

## Department of Mathematics

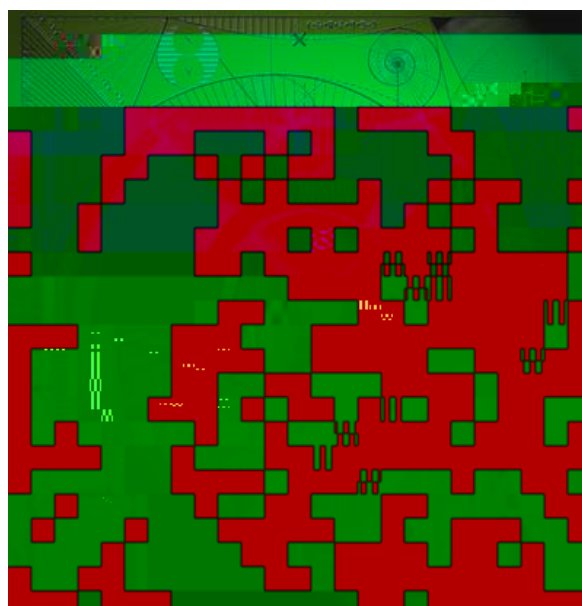
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# A Statistical Mechanical Approach for the Computation of the Climatic Response to General Forcings

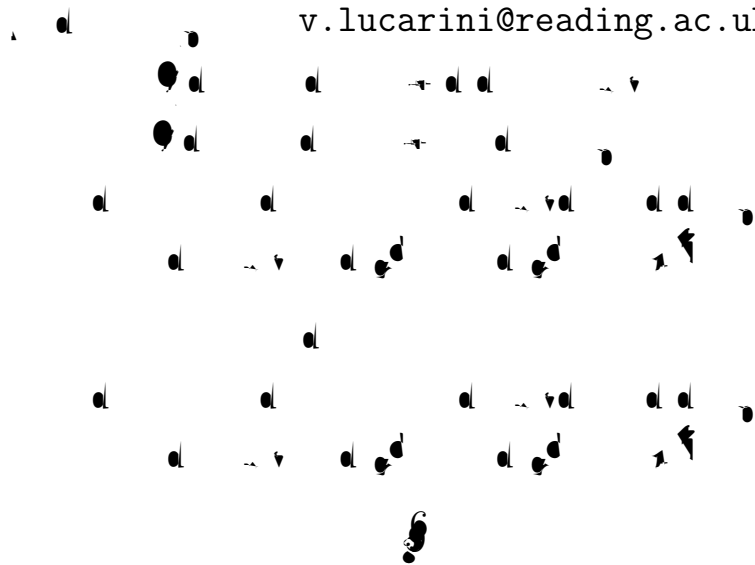
by

Valerio Lucarini and Stefania Sarno



# A Statistical Mechanical Approach for the Computation of the Climatic Response to General Forcings

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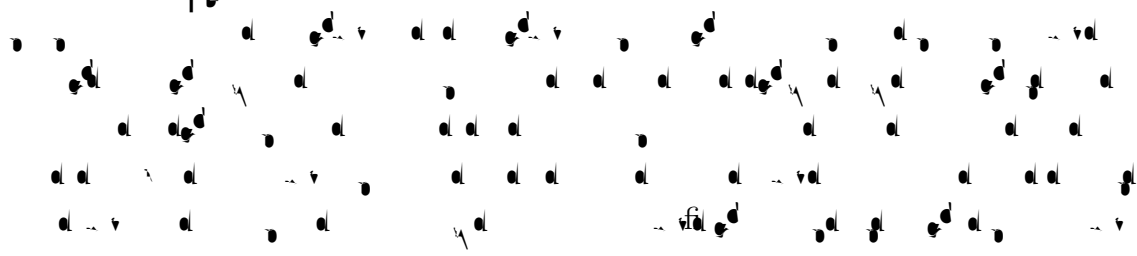
## **Abstract**

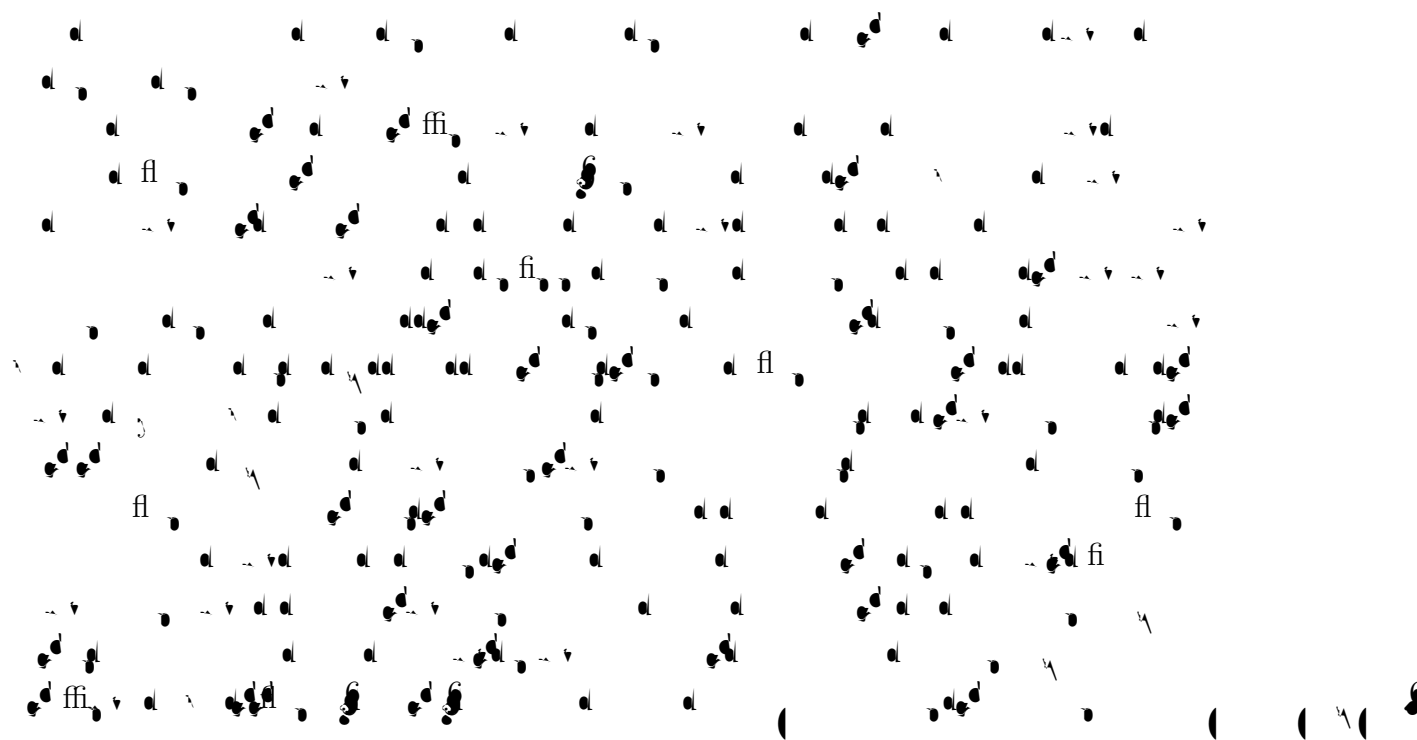
The climate belongs to the class of non-equilibrium forced and dissipative systems, for which most results of quasi-equilibrium statistical mechanics, including the fluctuation-dissipation theorem, do not apply. In this paper we show for the first time how the Ruelle linear response theory, developed for studying rigorously the impact of perturbations on general observables of non-equilibrium statistical mechanical systems, can be applied with great success to analyze the climatic response to general forcings. The crucial value of the Ruelle theory lies in the fact that it allows to compute the response of the system in terms of expectation values of explicit and computable functions of the phase space averaged over the invariant measure of the unperturbed state. We choose as test bed a classical version of the Lorenz 96 model, which, in spite of its simplicity, has a well-recognized prototypical value as it is a spatially extended one-dimensional

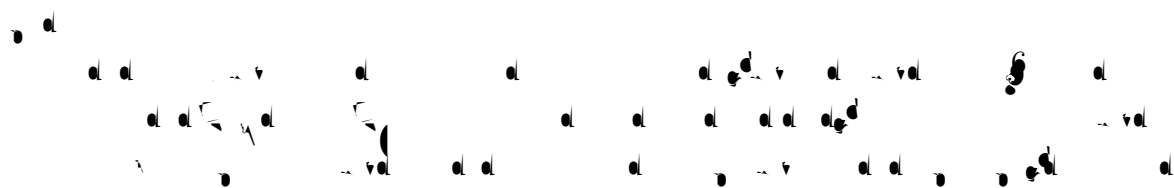
# Con ten

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# 1 Introduction











fi

66

fi

CO<sub>2</sub>

ff



$$\langle \Phi \rangle^{(1)} = \int_{-\infty}^{+\infty} \chi_{\Phi}^{(1)}(f) \times \delta(f - \omega) \chi_{\Phi}^{(1)}(f) ;$$

$$\chi_{\Phi}^{(1)} = \int_{-\infty}^{+\infty} t G_{\Phi}^{(1)}(t) dt :$$

$$\begin{aligned} & \sum_{\alpha} \chi_{\Phi_{\alpha}}^{(1)}(t) f_{\alpha}(t) \\ & \chi_{\Phi}^{(1)} = \sum_{\alpha} \chi_{\Phi_{\alpha}}^{(1)} \\ & f_{\beta} = \sum_{\alpha} \chi_{\Phi_{\alpha}}^{(1)} \langle \Phi_{\beta} \rangle^{(1)} \\ & G_{\Phi}^{(1)}(t) = \sum_{\alpha} \chi_{\Phi_{\alpha}}^{(1)} f_{\alpha}(t) \delta(t) \\ & \langle \Phi \rangle^{(1)} = \sum_{\alpha} \chi_{\Phi_{\alpha}}^{(1)} G_{\Phi}^{(1)}(t) \end{aligned}$$

## 2.2 Kramer-Kronig relation and Dm rDle

$$\begin{aligned} & G_{\Phi}^{(1)}(t) \in L^2 \\ & \chi_{\Phi}^{(1)} = \chi \end{aligned}$$

$$\int_{-\infty}^{+\infty} t \Theta(t) t^k dt = -i \frac{d^k}{dk} \left( P \frac{1}{t} \right) \approx k \frac{1}{(k+1)}$$

$$G_{\Phi}^{(1)}(t) \approx \bar{\alpha} \Theta(t) t^{\beta} + o(t^{\beta})$$

$$\chi_{\Phi}^{(1)} \approx \alpha \Gamma^{-\beta-1} \Gamma^{-\beta-1}$$

$$\frac{\pi}{2^{2p}} \chi_{\Phi}^{(1)}(v) = P \int_0^\infty \frac{v^{2p+1} \chi_{\Phi}^{(1)}(v)}{v^2 - v'^2} dv'$$

$$P \quad p \quad ; :: : \beta - = \beta$$

$p$  ;:::; $\beta=$   $\beta$

$\text{fi}$

$\text{fi}$

$p$

$!$

$$\chi^{(1)} \left( \frac{-P}{\pi} \int \epsilon v \frac{\chi^{(1)} v}{v} \right);$$

$\chi^{(1)}$

$\text{fi}$

$!$

$!$

$c_1 \quad c_2 !:^2 \quad o !^3$

$c_1$

$!\rightarrow \infty$



## 2.3 A practical formula for the linear dependence and consistency relation between dependence of different observable

$$\begin{aligned} & \text{if } f(t) \text{ is a function of } t, \text{ then } f(t) = f(t_0) + \int_{t_0}^t f'(t) dt \\ & \text{if } f(t) \text{ is a function of } t, \text{ then } f(t) = f(t_0) + \int_{t_0}^t f'(t) dt \end{aligned}$$

$$\begin{aligned} & \delta \Phi_\epsilon(t; t_0; x_0) = \Phi_\epsilon(t; t_0; x_0) - \Phi_0(t; t_0; x_0) \\ & \delta \Phi_\epsilon(t; t_0; x_0) = O(\epsilon) \end{aligned}$$

$$\chi_\Phi^{(1)} \equiv \lim_{\epsilon \rightarrow 0, T \rightarrow \infty} \chi_\phi^{(1)}; x_0; \epsilon; T$$

$$\begin{aligned} & \chi_\Phi^{(1)}; x_0; \epsilon; T = \frac{1}{T} \int_0^T \delta \Phi_\epsilon(t; x_0) dt \\ & \delta \Phi_\epsilon(t; x_0) = \Phi_\epsilon(t; x_0) - \Phi_0(t; x_0) \end{aligned}$$

$$\begin{aligned} & \chi_\Phi^{(1)} = \chi_\Phi^{(1)} - \chi_\Phi^{(1)} : \\ & \chi_\Phi^{(1)} = \chi_\Phi^{(1)} \end{aligned}$$

$$\begin{aligned} & \Phi(x) = \Gamma(x); \\ & \Gamma = F \cdot \nabla \Phi \end{aligned}$$

$$F(x) = X(x) f(t) \quad F(x) \in \mathbb{C}^n \quad t \in X(x) \quad \Phi(x)$$



$$i = \langle \Xi \rangle_0 \quad \langle \Xi \rangle_0 \quad \langle \Xi \rangle_0 / \quad \chi_\Phi^{(1)} \approx$$

$$G_\Phi^{(1)}(t) = \Theta(t) \langle \Xi \rangle_0 + t^0;$$

### 3 Application of the Response Theory to the Lorenz-96 Model

#### 3.1 Statistical properties of the Driven Lorenz-96 Model

$$\frac{dx_i}{dt} = x_{i-1}x_{i+1} - x_{i-2} - x_i - F$$

$$i = 1, \dots, N \quad x_{i+N} = x_{i-N} = x_i$$

$$E = \sum_i x_i^2 \quad E$$





### 3.2.1 Global per Drba ion

$$\rho_0(x) = \prod_{i=1}^N \rho_0(x_i) \quad \forall i \quad G_{E,a}^{(1)}(t)$$

$$G_{E,N}^{(1)}(t) = \int \rho_0(x) \Theta(t) \Lambda \Pi_0(t) E(x) dx$$

$$\int \rho_0(x) \Theta(t) \sim \cdot \nabla E(x) dt$$

$$\int \rho_0(x) \Theta(t) \sum_i @_i E(x) dt$$

$$E(x) t$$

$$E(x) t$$

$$\rho_0$$

$$E$$

$$G_{E,N}^{(1)}(t) = \int \rho_0(x) \Theta(t) \sum_i @_i (E|_{t=0} - tE|_{t=0} \circ t)$$

$$\int \rho_0(x) \Theta(t) \left[ \sum_i x_i - \left( \sum_i x_i - NF \right) t \circ t \right]:$$

$$\chi_{E,N}^{(1)}(t) = i \left( \sum_i \langle x_i \rangle_0 \right) t - \left( \sum_i \langle x_i \rangle_0 - NF \right) t^2 \circ t^{-2}$$

$$iN \frac{m}{t} - N \frac{F - m}{t^2} \circ t^{-2}$$

$$i$$

$$\begin{aligned}
 & \text{"} = x_i^2 \\
 & \text{fi} \\
 & \text{!} \rightarrow \infty
 \end{aligned}$$

$$\begin{aligned}
 & \chi_{\varepsilon, N}^{(1)} \left( i \frac{m}{!} - \frac{F - m}{!^2} \right) o !^{-2} : \\
 & \pi = \\
 & \chi
 \end{aligned}$$

$$\chi_{E,N}^{(1)} = \frac{F}{-}$$

$$\int_0^\infty \chi_{E,1}^{(1)} \frac{\pi}{l} ;$$

$$\chi_{M,\epsilon,1}^{(1)}$$

$$\chi_{M,1}^{(1)} \frac{\pi}{l^2} ;$$

$$\int_0^\infty \chi_{M,1}^{(1)} \frac{\pi}{l} ;$$

$$E_j \quad x_j$$

$$G_{E_j,1}^{(1)}(t) = \int \rho_0(x) \Theta(t) @_j \left[ -x_1^2 - (x_j - x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j - F) t - o(t) \right] \\ \Theta(t) \left[ \langle x_j \rangle_0 - \langle x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j - F \rangle_0 t - o(t) \right];$$

$$\langle x_j \rangle_0 - \langle x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j - F \rangle_0$$

$$\rho_0$$

$$\chi_{E_j,1}^{(1)} = i \frac{m}{l} - \frac{m}{l^2} - o(l^{-2}) ;$$

$$E_j \quad \chi_{E,1}^{(1)} = \sum_k \chi_{E_k,1}^{(1)} \quad E$$

$$\chi_{E_j,1}^{(1)} \approx \chi_{E,1}^{(1)}$$



$$\begin{aligned}
& \int_0^\infty \chi_{E_j,1}^{(1)} \left( \frac{\pi}{m} \right) \\
& \propto l^{-2} \\
& \chi_{E,1}^{(1)} \chi_{E_j,1}^{(1)} x_j \\
& = E_{j+1} E_{j+2} E_{j-1}
\end{aligned}$$

$$\chi_{\psi,1}^{(1)} \left( -\frac{\langle x_{j-1} x_{j+1} - x_{j-2} \rangle_0}{l^2} - \frac{F-m}{l^2} \right) o l^{-2} :$$

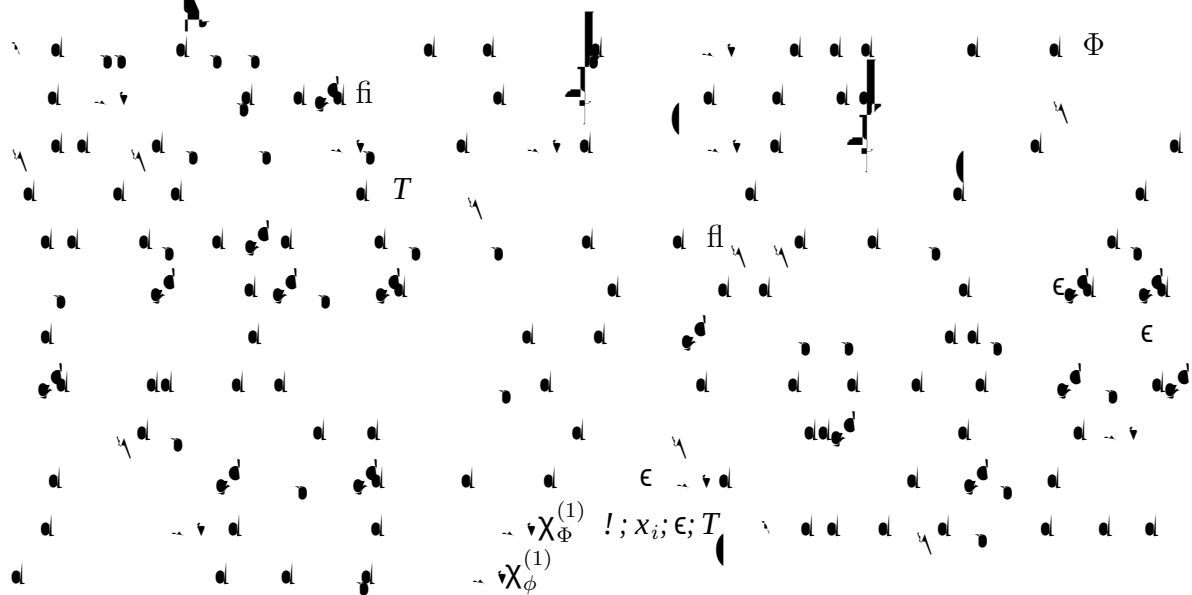
$$\begin{aligned}
& \chi_{\psi,1}^{(1)} \chi_{E_j,1}^{(1)} \chi_{E,1}^{(1)} \\
& \chi_{E,1}^{(1)} \chi_{E_{loc},1}^{(1)} E_{loc} E_j = x_j^2 = x_{j+1}^2 = x_{j+2}^2 = x_{j-1}^2 \\
& x_j
\end{aligned}$$

$$\chi_{x_j,1}^{(1)} \left( \frac{1}{l} - \frac{1}{l^2} \right)$$



## 4 Re DI

### 4.1 Simulation and Data Processing



$$\chi_{\Phi}^{(1)} \sim_{K \rightarrow \infty} \frac{1}{K} \sum_{i=1}^K \chi_{\Phi}^{(1)} ; x_i ; \epsilon ; T ;$$



$F$   $\Phi$   $K$   $ff$   $!$   $!$   $!_l$   $:\pi$   $!_h$   $\pi$   $:\pi$   $t$   $t$   $T$   $\pi=!$

$\epsilon \approx$   $ff$   $ff$   $\epsilon$   $:$   $K$   $\epsilon$   $K$

$f$   $t$   $(($   $K$   $T$

$$\chi_{\varepsilon,N}^{(1)} = N \chi_{E,N}^{(1)} \quad E=N$$

A complex musical score for a string quartet, featuring four staves with various musical notations including notes, rests, and dynamic markings. The notation is dense and includes many accidentals and dynamic markings such as  $! \approx$ .

Figure 1: A schematic diagram of a quantum circuit. It starts with a state  $|\psi\rangle$  on the left, followed by a unitary  $U$ . The circuit then branches into two paths. The top path has a unitary  $U_1$ , followed by a measurement  $M_1$ , and then a unitary  $U_2$ . The bottom path has a unitary  $U_3$ , followed by a measurement  $M_2$ , and then a unitary  $U_4$ . The two paths are recombined by a unitary  $U_5$ . The final state is  $|\phi\rangle$  on the right. The diagram is labeled with various symbols and numbers, including a large '1' in the center.

$$\sim -F - m_{\pi}^2$$

$$\chi_{\mu,N}^{(1)} = N \chi_{M,N}^{(1)} \quad \mu \leq M=N$$

A complex musical score for a string quartet, featuring four staves with various musical notations including notes, rests, and dynamic markings. The notation is dense and includes many accidentals and dynamic markings such as  $f$ ,  $mf$ ,  $pp$ , and  $ppp$ . The score is written in a standard musical notation style with a key signature of one flat and a time signature of 4/4. The notation is dense and includes many accidentals and dynamic markings such as  $f$ ,  $mf$ ,  $pp$ , and  $ppp$ . The score is written in a standard musical notation style with a key signature of one flat and a time signature of 4/4.

$$\begin{aligned}
& \chi_{\mu,N}^{(1)} \\
& \leq \\
& \chi_{\varepsilon,N}^{(1)}
\end{aligned}$$

$$\chi_{\varepsilon,N}^{(1)} \leq \begin{cases} \frac{\omega}{\omega_l} \chi_{\varepsilon,N}^{(1)}; & l \leq l; \\ \chi_{\varepsilon,N}^{(1)}; & l_l \leq l \leq l_h; \\ \frac{m}{\omega}; & l \geq l_h; \end{cases}$$

$$\begin{aligned}
& \rightarrow \\
& \chi_{\varepsilon,N}^{(1)}
\end{aligned}$$

$$\chi_{\varepsilon,N}^{(1)} \leq \begin{cases} \chi_{\varepsilon,N}^{(1)}; & l \leq l; \\ \chi_{\varepsilon,N}^{(1)}; & l_l \leq l \leq l_h; \\ -\frac{F-2m}{\omega^2}; & l \geq l_h; \end{cases}$$

$$\begin{aligned}
& \chi_{\mu,N}^{(1)}
\end{aligned}$$

$\pi$   
 $e F$   $m F$

$$\frac{m F}{F} \lambda \gamma F^{\gamma-1};$$

$$\frac{e F}{F} \lambda \frac{\gamma}{F^{\gamma}} m F \frac{\gamma}{F};$$

$X$   $e$   $m$

$$X_i x$$





$$\begin{aligned}
& \text{fi} \\
& \chi_{\mu,N}^{(1)} \quad E_j \quad \text{fi} \quad \chi_{x_j,1}^{(1)} \quad \chi_{M,1}^{(1)} \\
& M \quad \chi_j \\
& \text{fi} \\
& \chi_{M,1}^{(1)} \quad \chi_{\mu,N}^{(1)} \quad \chi_{x_j,1}^{(1)} \quad \chi_j
\end{aligned}$$

#### 4.4 Further implication of Kramer-Kronig relation and Dm

$$\begin{aligned}
& \text{fi} \\
& \text{th}
\end{aligned}$$

$\chi_{x_j,1}^{(1)}$   $\chi_{M,1}^{(1)}$   $\chi_{E_j,1}^{(1)}$   $\chi_{E,1}^{(1)}$   $E_{loc}$

$$\int_0^\infty \langle v | \chi_{\varepsilon,N}^{(1)} | v \rangle^2 dv = \frac{\pi}{2} m^2;$$

$$\int_0^\infty \langle v | \chi_{\varepsilon,N}^{(1)} | v \rangle^2 dv = \frac{\pi}{2} m^2;$$

$$\chi_{\varepsilon,N}^{(1)} | v \rangle^2$$

$$G_{\varepsilon,N}^{(1)}(t)$$

## 5 Practical Implication for Climate Change Scenarios

The following table shows the results of the numerical simulations for the different scenarios considered in this study. The first column represents the scenario, the second column the initial temperature  $T_S$ , the third column the final temperature  $T_F$ , the fourth column the change in temperature  $\Delta T$ , and the fifth column the change in the concentration of  $CO_2$ . The results are presented for two different values of the parameter  $\varepsilon$ ,  $\varepsilon = 0.1$  and  $\varepsilon = 0.5$ .

| Scenario            | $T_S$    | $T_F$    | $\Delta T$ | $CO_2$   |
|---------------------|----------|----------|------------|----------|
| $\varepsilon = 0.1$ | 288.15 K | 291.15 K | 3.0 K      | 1.0 ppmv |
| $\varepsilon = 0.5$ | 288.15 K | 294.15 K | 6.0 K      | 2.0 ppmv |

The results show that the change in temperature and the change in the concentration of  $CO_2$  are significantly larger for  $\varepsilon = 0.5$  than for  $\varepsilon = 0.1$ . This is due to the fact that the parameter  $\varepsilon$  is a measure of the strength of the interaction between the system and the environment. A larger value of  $\varepsilon$  implies a stronger interaction, which leads to a larger change in the system's properties.



A scatter plot showing the relationship between 'distance' (x-axis) and 'fi' (y-axis). The plot contains numerous data points, many of which are labeled with 'fi' and 'distance'.

*Hasselman programme* 6 *closure theory* fi

Handwritten musical notation on a single page. The notation consists of numerous small, dark, irregular marks and symbols scattered across the page, resembling a dense, abstract pattern. Some faint, larger symbols are visible, including the letters "ff" (fortissimo) and "fi" (fina), and the number "66". The overall appearance is that of a heavily scribbled or corrupted musical score.

The diagram is a complex network of interconnected nodes and lines, representing a system of relationships. The nodes are represented by various symbols, including letters, numbers, and chemical formulas. The connections are shown as lines of varying thickness and style, some solid, some dashed, and some with arrows. The overall structure is dense and interconnected, suggesting a highly complex system. The labels include:

- climate system** (on the left side)
- spectroscopy of the** (on the right side)
- CO<sub>2</sub>** (multiple instances, including one in the center and one at the bottom right)
- delta t** (at the bottom center)
- F** (multiple instances, including one in the upper left and one in the center)
- N** (multiple instances, including one in the upper right and one in the center)
- fi** (multiple instances, including one in the upper left and one in the center)
- 66** (multiple instances, including one in the upper left and one in the center)
- ffi** (multiple instances, including one in the upper right and one in the center)

The diagram is a complex network of interconnected nodes and lines, representing a system of relationships. The nodes are represented by various symbols, including letters, numbers, and chemical formulas. The connections are shown as lines of varying thickness and style, some solid, some dashed, and some with arrows. The overall structure is dense and interconnected, suggesting a highly complex system.



Handwritten musical notation on a page, featuring various notes, rests, and dynamic markings. The notation is written in black ink on a white background. The page contains several staves of music, with notes and rests connected by stems. Dynamic markings such as *ff* (fortissimo) and *fi* (finito) are visible. The notation is dense and covers most of the page area.

Handwritten musical notation on a page, featuring various notes, rests, and dynamic markings. The notation is written in black ink on a white background. The page contains several staves of music, with notes and rests connected by stems. Dynamic markings such as *ff* (fortissimo) and *fi* (finito) are visible. The notation is dense and covers most of the page area.

- <http://www-pcmdi.llnl.gov/>  
climateprediction.net <http://cmip-pcmdi.llnl.gov/cmip5/>

*VL acknowledges the financial support of the EU-ERC research grant NAMASTE and of the Walker Institute Research Development Fund.*

## Reference

This image shows a page of musical notation, likely a score for a piece of music. The notation is written on multiple staves, with various musical symbols including notes, rests, and dynamic markings. The page is numbered with measure numbers 80, 86, 90, and 29. The notation is complex, featuring many notes and rests, and is set against a background of musical staves. The page is titled "Reference" at the top left.

80

86

90

29

This image shows a page of musical notation, likely a score for a piece. The notation is written on multiple staves, with various musical symbols including notes, rests, and bar lines. The page is numbered with several measure numbers: 36, 14, 66, 461, 61, 136, 66, 22, 131, 12, 57, and 66. The notation is dense and complex, suggesting a high level of musical sophistication. The page is oriented vertically, with the musical staves running from top to bottom. The background is white, and the notation is in black ink.



Handwritten musical notation on a page. The notation includes various notes, rests, and dynamic markings. The number 155 is written in the center, and the number 275 is written below it. The notation is written in a cursive, handwritten style.

206

81

fi

ff

131

234

130

56

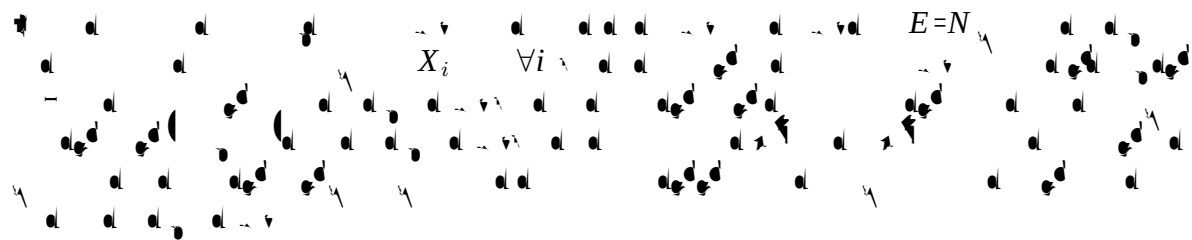
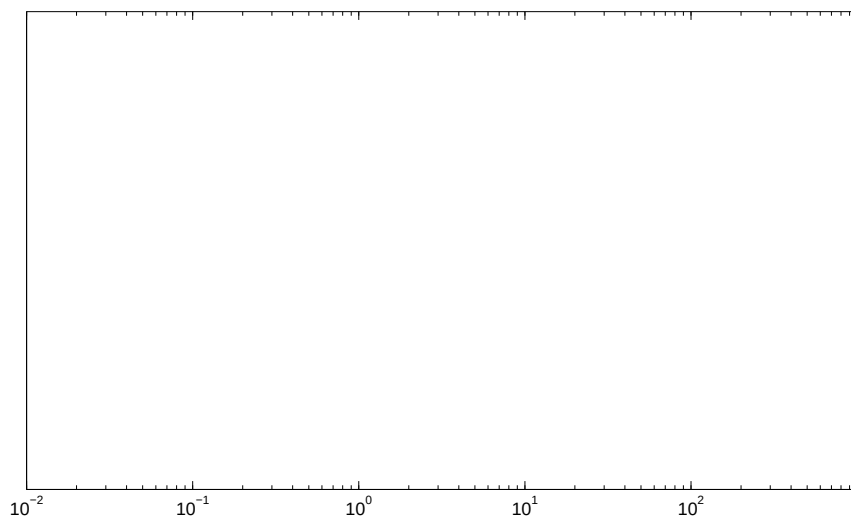
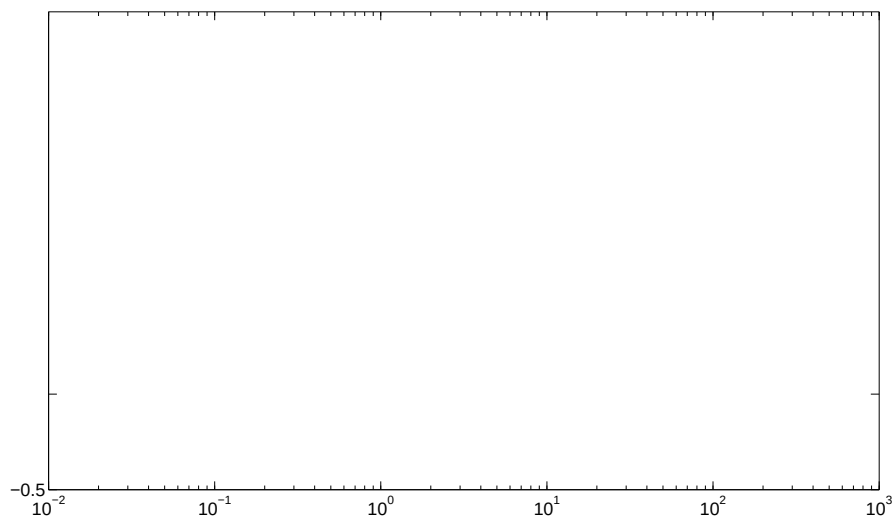
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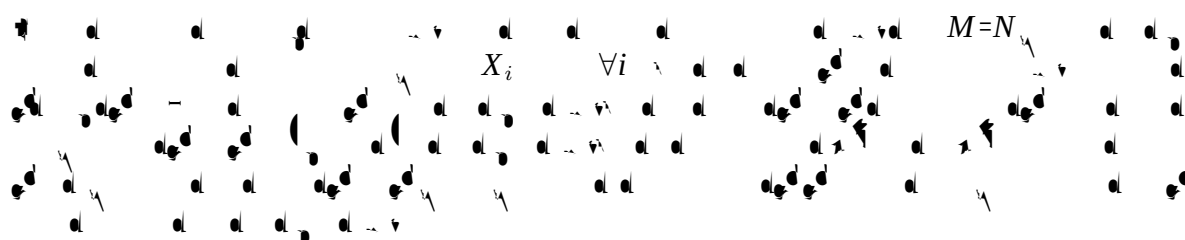
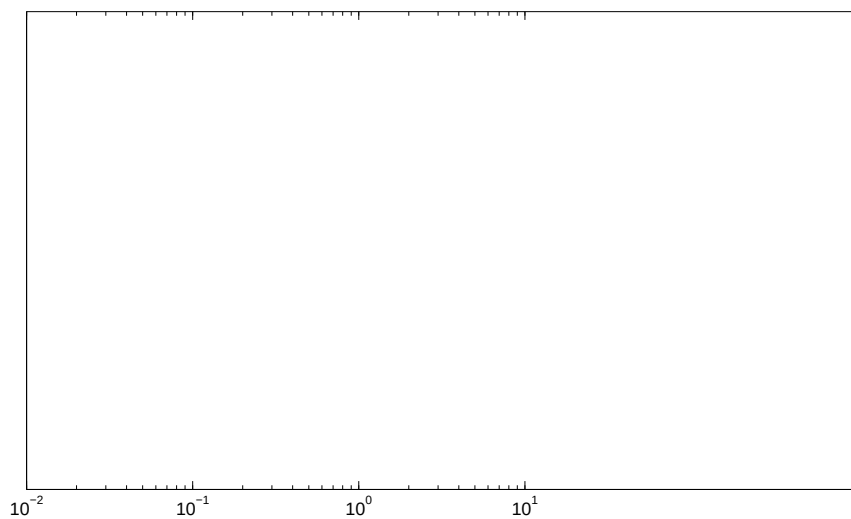
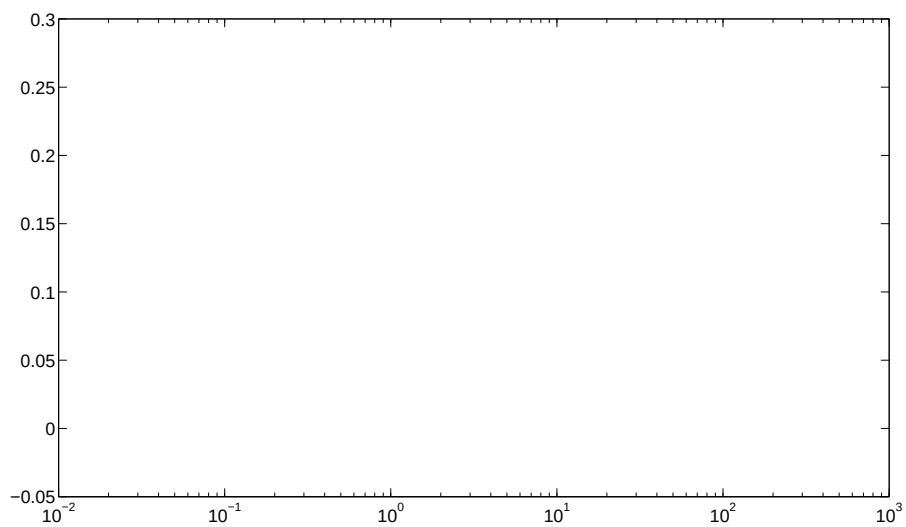
58

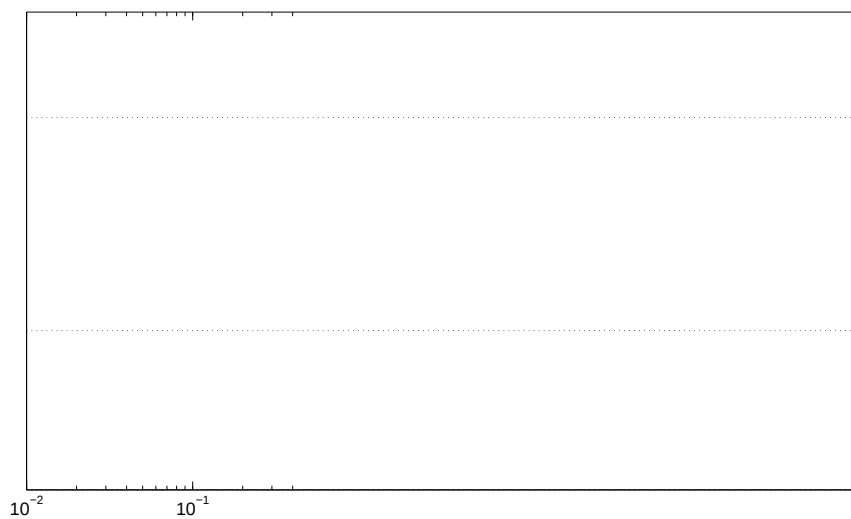
54











$$E=N \quad \quad \quad X_i \quad \quad \quad M=N \quad \quad \quad \forall i \quad \quad \quad \text{fi}$$

