we consider the problem

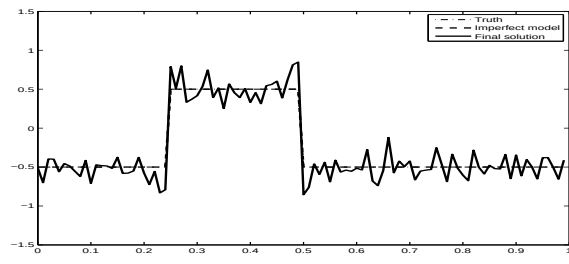
$$t + [\quad]_x = 0 \quad (26)$$

where $[\quad]$ is given by

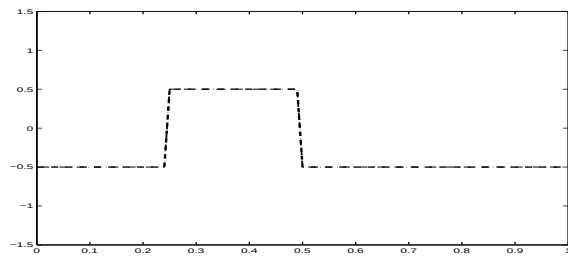
$$[\quad] = \quad (27)$$







is obtained if we use the correct initial conditions and the (imperfect) model. It represents the best solution that we



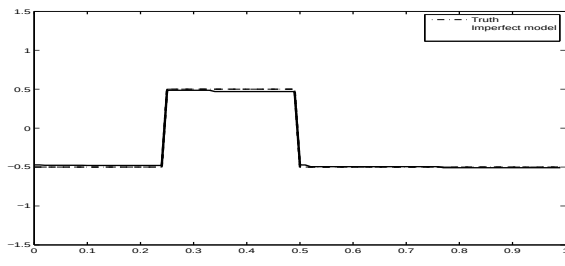
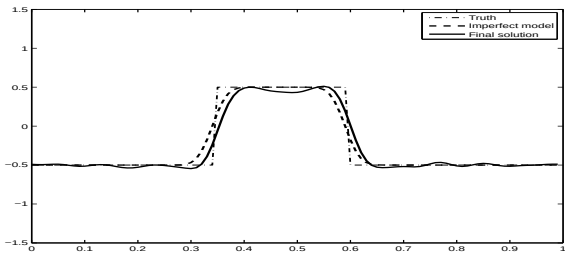
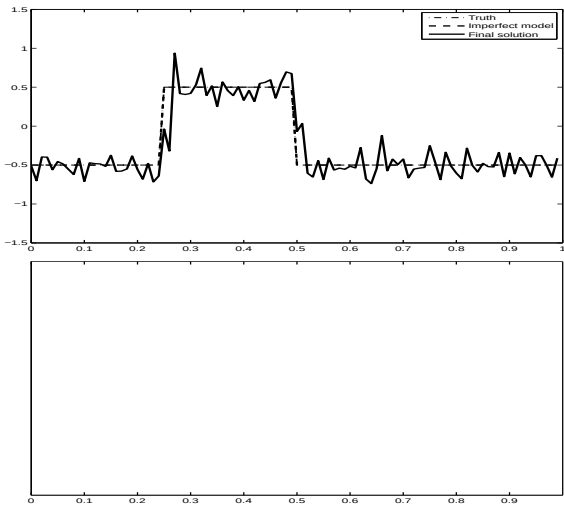


Table I. Comparison between errors in the analysis in standard 4DVar, L_1 -norm regularisation and mixed TV L_1 - L_2 -norm regularisation measured in the L_2 -norm



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