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Models for Evolving Networks: with Applications in Telecommunication and Online Activities

by

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Abstract

Driven by a range of modern applications that includes telecommunications, e-business and on-line social interaction, recent ideas in complex networks can be extended to the case of time-varying connectivity. Here we propose a general framework for modelling and simulating such dynamic networks, and we explain how the long time behaviour may reveal important information about the mechanisms underlying the evolution.

1 Introduction

Many applications [illegible]

- networks of mobile phone users with a link denoting current interaction [24, 26],
- transportation networks defined over a dynamic infrastructure [6],
- networks describing transient social interactions [25],
- correlated neural activity in response to a functional task [11].

Our aims here are (a) to set out and study some general options for describing such evolving networks in a stochastic setting and (b) to discuss practical challenges in interpreting and calibrating suitable models. In particular, we show how ideas from [11] can be extended to produce a wider class of models.

In section 2 we introduce a range of models under successively less restrictive assumptions. Section 3 focusses on a particularly promising model class and in section 4 we make some observations about long-time behaviour. In section 5 we give some illustrative simulations on synthetic data and then show computational results on a real evolving mobile phone network. Concluding remarks appear in section 6.

2 Stochastic models

For simplicity, we consider here undirected, unweighted graphs, defined on a set V of exactly $n \geq 2$ vertices, with no self-loops. Extensions to directed graphs and self-loops follow naturally. Any such graph G may be represented by a symmetric $(n \quad n)$ adjacency matrix, A, with elements $A_{ij} = A_{ji} = 1$ if the edge $e = (i \ j)$ is present, and zero otherwise.

Let S_n denote the set of all such graphs defined over these n vertices. We have $|S_n| = 2^{\frac{n(n-1)}{2}} = M(n)$, say. An evolving graph over discrete time steps is simply an ordered sequence, $\ G_k\}$ for $k = 0 \ 1 \ 2$, within S_n. We think of the evolving graph as taking the particular state G_k at kth time step, that is, at time t_k from some monotonically increasing time sequence.

To introduce a stochastic element, suppose we have a set of conditional probabilities, defined for all possible networks, $G_{k+1} \quad S_n$, given all of the networks earlier within the evolving sequence: say

$$P(G_{k+1}|G_k \ G_{k-1} \quad)$$

This set determines a probability distribution for the next element, G_{k+1}, in the sequence, given its history to da

3.1 Births and deaths dependent upon degree

Suppose the edge e that connects vertices v_i and v_j is not in G_k. Let d_i and d_j denote the degree of vertices v_i and v_j within G_k, respectively. Then let us define

$$\alpha(e) = F_\alpha(d_i \;\; d_j)$$

where F_α is any continuous mapping from pairs of integers onto the interval [0,1]. In the undirected edge case that we consider in this work, symmetry demands $F_\alpha(z_1 \;\; z_2) = F_\alpha(z_2 \;\; z_1)$ for all nonnegative integers z_1 and z_2. For example, F_α might be monotonically increasing in both arguments, meaning that edges are more likely to appear between vertices of higher degree. Such a case is given by

$$F_\alpha(d_i \;\; d_j) = \frac{d_i d_j + a}{d_i d_j + a + \;}$$

for positive reals a and . This mirrors the concepts of preferential attachment and assortativity found in static models [3, 21].

Similarly suppose e is in G_k, and connects vertices v_i and v_j. Then we may define

$$(e) = F_\omega(d_i \;\; d_j)$$

where$i$

where F_α is any continuous mapping from triples of integers onto the interval [0,1]. Note that r_{ij} min d_i $d_j\}$. As before we require $F_\alpha(z_1\ z_2\ z_3) = F_\alpha(z_2\ z_1\ z_3)$ for all nonnegative integers z_1 and z_2. For example F_α may be monotonically decreasing in both $z_1 - z_3$ and $z_2 - z_3$, but increasing in z_3, so that that edges are likely to appear between vertices that have many adjacent vertices in common. Such a case is given by

$$F_\alpha(d_i\ d_j\ r_{ij}) = \square\frac{1 + r_{ij}}{1 + d_i d_j}$$

Similarly, suppose e is in G_k, and connects vertices v_i and v_j. Then we may define

$$(e) = F_\omega(d_i\ d_j\ r_{ij})$$

where F_ω is any continuous mapping from from triples of integers onto the interval [0,1], with $F_\omega(z_1\ z_2\ z_3) = F_\omega(z_2\ z_1\ z_3)$. For example F_ω may be monotonically decreasing in z_3 meaning that edges are less likely to disappear between vertices of with many common adjacencies. Such a case is given by

$$F_\omega(d_i\ d_j\ r_{ij}) = \frac{1 + \square\overline{d_i d_j}}{1 + r_{ij}}$$

3.3 Births and deaths dependent upon edge range

In some circumstances, it is reasonable to assume that connections between vertices are determined in part by their relative locations in some physical or abstract space [18, 23, 29]. This concept of location in space may be more than geographical; there is evidence for a more general 'social distance' metric that, in principle, could be inferred form the network data [30]. Specifically, for range dependent graphs [9, 10, 12, 14, 15] the vertices are considered to have an underlying (generally unknown) ordering on the integer lattice, and the range of any possible edgegb 5i770egbo 8482]TJ/g770ebo o

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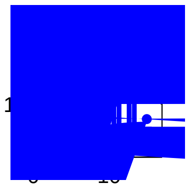
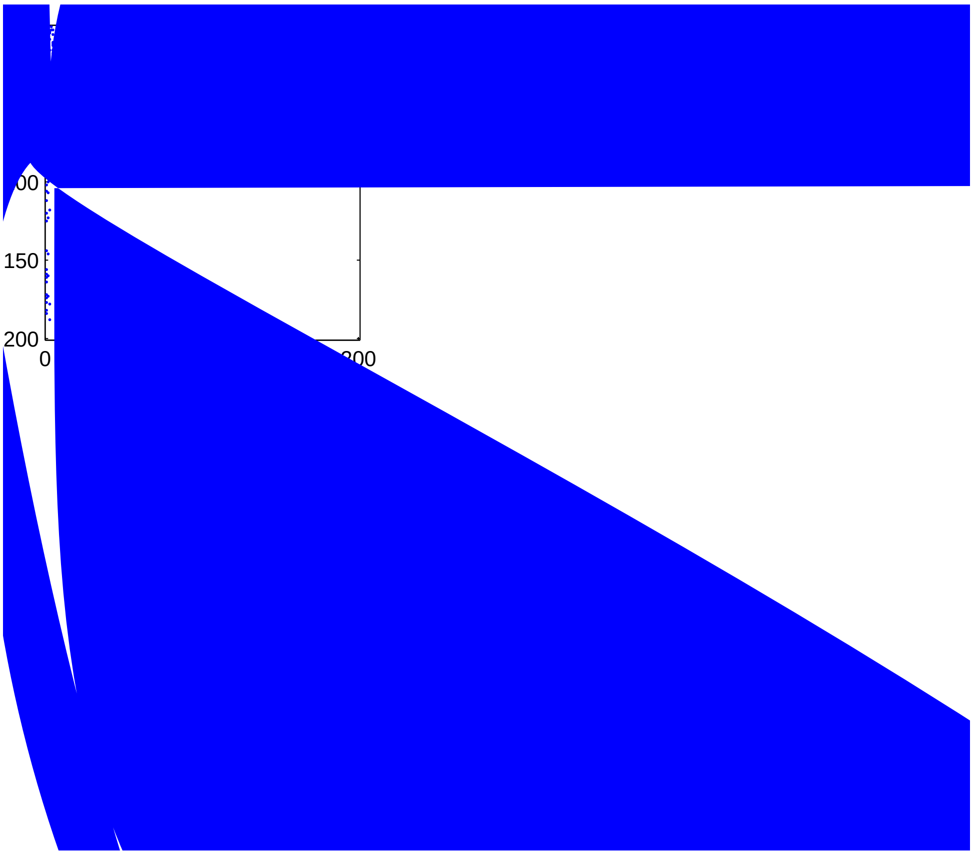
Suppose this model does not use some additional (imposed) knowledge that di□erentiates between the vertices. Of course range-dependent evolving graphs employ an imposed ordering of the vertices for example; whilst Barabási style aggregative graphs allow vertices to become active in some predefined order, externally imposed. But for evolving graphs having no such vertex discrimination, symmetry demands that $G^{\star}$ is invariant to any permutations of the vertices. Hence all possible edges in $G^{\star}$ are equally likely: $G^{\star}$ is an Erdös-Rényi random graph, with a Poisson distribution of vertex p





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As a final computational test, we consider an evolving network from [5]. This data comes from a "Reality Mining" study that used mobile phones to follow 106 subjects at MIT over the course of the 2004–2005 academic year. Pairwise calls, SMS activity and proximity information were recorded. Here we consider just the voice call component of the data, summarized into weekly activity. So a link between nodes i and j in the kth adjacency matrix indicates that at least one phone call took place between subjects i and j in week k. This represents an an evolving network over 52 time points. This network sequence was also used in [13], a visualization of the complete data set can be found there. Of course, understanding the mechanisms that drive this type of dynamic network has immediate benefits for designing mobile phone contracts, identifying and marketing to specific customer groups, and predicting future patterns of network useage.

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8 this would correspond to a Toeplitz structure (common values along each super-diagonal). No such obvious pattern is observed, although there is an indication

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