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by

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Abstract. In this paper we develop and apply methods for the spectral analysis of non-self-adjoint tridiagonal in nite and nite random matrices, and for the spectral analysis of analogous deterministic matrices which are pseudo-ergodic in the sense of E. B. Davies (Commun. Math. Phys. 216 (2001), 687{704). As a major application to illustrate our methods we focus on the \hopping sign model" introduced by J. Feinberg and A. Zee (Phys. Rev. E 59 (1999), 6433{6443), in which the main objects of study are random tridiagonal matrices which have zeros on the main diagonal and random ± 1's as the other entries. We explore the relationship between spectral sets in the nite and in nite matrix cases, and between the semi-in nite and bi-in nite matrix cases, for example showing that the numerical range and p-norm ″-pseudospectra (″ > 0, $p \in [1;\infty]$) of the random nite matrices converge almost surely to their in nite matrix counterparts, and that the nite matrix spectra are contained in the in nite matrix spectrum . We also propose a sequence of inclusion sets for which we show is convergent to , with the n"

is the order n tridiagonal matrix given, for $n \geq 2$, by

$$A_n^b = \begin{pmatrix} 0 & 1 & & & \\ b_1 & 0 & 1 & & \\ & b_2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & b_{n-1} & 0 \end{pmatrix},$$

where $b = (b_1, \ldots, b_{n-1}) \in \mathbb{C}^{n-1}$ and each $b_j = \pm 1$. (For $n = 1$ we set $A_n^b = (0)$.)

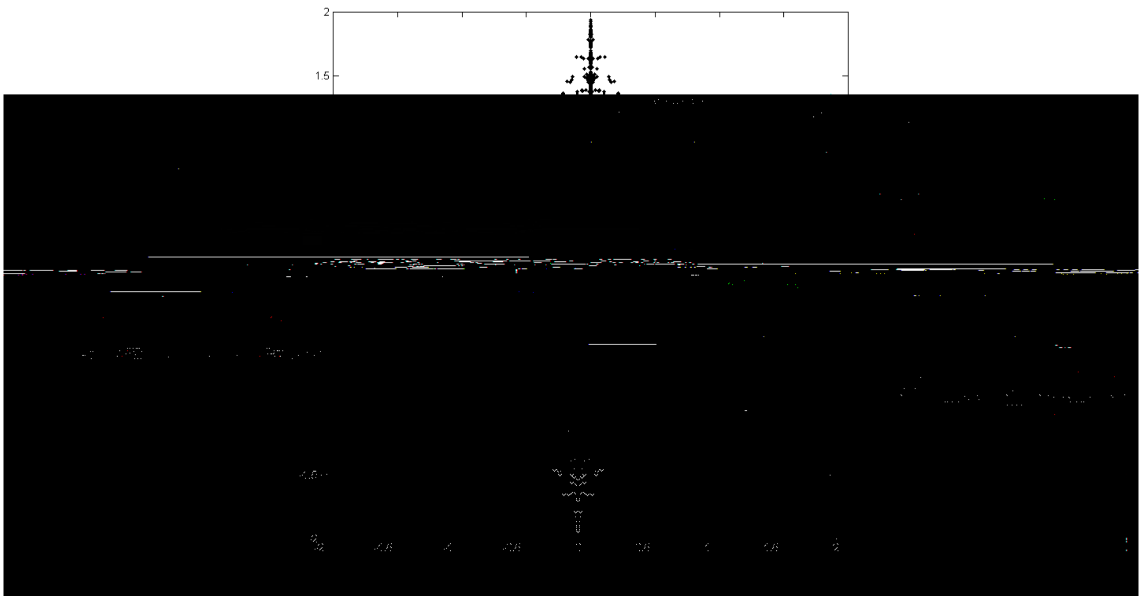


Figure 1: A plot of spec A_n^b, the set of eigenvalues of A_n^b, for a randomly chosen $b \in \{\pm 1\}^{n-1}$, with $n = 5000$ and the components b_j of b independently and identically distributed, with each b_j equal to 1 with probability 1/2. Note the symmetry about the real and imaginary axes by Lemma 3.4 below, and that the spectrum is contained in the square with corners at ± 2 and ± 2i by Lemma 3.1 below.

The objectives we set ourselves in this paper are to understand the behaviour of the spectrum and pseudospectrum of the matrix A_n^b, the spectrum and pseudospectrum of the corresponding semi-infinite and bi-infinite matrices, and the relationship between these spectral sets in the finite and infinite cases. Emphasis will be placed on asymptotic behaviour of the spectrum and pseudospectrum of the finite matrix A_n^b as $n \to \infty$, and we will be interested particularly in the case when the b_j are random variables, for example independent and identically distributed (iid), with $\Pr(b_j = 1) = 0.5$ for each j. (A visualisation of spec A_n^b for a realisation of this random matrix with $n = 5000$ is shown in Figure 1; cf. [17].) To be more precise, we will focus on the case when the vector $b \in \{\pm 1\}^{n-1}$ is the first $n-1$ terms of an infinite sequence $(b_1, b_2, \ldots)$, with each $b_j = \pm 1$, which is *pseudo-ergodic* in the sense introduced by Davies [14], which simply means that every finite sequence of ± 1's appears somewhere in $(b_1, b_2, \ldots)$ as a consecutive sequence. If the b_j are random variables then, for a large class of probability distributions for the b_j, in particular if each b_j is iid with $\Pr(b_j = 1) \in (0, 1)$ for each j, it is clear that the sequence $(b_1, b_2, \ldots)$ is pseudo-ergodic almost surely (with probability one). Thus, although pseudo-ergodicity is a purely deterministic property, our results assuming pseudo-ergodicity have immediate and significant corollaries for the case when A_n^b is a random matrix.

acts on $\ell^p(\mathbb{Z})$, again focusing on the case when $b \in \{\pm 1\}^{\mathbb{Z}}$ is pseudo-ergodic. The action of A^b is

A key result we obtain on the spectra of our infinite matrices, in large part through limit operator arguments described in Section 2, is the following (cf. [14]): if $b, c, d \in \{\pm 1\}^{\mathbb{N}}$, $\tilde{b}, e, \tilde{d} \in \{\pm 1\}^{\mathbb{Z}}$, and b, $\tilde{b}$, cd, and $e\tilde{d}$ are all pseudo-ergodic, then

$$\operatorname{spec} A_+^b = \operatorname{spec} A_+^{c,d} = \operatorname{spec} A^{\tilde{b}} = \operatorname{spec} A^{e,\tilde{d}} = \Sigma := \bigcup_{e \in \{\pm 1\}^{\mathbb{Z}}} \operatorname{spec} A^e = \bigcup_{e \in \{\pm 1\}^{\mathbb{Z}}} \operatorname{spec}_{\mathrm{point}}^{\infty} A^e. \quad (5)$$

One surprising aspect of this formula is that the semi-infinite and bi-infinite matrices share the same spectrum, in contrast to many of the cases discussed in [39], this connected to the symmetries that we explore in Section 3.

We do not know a simple test for membership of the set Σ given by this characterisation (though see Figures 2 and 3 below for plots of known subsets of Σ, and see Section 4.3 for an algorithm for computing approximations to Σ). But this result implies that $\operatorname{spec} A^b \subset \Sigma$ for every $b \in \{\pm 1\}^{\mathbb{Z}}$ which gives the possibility of determining subsets of Σ by computing $\operatorname{spec} A^b$ for particular choices of b. In particular, as recalled in Section 2, when b is n-periodic for some $n \in \mathbb{N}$, i.e. $b_{j+n} = b_j$ for $j \in \mathbb{Z}$, $\operatorname{spec} A^b$ can be computed by calculating eigenvalues of an order n matrix (a periodised version of A_n^b). We compute $\pi_n \subset \Sigma$, for $n = 5, 10, \ldots, 30$ in Section 2, where π_n denotes the union of $\operatorname{spec} A_b$ over all n-periodic $b \in \{\pm 1\}^{\mathbb{Z}}$. We speculate at the end of the paper that

$$\pi_\infty := \bigcup_{n \in \mathbb{N}} \pi_n \quad (6)$$

is dense in Σ, and it has been shown recently in [9] that certainly π_∞ is dense in the unit disc $\mathbb{D} = \{z : |z| < 1\}$, which implies that $\overline{\mathbb{D}} \subset \Sigma$, as established slightly earlier directly from (5) in [5]. (Throughout, $\overline{S}$ denotes the closure of a set $S \subset \mathbb{C}$: for an element $z \in \mathbb{C}$, $\bar{z}$ denotes the complex conjugate.)

To obtain a first upper bound on Σ we compute the ℓ^2-numerical range, $W(A^b)$, of A^b when b is pseudo-ergodic. We show that, if $b, c, d \in \{\pm 1\}^{\mathbb{N}}$, $\tilde{b}, e, \tilde{d} \in \{\pm 1\}^{\mathbb{Z}}$, and b, $\tilde{b}$, cd, and $e\tilde{d}$ are all pseudo-ergodic, then

$$W(A_+^b) = W(A_+^{c,d}) = W(A^{\tilde{b}}) = W(A^{e,\tilde{d}}) = \Delta := \{z = a + \mathrm{i}b : a, b \in \mathbb{R}, |a| + |b| < 2\}.$$

Since the spectrum is necessarily contained in the closure of the numerical range, this implies that

$$\overline{\mathbb{D}} \subset \Sigma \subset \overline{\Delta}.$$

We point out that the numerical range of A_n^b converges to that of A^b, in particular that $W(A_n^b) \nearrow \Delta$ as $n \to \infty$, if b is pseudo-ergodic. (Here and throughout, for $T_n \subset \mathbb{C}$ and $T \subset \mathbb{C}$, the notation $T_n \nearrow T$ means that $T_n \subset T$ for each n and that $\operatorname{dist}(T, T_n) \to 0$ as $n \to \infty$, with $\operatorname{dist}(T, T_n)$ the Hausdorff distance defined in (16) below.)

The largest part of the paper (Section 4) is an investigation in

We can prove neither of these last two conjectures about spectral asymptotics. On the other hand, our theoretical results for the pseudospectrum are fairly complete. We show first in Theorem 3.6 a pseudospectral version of (5), that, if $b, c, d \in \{\pm 1\}^{\mathbb{N}}$, $\tilde{b}, \tilde{c}, \tilde{}$

implications in a final Theorem 5.1, in the same section summarising succinctly what we have established about the spectral sets and ${}^{p}_{\varepsilon}$ (Theorem 5.2), and outlining a number of open problems.

In the course of this investigation, focused on a particular operator and matrix class, we develop results for the larger classes of tridiagonal or banded finite and infinite matrices. In particular, Theorem 4.4 shows that, for $p \in [1, \infty]$, $\varepsilon > 0$, the ℓ^p ε-pseudospectrum of a general, semi-infinite tridiagonal matrix is contained, for $\varepsilon' > \varepsilon$, in the ℓ^p ε'-pseudospectrum of its $n \times n$ finite section if n is sufficiently large. It also shows corresponding results relating the pseudospectra of a general bi-infinite matrix to that of its finite sections. In Section 2 we employ recent work [7, 8] on limit operator methods for the study of spectral sets for very general classes of infinite matrices. We make explicit in Theorems 2.1 and 2.9

important property of the lower norm is that

$$|\nu(A) - \nu(B)| \le \|A - B\|, \tag{10}$$

for any bounded linear operators A and B on X.

In the case when, for some $N \in \mathbb{N}$, $X = \mathbb{C}^N$ and B is an $N \times N$ matrix, (i)-(v) are equivalent additionally to $\mathrm{spec}_\varepsilon B = \{\lambda \in \mathbb{C} : \nu(B - \lambda I) < \varepsilon\} = \mathrm{spec}_{\mathrm{point},\varepsilon} B$. If $\|\cdot\| = \|\cdot\|_2$, then, for every $N \times N$ matrix A, $\nu(A) = s_{\min}(A)$, the smallest singular value of A. Thus these definitions are additionally equivalent to [40]

$$\mathrm{spec}_\varepsilon B = \{\lambda \in \mathbb{C} : s_{\min}(B - \lambda I) < \varepsilon\}. \tag{11}$$

Note that (10) implies that

$$|s_{\min}(B - \lambda I) - s_{\min}(B - \mu I)| \le |\lambda - \mu|, \quad \lambda, \mu \in \mathbb{C}. \tag{12}$$

It is equation (11) that we use for the numerical computations of pseudospectra in Section 4.3.

An alternative definition of the pseudospectrum is to replace the strict inequality $>$ in (i) by $\ge$, so that the ε-pseudospectrum is defined to be

$$\mathrm{Spec}_\varepsilon B = \mathrm{spec}\, B \cup \{\lambda \in \mathbb{C} : \|(B - \lambda I)^{-1}\| \ge \varepsilon^{-1}\}.$$

This has the attraction that $\mathrm{Spec}_\varepsilon B$, like $\mathrm{spec}\, B$, is a compact set for $\varepsilon > 0$. An interesting question is whether $\overline{\mathrm{spec}_\varepsilon B} = \mathrm{Spec}_\varepsilon B$, which hinges on the question of whether or not it is possible for the norm of the resolvent of B, $\|(B - \lambda I)^{-1}\|$, to take a finite constant value on a open set $G \subset \mathbb{C}$. Let us say that the Banach space X has the *strong maximum property* if, for every open set $G \subset \mathbb{C}$, every bounded linear operator B on X, and every $M > 0$, it holds that

$$(\|(B - \lambda I)^{-1}\| \le M, \ \forall \lambda \in G) \ \Rightarrow \ (\|(B - \lambda I)^{-1}\| < M, \ \forall \lambda \in G).$$

If X has the strong maximum property, then no bounded linear operator on X can have a resolvent norm with a constant finite value on an open subset of $\mathbb{C}$, and it is easy to see that

For $S, T \subset \mathbb{C}$, let

$$\operatorname{dist}(S, T) := \max(\sup\{\operatorname{dist}(z, S) : z \in T\}, \sup\{\operatorname{dist}(z, T) : z \in S\}). \tag{16}$$

(This notion of distance, when applied to compact subsets of $\mathbb{C}$, is an instance of the Hausdorff distance between compact subsets of a metric space.) Given a sequence $T_n \subset \mathbb{C}$ and $T \subset \mathbb{C}$, let us write $T_n \to T$ if $\operatorname{dist}(T_n, T) \to 0$ as $n \to \infty$. Additionally, let us write $T_n \nearrow T$ if $T_n \to T$ and $T_n \subset T$ for each n, and write $T_n \searrow T$ if $T_n \to T$ and $T \subset T_n$ for each n. It is an easy calculation to show that

$$\operatorname{spec}_\varepsilon B \searrow \operatorname{spec} B \quad \text{as} \quad \varepsilon \to 0^+. \tag{17}$$

Similarly, it holds for $\varepsilon > 0$ that $\operatorname{spec}_{\varepsilon'} B \searrow \operatorname{Spec}_\varepsilon B$, as $\varepsilon' \to \varepsilon^+$, and $\operatorname{spec}_{\varepsilon'} B \nearrow \operatorname{spec}_\varepsilon B$, as $\varepsilon' \to \varepsilon^-$. Thus, in the case where X has the strong maximum property so that $\overline{\operatorname{spec}_\varepsilon B} = \operatorname{Spec}_\varepsilon B$, it holds for $\varepsilon > 0$ that

$$\operatorname{spec}_{\varepsilon'} B \searrow \operatorname{spec}_\varepsilon B, \text{ as } \varepsilon' \to \varepsilon^+, \quad \text{and} \quad \operatorname{spec}_{\varepsilon'} B \nearrow \operatorname{spec}_\varepsilon B, \text{ as } \varepsilon' \to \varepsilon^-, \tag{18}$$

so that $\operatorname{spec}_\varepsilon B$ depends continuously on ε.

The spectrum and ε-pseudospectra are connected to the numerical range. In the case that X is a Hilbert space with inner product $(\cdot, \cdot)$, and where B is a bounded linear operator on X, the and note that $V_j M_b = M_{V_j b} V_j$, for $j \in \mathbb{Z}$, $b \in \ell^\infty(\mathbb{Z})$. In terms of these notations, the operators A^b and $A^{b,c}$, corresponding to the infinite matrices (

which is not only bounded but also quasi-periodic, i.e. for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, $x_{k+n} = \lambda x_k$, $k \in \mathbb{Z}$. It is easy to see that this implies that

$$\operatorname{spec} A^{b,c} = \bigcup_{|\lambda|=1} \operatorname{spec}\left(A_n^{b,c} + B_{n,\lambda}^{b,c}\right), \tag{23}$$

where $A_n^{b,c}$ is given by (3) (with $A_1^{b,c} := (0)$) and $B_{n,\lambda}^{b,c}$ is the $n \times n$ matrix whose entry in row i, column j is $\delta_{i,n}\delta_{j,1}\, c_n \lambda + \delta_{i,1}\delta_{j,n}\, \lambda^{-1} b_n$, where δ_{ij} is the Kronecker delta. We will abbreviate $B_{n,\lambda}^{b,c}$ as $B_{n,\lambda}^{b}$ in the case that $c = (1, \dots, 1)$.

An important case where A^b is self-similar is where A^b is *pseudo-ergodic* in the sense of Davies [14]. The following is a specialisation of the definition from [14].

Definition 2.2 *Call $b \in \{\pm 1\}^{\mathbb{Z}}$ and the operator A^b* pseudo-ergodic *if, for every $N \in \mathbb{N}$ and every $w \in \{\pm 1\}^N$, there exists $J \in \mathbb{Z}$ such that $b_{n+J} = w_n$, for $n = 1, \dots, N$.*

We see from this definition that A^b is pseudo-ergodic if and only if every finite sequence of ± 1's appears somewhere in the bi-infinite sequence b. The significance of this definition is that, for many cases where the entries b_n are random variables, the sequence b is pseudo-ergodic with probability one. In particular, the following lemma follows easily from the Second Borel Cantelli Lemma (e.g. [3, Theorem 8.16]), the argument sometimes called the 'Infinite Monkey Theorem'.

Lemma 2.3 *If the matrix entries b_n, for $n \in \mathbb{Z}$, are iid random variables taking the values ± 1 with $\Pr(b_n = 1) \in (0, 1)$, then A^b is pseudo-ergodic with probability one.*

The link to limit operators is provided by the following lemma (see [14, Lemma 6], [26, Corollary 3.70] or [8, Theorem 7.6]):

Lemma 2.4 *For $b \in \{\pm 1\}^{\mathbb{Z}}$, A^b is* pseudo-ergodic *if and only if $\sigma^{\mathrm{op}}(A^b) = \{A^c : c \in \{\pm 1\}^{\mathbb{Z}}\}$.*

Combining this lemma with Theorem 2.1 gives the following characterisation of the spectrum and pseudospectrum of A^b in the case when b is pseudo-ergodic:

Theorem 2.5 *If $b \in \{\pm 1\}^{\mathbb{Z}}$ and A^b is pseudo-ergodic, then*

$$\operatorname{spec} A^b = \operatorname{spec}_{\mathrm{ess}} A^b = \bigcup_{c \in \{\pm 1\}^{\mathbb{Z}}} \operatorname{spec} A^c = \Sigma := \bigcup_{c \in \{\pm 1\}^{\mathbb{Z}}} \operatorname{spec}^{\infty}_{\mathrm{point}} A^c \tag{24}$$

and

$$\operatorname{spec}^p_\varepsilon A^b = \Sigma^p_\varepsilon := \bigcup_{c \in \{\pm 1\}^{\mathbb{Z}}} \operatorname{spec}^p_\varepsilon A^c, \tag{25}$$

for $\varepsilon > 0$ and $p \in [1, \infty]$.

Limit operator ideas, the "Infinite Monkey" argument and the validity of the first two "=" signs in (24) are not new in the spectral theory of random matrices (see e.g. [4, 13, 14, 19, 33]). Equation (25) is previously shown, for a general class of pseudo-ergodic operators for the case $p = 2$ in [14]. What is more recent is the third "=" sign iis theo8(2)]TJ -8.712 -3.616 Td [(and)]TJ/F11 9.9626 Tf 19.35 0 Td [

for the pseudospectrum $\mathrm{spec}^p_{\varepsilon} A^b$. In particular, $\mathrm{spec}\, A^c$ if $c \in {}_n$, for some $n \in \mathbb{N}$, where ${}_n := \{c \in \{ 1\}^{\mathbb{Z}} : c$ is n-periodic$\}$. Thus

$$ {}_n := \bigcup_{c \in {}_n} \mathrm{spec}\, A^c = \bigcup_{c \in {}_n} \mathrm{spec}^{1}_{\mathrm{point}}\, A^c ; \qquad (26)$$

for every $n \in \mathbb{N}$: this is informative as ${}_n$ can be computed explicitly by (23) as the union of

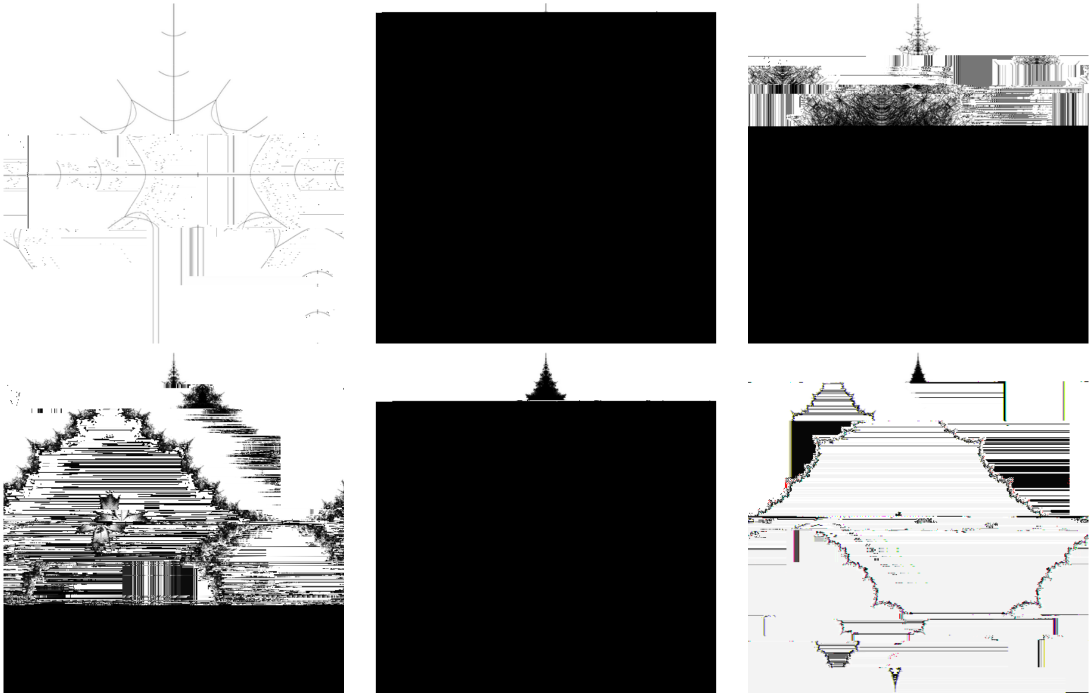


Figure 2: Our figure shows the sets $_n$, as defined in (26), for $n = 5, 10, \ldots, 30$, computed using the characterisation (23), which is made explicit for $n = 1, 2$ and 3 in Lemma 2.6. In particular, $_1 = [-2, 2] \cup \mathrm{i}[-2, 2]$ and, for each n, $_1 \subset \,_n$ and, by Lemma 3.1, $_n \subset \overline{} = \{x + \mathrm{i}y : x, y \in \mathbb{R}, |x| + |y| \leq 2\}$.

Theorem 2.7 *[5, Proposition 2.1]* $\overline{\mathbb{D}}$.

Recently [9], an alternative proof of this theorem has been obtained, through a construction that shows that $_1$ is dense in $\mathbb{D}$. It is an open (and interesting) question as to whether $_1$ is dense in . An interesting, related, case where the union of the spectra of all periodic operators is shown to be dense in the spectrum of the pseudo-ergodic case is studied in [29], but there are other pseudo-ergodic bi-infinite tridiagonal examples where this is not true.

The above results concern bi-infinite matrices, but similar results apply to the semi-infinite matrices A_+^b and $A_+^{b,c}$. We say that the operator induced by the bi-infinite matrix $B = (b_{ij})_{i,j \in \mathbb{N}}$ is a *limit operator* of the operator induced by the banded semi-infinite matrix $A_+ = (a_{ij})_{i,j \in \mathbb{N}}$ if, for a sequence $h_1, h_2, \ldots$ of integers with $h_k \to +\infty$, it holds that

$$a_{i+h_k, j+h_k} \to b_{ij} \mathrm{A}$$

3 The Numerical Range and Symmetry Arguments

Let us first introduce some properties of and notation related to adjoint operators. Given a banded bi-infinite matrix $A = (a_{ij})_{i,j \in \mathbb{Z}}$, with $\sup_{ij} |a_{ij}| < \infty$, A^* will denote th-354(denot 0 Td [19sr9307.9626 Tf 6.345 0 Td [

that $W(A^b$

Lemma 3.3 *For $n \in \mathbb{N}$, $a \in \{\pm 1\}^n$, and $b, c \in \{\pm 1\}^{n-1}$,*

$$D_n^a A_n^{b,c} D_n^a = A_n^{bd,cd},$$

where $d = (a_1 a_2, \ldots, a_{n-1} a_n)$, so that

$$\operatorname{spec} A_n^{b,c} = \operatorname{spec} A_n^{bd,cd} = \operatorname{spec} A_n^{bc}.$$

Further, for $p \in [1, \infty]$ and $\varepsilon > 0$, where $q \in [1, \infty]$ is given by $p^{-1} + q^{-1} = 1$,

$$\operatorname{spec}_\varepsilon^p A_n^{b,c} = \operatorname{spec}_\varepsilon^p A_n^{bd,cd} = \operatorname{spec}_\varepsilon^p A_n^{bc} = \operatorname{spec}_\varepsilon^q A_n^{b,c}.$$

Moreover, for $1 \le p \le r \le 2$ and $\varepsilon > 0$, $\operatorname{spec}_\varepsilon^r A_n^{b,c} \subset \operatorname{spec}_\varepsilon^p A_n^{b,c}$.

A first application of the above lemmas is the following symmetry result (cf. [22]).

Lemma 3.4 *For $b \in \{\pm 1\}^{\mathbb{Z}}$, $\varepsilon > 0$, and $p \in [1, \infty]$, $\operatorname{spec} A^b$, $\operatorname{spec}_{\mathrm{ess}} A^b$, $\operatorname{spec}_\varepsilon^p A^b$, $\operatorname{spec} A_n^b$, and $\operatorname{spec}_\varepsilon^p A_n^b$ are invariant under reflection in the real and imaginary axes. Further, where $S(b)$ denotes any one of these sets, it holds that $S(-b) = \mathrm{i} S(b)$. The set Σ, which is the set $\operatorname{spec} A^b = \operatorname{spec}_{\mathrm{ess}} A^b$ in the case that b is pseudo-ergodic, and, for $\varepsilon > 0$ and $p \in [1, \infty]$, the set Σ_ε^p, which is the set $\operatorname{spec}_\varepsilon^p A^b$ for b pseudo-ergodic, are invariant under reflection in either axis and under rotation by 90^0.*

Proof. We prove the results for A^b using Lemma 3.2; the proof for A_n^b using Lemma 3.3 is similar. That the entries of the matrix A^b are real implies the symmetry about the real axis. Defining $a \in \{\pm 1\}^{\mathbb{Z}}$ by $a_k = (-1)^k$, $k \in \mathbb{Z}$, so that $d = aV_{-1}a$ is the constant sequence $d = (\ldots, -1, -1, \ldots)$, it follows from Lemma 3.2 that $M_a A^b M_a^{-1} = -A^b$, which implies that the sets $\operatorname{spec} A^b$, $\operatorname{spec}_{\mathrm{ess}} A^b$, and $\operatorname{spec}_\varepsilon^p A^b$ are also invariant under reflection in the origin, so that they are also invariant under reflection in the imaginary axis. Defining, instead, $a \in \ell^\infty(\mathbb{Z})$ by $a_k = \mathrm{i}^k$, we obtain, similarly, that $M_a A^b M_a^{-1} = \mathrm{i} A$

Similarly, for $x \in \ell^2_e(\mathbb{Z})$ and $k \in \mathbb{N}$, $(A^{b;c}x)_{-k} = b_{-k-1}x_{-k-1} + c_{-k}x_{-k+1} = c_k x_{k+1} + b_{k-1}x_{k-1} = (A^{b;c}x)_k$, so that $A^{b;c} : \ell^p_e(\mathbb{Z}) \to \ell^p_e(\mathbb{Z})$. Further, for $k \in \mathbb{N}$, $(EA^b_+ x)_k = b_{k-1}x_{k-1} + x_{k+1} = b_{k-1}x_{k-1} + c_k x_{k+1} = (A^{b;c}_o x)_k$, so that (29) holds. Since E and P are isometric isomorphisms and $E = P^{-1}$, it follows that $\operatorname{spec} A^b_+ = \operatorname{spec} A^{b;c}_o$ and that $\operatorname{spec}^p_\varepsilon A_+ = \operatorname{spec}^p_\varepsilon A^{b;c}_o$, for $\varepsilon > 0$ and $p \in [1, \infty]$. The remaining results follow from Lemma 2.8 and Lemma 3.2. ■

Putting the results from the previous section and this section together gives the following characterisations of the spectrum, essential spectrum, and pseudospectrum in the pseudo-ergodic case.

Theorem 3.6 *If $b, c, d \in \{\pm 1\}^{\mathbb{N}}$, $e, f, g \in \{\pm 1\}^{\mathbb{Z}}$, and b, cd, e, and fg are pseudo-ergodic, then $\operatorname{spec} A^b_+ = \operatorname{spec} A^{c;d}_+ = \operatorname{spec} A^e = \operatorname{spec} A^{f;g} = \operatorname{spec}_{\mathrm{ess}} A^b_+ = \operatorname{spec}_{\mathrm{ess}} A^{c;d}_+ = \operatorname{spec}_{\mathrm{ess}} A^e = \operatorname{spec}_{\mathrm{ess}} A^{f;g} = \Sigma$ and, for $\varepsilon > 0$ and $p \in [1, \infty]$, where $q \in [1, \infty]$ is given by $p^{-1} + q^{-1} = 1$, $\operatorname{spec}^p_\varepsilon A^b_+ = \operatorname{spec}^p_\varepsilon A^{c;d}_+ = \operatorname{spec}^p_\varepsilon A^e = \operatorname{spec}^p_\varepsilon A^{f;g} = \Sigma^p_\varepsilon = \Sigma^q_\varepsilon$. Further, for $1 \le p \le r \le 2$ and $\varepsilon > 0$, $\Sigma^r_\varepsilon \subset \Sigma^p_\varepsilon$.*

Proof. From Lemma 3.2

4.1 That the finite matrix spectral sets are contained in the infinite matrix counterparts

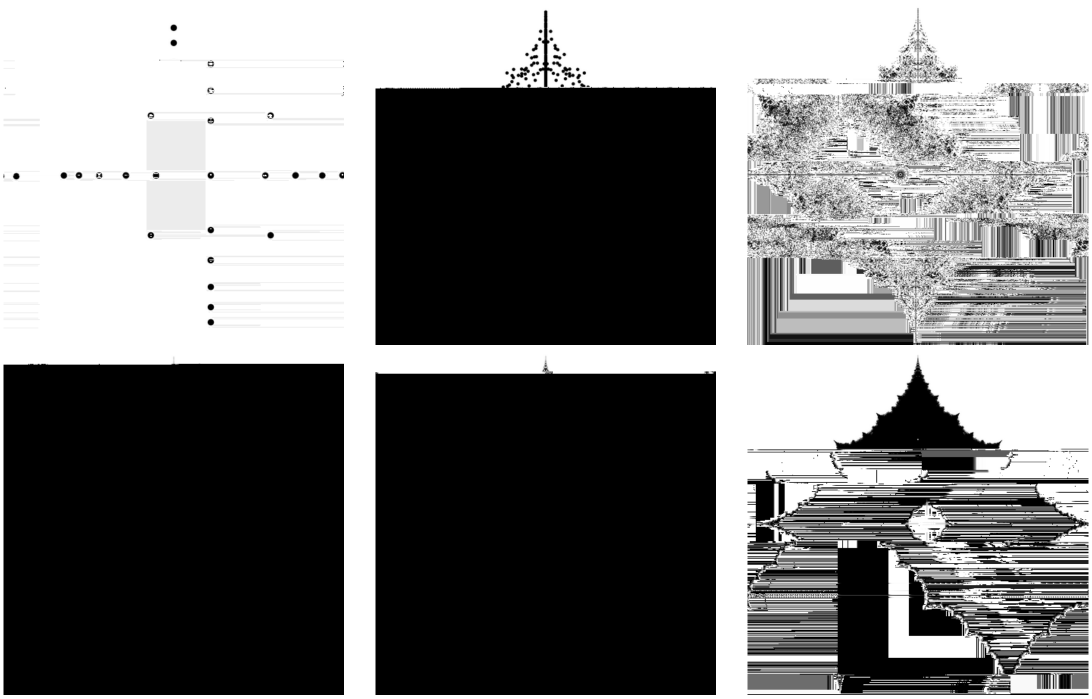
For $n \in \mathbb{N}$, introduce the $n \times n$ matrices

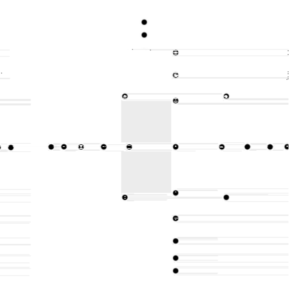
$$I_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \quad \text{and} \quad J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

so that I_n is the order n identity matrix. The proof of the following result uses a similar construction to that of the bi-infinite matrix $A^{b,c}$ in the proof of Lemma 3.5.

Theorem 4.1 *If b is pseudo-ergodic then, for $n \in \mathbb{N}$,*

$$\operatorname{spec} A_n^b \subset \sigma_n := \bigcup_{f \in \{\pm 1\}^{n-1}} \operatorname{spec} A_n^f \subset \pi_{2n+2} \subset \operatorname{spec} A^b = \Sigma.$$





and hence, using repeated re ections, i.e. by putting

$$A^{c;d} := \begin{pmatrix} 0 & \ddots & 1 \end{pmatrix}$$

The inclusion $_n$ $_{2n+2}$ is illustrated for $n = 4$ in Figure 4.

An interesting question, alluded to already in Section 2, is whether $_1$, which is contained in , or $_1$, which is a countable subset of $_1$, are dense in , the spectrum of A^b for b pseudo-ergodic. Of course, we do not know what is, so that this question is di cult to resolve! We do know however (Theorem 2.7) that the unit disc $\mathbb{D}$, and we can consider the question as to whether $_1$ or $_1$ are dense in $\mathbb{D}$. Recall that the sets $_n$, for $n = 5; 10; \ldots; 30$, are plotted already in Figure 2. Studying Figures 2 and 3, it appears that there is a \hole" in both $_n$ and $_n$ around the origin, though these holes appear to be reducing in size as n increases. And in fact, as mentioned already in Section 2, it has been shown recently that $_1$ is dense in $\mathbb{D}$. Further, it appears to us plausible, comparing the two gures, to conjecture that $_1$ is dense in $_1$ and so dense in $\mathbb{D}$.

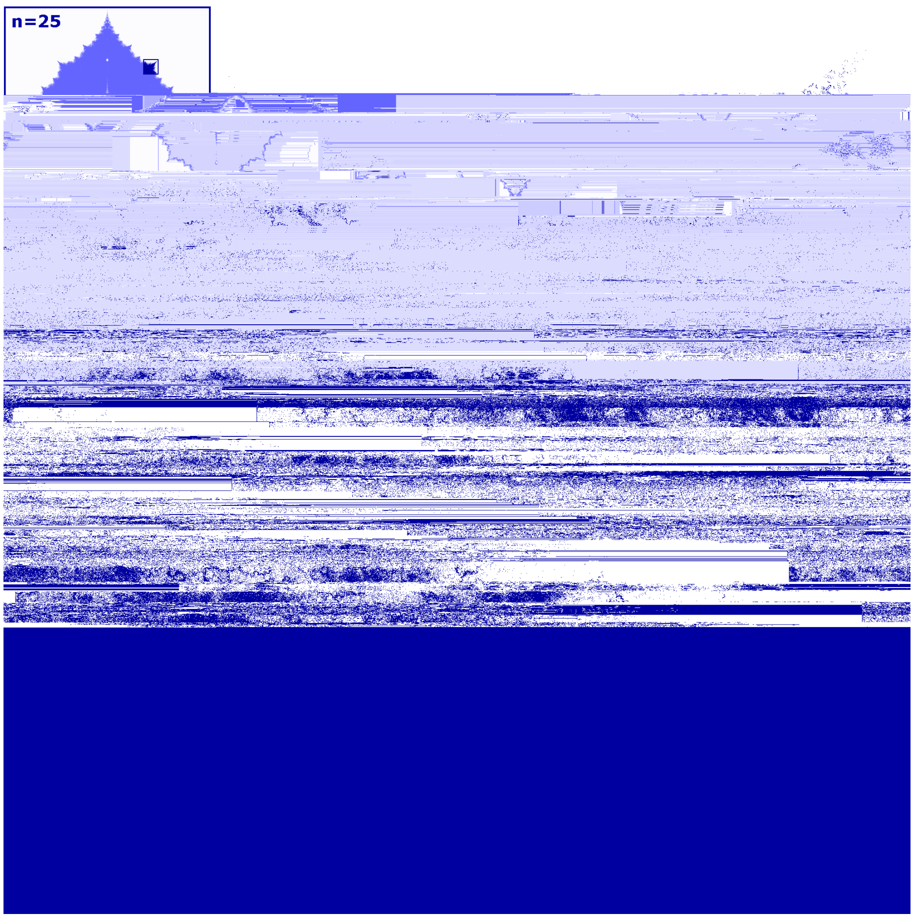


Figure 5: This is a zoom into $_{25}$ { the 5th picture of Figure 3. The location of this zoom is near the point $1 + i$, which is the midpoint of the northeast edge of the square $W(A^b) =$. The picture clearly suggests self-similar features of the set $_{25}$.

Figure 5, taken from [5], zooms into the part of the set $_{25}$ around 1 + i. Intriguingly this set, the collection of all eigenvalues of a set of 2^{24} matrices of size 25 25 (25 $2^{24} = 419; 430; 400$ eigenvalues in all!), appears to have a self-similar structure. We have no explanation for these beautiful geometrical patterns, and it is not clear to us how to gain insight into the geometry of this set.

In the next theorem and corollary we show the analogue of Theorem 4.1 for pseudospectra.

Theorem 4.2 *If b*

4.2 Convergence of the finite matrix spectral sets to their infinite matrix counterparts

As we have remarked at the beginning of this section, it is not clear that the spectrum of a general banded matrix should have anything to do with the spectra of its finite submatrices. In

The previous subsection already provides potential methods for computing these sets. We have that, if $b \in \{\pm 1\}^n$ is pseudo-ergodic, then

$$\Sigma^2_\varepsilon = \lim_{n\to\infty} \mathrm{spec}^2_\varepsilon A^b_n. \tag{33}$$

This then implies, by (18), that

$$\Sigma = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \mathrm{spec}^2_\varepsilon A^b_n. \tag{34}$$

In principle, these equations can be used as the basis of algorithms for computing Σ^p_ε and Σ. In particular, to approximate Σ^2_ε one uses the sequence of sets $\mathrm{spec}^2_\varepsilon A^b_n$, $n = 1, 2, \ldots$, which can be computed as described in Section 1.2. The difficulty with this scheme is that one has no idea of the rate of convergence of $\mathrm{spec}^2_\varepsilon A^b_n$

the notation introduced in Corollary 4.3, it must hold that $\bigcup_{\ell\in\mathbb{Z}}\operatorname{spec}_\epsilon A^b_{\ell;\ell+n-1}\subset\Sigma^{\epsilon}_{n,\epsilon}$, for every $\epsilon>0$. For small values of n, ε_n in the above theorem can be calculated explicitly, in particular

$$\varepsilon_1 = 2 \text{ and } \varepsilon_2 = \sqrt{2}. \tag{35}$$

Example 4.8 As a first example of application of the above theorem, consider the case when $b_m = 1$ for each m. Then $A^b_{\ell;\ell+n-1} = A^b_{1;n} = A^b_n$ for each ℓ. Further, this matrix is self-adjoint, so that $\operatorname{spec}_\epsilon A^b_n = \operatorname{spec} A^b_n + \epsilon\mathbb{D}$, for every $\epsilon > 0$. Thus the statements of the theorem reduce to

$$\operatorname{spec} A^b \subset \operatorname{spec} A^b_n + \varepsilon_n\overline{\mathbb{D}} \quad\text{and}\quad \operatorname{spec}_\epsilon A^b \subset \operatorname{spec} A^b_n + (\epsilon+\varepsilon_n)\mathbb{D}, \quad \epsilon > 0. \tag{36}$$

In this simple case we can compute the above sets explicitly, to check that the above inclusions hold, finding that $\operatorname{spec} A^b = [-2,2]$, $\operatorname{spec}_\epsilon A^b = [-2,2]+\epsilon\mathbb{D}$, and $\operatorname{spec} A^b_n = \left\{2\cos\frac{j\pi}{n+1} : j = 1,\dots,n\right\}$.

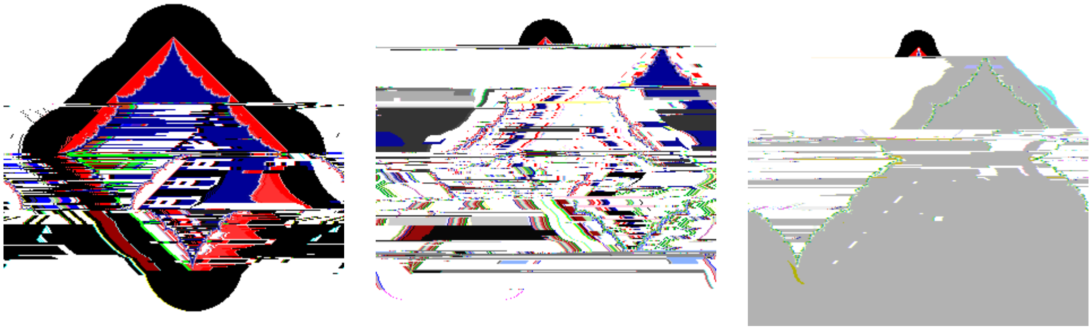


Figure 6: Plots, for $n = 6, 12$ and 18, of the sets $\overline{{}^{2}_{n,\varepsilon_n}}$, which are inclusion sets for $=$ spec A^b, when $b \in \{\pm 1\}^{\mathbb{Z}}$ is pseudo-ergodic. Also shown, overlaid in red, is the square , with corners at ± 2 and ± 2i, which is $W(A^b)$, the numerical range of A^b. Overlaid on top of that in blue is the set $_{30} \cup \mathbb{D}$ which, by definition and Theorem 2.7, is a subset of .

In Figure 6 we plot $\overline{{}^{2}_{n,\varepsilon_n}}$, for $n = 6, 12$, and 18. Each of these sets contains , by Theorem 4.9, and note that each set is invariant under reflection in either axis or under rotation by 90^0, by Lemma 3.4. On the same figure we plot the square $\overline{}$ which, by Lemma 3.1, also contains . It appears that, for n 18, ${}^{2}_{n,\varepsilon_n}$. If this were to hold for all $n \in \mathbb{N}$ then it would follow, from Theorem 4.9, which tells us that $\overline{{}^{2}_{n,\varepsilon_n}}$ & , and Lemma 3.1, which tells us that $\overline{}$, that $= \overline{}$. It seems impossible from these plots to take an educated guess as to whether or not ${}^{2}_{n,\varepsilon_n}$ holds for all n, not least because the convergence rate of $\overline{{}^{2}_{n,\varepsilon_n}}$ to may be slow: Theorem 4.9 tells us that dist$(\overline{{}^{2}_{n,\varepsilon_n}};\ )$ dist$({}^{2}_{\varepsilon_n};\ )$ but it follows from (13) that dist$({}^{2}_{\varepsilon_n};\ )$ ε_n 2 $=(n+2)$.

We have not been able to produce similar plots to those in Figure 6 for much larger values of n because of the large computational cost. But it is feasible to compute $S_n(\ )$ for a single for larger n. We have carried out this computation for $= 1.5 + 0.5$i, a quarter of the way along one of the sides of . Computing in standard double-precision floating point arithmetic we find that

$$S_{34}(1.5 + 0.5\mathrm{i}) = 0.17201954132506\ldots > \varepsilon_{34} = 0.169830415547956\ldots. \tag{42}$$

This implies that $1.5 + 0.5\mathrm{i} \notin {}^{2}_{34,\varepsilon_{34}}$ and so $1.5 + 0.5\mathrm{i} \notin$, which of course implies that is a strict subset of $\overline{}$. In fact, in view of (41) and the symmetries of noted in Lemma 3.2, the inequality (42) implies more, namely that

$(:$

Theorem 5.1 *Suppose that the entries of* $b \in \{\pm 1\}^{\mathbb{Z}}$ *are iid random variables, with* $\Pr(b_m = 1) \in (0,1)$. *Then:*

(i) $\operatorname{spec} A^b$, $\operatorname{spec} A^b_+$, *with* $\operatorname{spec}_{\mathrm{ess}} A^b = \operatorname{spec} A^b = \operatorname{spec}_{\mathrm{ess}} A^b_+ = \operatorname{spec} A^b_+ =$ *almost surely.*

(ii) $W(A^b_+)$ $W(A^b)$, *with* $W(A^b) = W(A^b_+) =$ *almost surely.*

(iii) *For* $n \in \mathbb{N}$, $\operatorname{spec} A^b_n$ *and* $W(A^b_n)$, *and, as* $n \to \infty$, $W(A^b_n) \nearrow$, *almost surely.*

(iv) *For* $\varepsilon > 0$ *and* $p \in [1,\infty]$, $\operatorname{spec}^p_\varepsilon A^b$ $^p_\varepsilon$, $\operatorname{spec}^p_\varepsilon A^b_+$ $^p_\varepsilon$, *with* $\operatorname{spec}^p_\varepsilon A^b = \operatorname{spec}^p_\varepsilon A^b_+ =$ $^p_\varepsilon$ *almost surely.*

(v) *For* $\varepsilon > 0$, $p \in [1,\infty]$, *and* $n \in \mathbb{N}$, $\operatorname{spec}^p_\varepsilon A^b_n$ $^p_\varepsilon$ *and, as* $n \to \infty$, $\operatorname{spec}^p_\varepsilon A^b_n \nearrow$ $^p_\varepsilon$, *almost surely.*

Similarly, if $b, c \in \{\pm 1\}^{\mathbb{Z}}$, *and the entries of* bc *are iid random variables, with* $\Pr(b_m c_m = 1) \in (0,1)$, *then (i)-(v) hold with* A^b, A^b_+, A^b_n, *replaced by* $A^{b,c}$, $A^{b,c}_+$, *and* $A^{b,c}_n$, *respectively.*

Proof. To see that (i)-(v) hold, note that, by Lemma 2.3 and the remarks at the end of Section 2, the condition of the theorem imply that b and also $b_+ := (b_1, b_2, \ldots)$ are pseudo-ergodic with probability one. Then (i) follows from the de nition of in Theorem 2.5, and from Lemma 3.5 and Theorem 3.6. That (ii) and (iii) hold follows from Lemma 3.1, Theorem 4.1 and Corollary 4.6. That (iv) holds follows from the de nition of $^p_\varepsilon$ in Theorem 2.5, and from Lemma 3.5 and Theorem 3.6. Finally, (v) follows from corollaries 4.3 and 4.5. That (i)-(v) hold for the case where A^b, A^b_+, A^b_n are replaced by $A^{b,c}$, $A^{b,c}_+$, and $A^{b,c}_n$, respectively, and the entries of bc are iid random variables, with Pr(b

4/F8iables,)]TJ0.889 Td 5.5 [(T-81 969626 T> 0 G8 (" Td [(4/F8iables,) -1 (A10 738 Tf:6.) -454934 0Se) 50082-340 (Figur) 5008s RG [-306 (4.3)]TJ0 g 0 G5

Proof. Part (i) follows from Theorem 2.7 (taken from [5]) and Lemma 3.1, and that is a strict subset of holds, as discussed at the end of 4.3, provided $= \varepsilon_{34} \quad S_{34}(1.5 + 0.5i) > 0$. Part (ii) is Theorem 4.1, with $^p_\varepsilon \quad ^p_{\varepsilon_0}$ because $^p_\varepsilon = \mathrm{spec}^p_\varepsilon A^b$ if $b \in \{\ 1\}^{\mathbb{Z}}$ is pseudo-ergodic (Theorem 3.6). Part (iii) is Lemma 3.4, (iv) is from 3.2, (v) is part of Theorem 4.9, and (vi) is from the end of Section 4.3. ■

It is clear from the above results that we understand well, in Theorem 5.1, the interrelation between the numerical ranges and pseudospectra of the semi-infinite, bi-infinite, and finite random matrix cases, and have shown that the almost sure spectrum is the same set for the semi-infinite and bi-infinite cases, and contains the spectrum in the finite matrix case. Interesting open questions are whether or not, similarly to the analogous results for the pseudospectra, $\mathrm{spec}\, A^b_n \nearrow$ almost surely as $n \to \infty$, which would imply that $_\infty$ is dense in , so that $_\infty$ is dense in . (That $_\infty$ is dense in was conjectured in [5].) Note that, if it does hold that $\mathrm{spec}\, A^b_n \nearrow$ almost surely, then both Figs 2 and 3 are visualisations of sequences of sets converging to .

Regarding the geometry of (and of the pseudospectra $^p_\varepsilon$), we have some information in Theorem 5.2, including in the last part of this theorem establishing a computable sequence of sets converging from above to (a sequence of three of these plotted in Figure 6). However there is much that is not known. Is connected (which would imply, by general results on pseudospectra [40, Theorem 4.3], that also $^p_\varepsilon$ is connected)? In fact, is simply-connected? What is the geometry of the boundary of , and the geometry of the sets $_n$, the finite-dimensional analogues of (cf. Figure 5)? We have conjectured in [5] that is a simply-connected set which is the closure of its interior and which has a fractal boundary, which is plausible from, or at least consistent with, Figure 6, if it holds that $\overline{\ _\infty} =$. Our methods and results provide no information about what is a usual concern of research on random matrices, to obtain asymptotically in the limit as $n \to \infty$ the pdf of the density of eigenvalues, except, of course, that we have shown in Theorem 5.1(iii) that the support of this pdf is a subset of .

There are many possibilities for applying the methods introduced in this paper to much larger classes of random (or pseudo-ergodic) operators. For some steps in this direction we refer the reader to [30, 6, 9].

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