Eigenvalue problem meets Sierpinski triangle: computing the spectrum of a non-self-adjoint random operator

by

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Abstract. The purpose of this paper is to prove that the spectrum of the non-self-adjoint one-particle Hamiltonian proposed by J. Feinberg and A. Zee (Phys. Rev. E 59 (1999), 6433–6443) has interior points. We do this by first recalling that the spectrum of this random operator is the union of the set of ℓ^∞ eigenvalues of all infinite matrices with the same structure. We then construct an infinite matrix of this structure for which every point of the open unit disk is an ℓ^∞ eigenvalue, this following from the fact that the components of the eigenvector are polynomials in the spectral parameter whose non-zero coefficients are ± 1's, forming the pattern of an infinite discrete Sierpinski triangle.

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1 Introduction and Notations

In this paper we study infinite matrices of the form

$$\begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 0 & 1 & & & \\ & b_{-1} & 0 & 1 & & \\ & & b_0 & 0 & 1 & \\ & & & b_1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix} \qquad (1)$$

with $b_k \in \{\pm 1\} := \{-1, +1\}$ for all $k \in \mathbb{Z}$

Physicists have studied the operator A^b as the (non-self-adjoint) Hamiltonian of a particle hopping (asymmetrically) on a 1-dimensional lattice [15, 16, 9, 22]. Applications of such and related Hamiltonians, especially examples with random diagonals, include vortex line pinning in superconductors and growth models in population biology. The particular model (1) was proposed by Feinberg and Zee in [15], and some properties of its spectrum have been studied in [9, 22] (also see Paragraph 37, in particular Figure 37.7c, in [38]).

In all these studies the focus is on the case of a random sequence $b \in \{\pm 1\}^{\mathbb{Z}}$. A related but completely deterministic concept is that of a pseudo-ergodic sequence. In accordance with Davies [11], we call $b \in \{\pm 1\}^{\mathbb{Z}}$ *pseudo-ergodic* if every finite pattern of ± 1's can be found somewhere (as a string of consecutive entries) in b. If b is pseudo-ergodic (which is almost surely the case if all b_k, $k \in \mathbb{Z}$, are independent (or at least not fully correlated) samples from a random variable with values in $\{\pm 1\}$ and nonzero probability for both $+1$ and -1) then, as a consequence of [7] (also see [6, 8, 29, 30] and cf. [11]), it holds that

$$\operatorname{spec} A^b \;=\; \operatorname{spec}_{\mathrm{ess}} A^b \;=\; \bigcup_{c \in \{\pm 1\}^{\mathbb{Z}}} \operatorname{spec} A^c \;=\; \bigcup_{c \in \{\pm 1\}^{\mathbb{Z}}} \operatorname{spec}^{\infty}_{\mathrm{point}} A^c. \tag{2}$$

The contribution of [7] is the third "=" sign in (2) that enables, or at least simplifies, the explicit computation of the spectra of particular pseudo-ergodic operators in [6, 8, 29]. The first "=" sign in (2) follows immediately from the second; the second comes from the Fredholm theory of much more general operators and is typically expressed in the language of so-called limit operators [34, 35, 27, 8]. (A similar equality, often with the closure taken of the union of spectra, can be found in the literature on spectral properties of Schrödinger and more general Jacobi operators [32, 4, 10, 11, 21, 1, 31, 17, 18, 19, 20, 33, 26, 25, 36, 37]. The three last papers also shed some light on the role of limit operators in the study of the absolutely continuous spectrum.)

Note that, by (2), the spectrum of A^b does not depend on the actual sequence b – as long as it is pseudo-ergodic. In [6] we obtain information about the spectrum, pseudospectrum and numerical range of the bi-infinite matrix operator A^b, its contraction A^b_+ to the positive half axis (a semi-infinite matrix) and the finite sections A^b_n which, for $n \in \mathbb{N}$, are $n \times n$ submatrices of (1). Explicitly and precisely, these related matrices are

$$A^b_+ \;=\; \begin{pmatrix} 0 & 1 & & & \\ b_1 & 0 & 1 & & \\ & b_2 & 0 & 1 & \\ & & b_3 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad A^b_n \;=\; \begin{pmatrix} 0 & 1 & & \\ b_1 & 0 & \ddots & \\ & b_2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & b_{n-1} & 0 \end{pmatrix},$$

where in the case $n = 1$ we set $A^b_1 = (0)$. We explore in some detail in [6] the interrelations between the spectra and pseudospectra of A^b, A^b_+ and A^b_n. Here, for $\varepsilon > 0$ and a bou

b) *Provided the "positive" part of the sequence b (by which we mean $(b_k)_{k \in \mathbb{N}}$) is itself pseudo-ergodic (contains every finite pattern of ± 1's), then, for all $\varepsilon > 0$ one has*

$$\mathrm{spec}_\varepsilon A^b = \mathrm{spec}_\varepsilon A^b_+ .$$

c) *loc A^b calglcq of*

literature) whether $\mathrm{spec}\,A^b$ has positive Lebesgue measure, in particular whether it has interior points. Related to this question, Holz et al. [22, Sections I, V, VI], conjecture that $\mathrm{clos}(\pi_\infty) = \mathrm{spec}\,A^b$ has a fractal dimension in the range $(1,2)$, and so has zero Lebesgue measure.

The purpose of the current paper is to shed light on these questions by constructing a sequence $c \in \{\pm 1\}^{\mathbb{Z}}$ for which $\mathrm{spec}^{\infty}_{\mathrm{point}}\,A^c$ contains the open unit disk. As a consequence of formula (2) and the closedness of spectra, this shows that $\mathrm{spec}\,A^b$ contains the closed unit disk and therefore has dimension 2 and a positive Lebesgue measure. This is the main result of the next section. Intriguingly we will see that the sequence constructed, while rather irregular, is such that each λ in the unit disk is an eigenvalue of A^c in $\ell^\infty(\mathbb{Z})$, and with a vector whose components are polynomials in λ with coefficients forming the regular self-similar pattern of a discrete Sierpinski triangle.

We will finish the paper with our own conjecture on the geometry of $\mathrm{clos}(\pi_\infty)$ and $\mathrm{spec}\,A^b$.

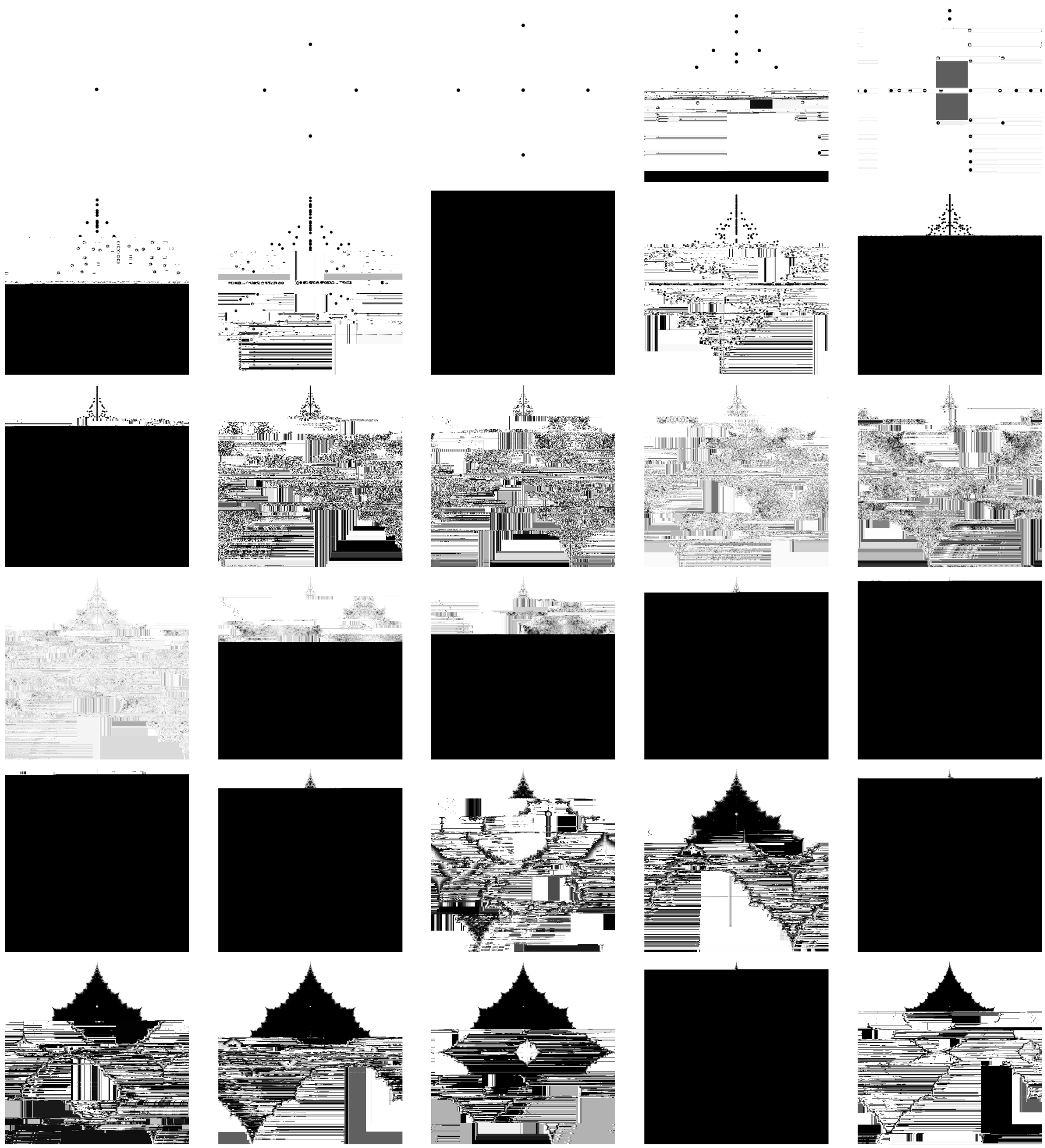

Figure 1.2: Our figure shows the sets $\pi_n := \pi_{n,0}$ of all $n \times n$ matrix eigenvalues, as defined in (4), for $n = 1, \dots, 30$. Note that in the first pictures (with only a few eigenvalues), we have used heavier pixels for the sake of visibility. By (5), each of the sets with $n = 1, 2, \dots, 14$ in this figure is contained, respectively, in the set number $2n + 2$ of Figure 1.1.

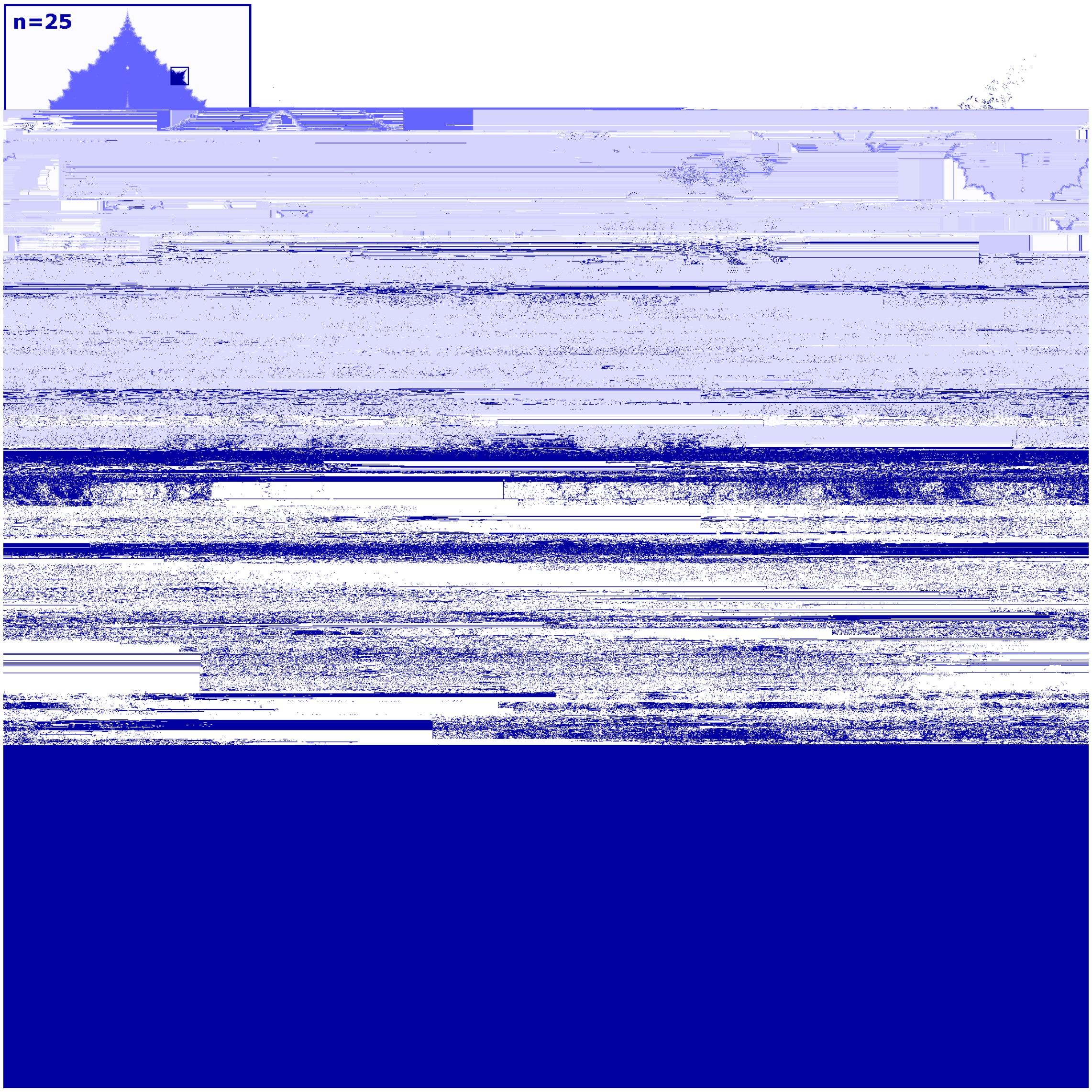

Figure 1.3: This is a zoom into π_{25} – the 25th picture of Figure 1.2. The location of this zoom is near the point $1 + i$, which is the midpoint of the northeast edge of the square $\operatorname{clos}(W(A^b)) = \operatorname{conv}\{2, -2, 2i, -2i\}$. The picture clearly suggests self-similar features of the set π_{25}.

2 A sequence c for which $\operatorname{spec} A^c$ contains the unit disk

The formula (2) for the spectrum of A^b when $b \in \{\pm 1\}^{\mathbb{Z}}$ is pseudo-ergodic motivates the following approach to decide whether a given point $\lambda \in \mathbb{C}$ is in $\operatorname{spec} A^b$ or not: look for a sequence $c \in \{\pm 1\}^{\mathbb{Z}}$ such that $\lambda \in \operatorname{spec}^{\infty}_{\text{point}} A^c$, in other words, such that there exists a non-zero $u \in \ell^{\infty}(\mathbb{Z})$ with

$A^c u = \quad u$, i.e.

$$u_{i+1} \;=\; \quad u_i \qquad c_i\, u_{i \quad 1} \tag{6}$$

for i 2 Z. If such a sequence c exists then $\quad 2$

and so on. Explicitly, it is easy to check that, for i 3, the solution of (6) with initial conditions $u_0 = 0$ and $u_1 = 1$ is given by the characteristic polynomial

$$u_i = \begin{array}{cccc} & 1 & & \\ c_2 & & \ddots & \\ & \ddots & \ddots & 1 \\ & & c_{i\ 1} & \end{array} :$$

Thus, for i 3, u_i is a polynomial of degree i 1 in with coe cients depending on $c_2; :::; c_{i\ 1}$. We will aim to achieve that u be a bounded sequence at least for $j\ j < 1$. With this in mind we should try to keep the coe cients of these polynomials small. Precisely, our strategy will be to try to choose $c_1; c_2; ::: 2\ f\ 1g$ such that each u_i is a polynomial in with coe cients in $f\ 1; 0; 1g$. The following table, where we abbreviate 1 by , +1 by +, and 0 by a space, suggests that this seems to be possible.

i	c_i	j → 1	2	3	4	5	6	7	8 coe cients of $^{j-1}$ in the polynomial u_i	9	10	11	12	13	14	15	16	...
1	+	+																
2	+		+															
3	–	–		+														
4	–				+													
5	+	–		+		+												
6	–		–				+											
7	+	–				+		+										
8	–								+									
9	+	–				+		+		+								
10	+		–				+				+							
11	–	+		–		–				–		+						
12	+				–								+					
13	–	–		+						+		–		+				
14	–		–								+				+			
15	+	–								+				+		+		
16	–																+	
⋮	⋮									⋮								

(7)

For $i; j\ 2$ N, denote the coe cient of $^{j\ 1}$ in the polynomial u 0 9.464 l SQq1 87038 -.4904 Td [iN 1

Proposition 2.1 *Define the sequence $c \in \{\pm 1\}^{\mathbb{Z}}$, for positive indices by $c_1 = 1$ and by the requirement that*

$$c_{2i} = -c_{2i-1} c_i \quad \text{and} \quad c_{2i+1} = -c_{2i}; \qquad i = 1; 2; \ldots;$$

Now suppose $i+j$ is odd. Then, by (10) and the inductive hypothesis,

$$\begin{aligned} p_{2i-1,2j-1} &= p_{2i-2,2j-2} - c_{2i-2}p_{2i-3,2j-1} \\ &= 0 - c_{2i-3}c_{i-1}p_{i-1,j} = c_{2i-1}p_{i-1,j}, \end{aligned}$$

since $c_{2i-1} = -c_{2i-2} = -c_{2i-3}c_{i-1}$. By (10) and the inductive hypothesis and noting that $i-1+j$ is even,

$$\begin{aligned} p_{2i,2j} &= p_{2i-1,2j-1} - c_{2i-1}p_{2i-2,2j} \\ &= c_{2i-1}p_{i-1,j} - c_{2i-1}p_{i-1,j} = 0. \end{aligned}$$

This completes the proof of (iv), and (v) follows from (iv) by a simple induction argument.

To see that (vi) is true, observe first that, from (i), (iii), and (iv) (and cf. Remark 2.2), it holds for $i', j' \in \mathbb{N}$ that $(i',j') \in S$ iff, for some $i,j \in \mathbb{N}$ either (a) $(i',j') = (2i,2j)$ and $(i,j) \in S$; or (b) $(i',j') = (2i-1,2j-1)$ and $(i,j) \in S$ or $(i-1,j) \in S$. From this it follows that $S = T(S)$.

Define a metric d on Ω by

$$d(\Gamma,\Delta) := \sum_{(i,j)\in(\Gamma\cup\Delta)\setminus(\Gamma\cap\Delta)} 2^{-i-j}, \quad \Gamma,\Delta \in \Omega.$$

Then

$$\begin{aligned} d(T(\Gamma),T(\Delta)) &\leq \sum_{(i,j)\in(\Gamma\cup\Delta)\setminus(\Gamma\cap\Delta)} \left(2^{-2i-2j} + 2^{-(2i-1)-(2j-1)} + 2^{-(2i+1)-(2j-1)}\right) \\ &= \sum_{(i,j)\in(\Gamma\cup\Delta)\setminus(\Gamma\cap\Delta)} 2^{-i-j}\left(2^{-i-j} + 2^{2-i-j} + 2^{-i-j}\right) \leq \frac{3}{4}d(\Gamma,\Delta), \end{aligned} \tag{14}$$

if $(1,1) \notin (\Gamma\cup\Delta)\setminus(\Gamma\cap\Delta)$. Let $\Omega_1 := \{\Gamma \in \Omega : (1,1) \in \Gamma\}$. Then $T(\Omega_1) \subset \Omega_1$ and, by (14), T is a contraction mapping on Ω_1. Thus, by the contraction mapping theorem, T has a unique fixed point in Ω_1, which is the set S, and, if $\Gamma_1 \in \Omega_1$ and $\Gamma_{n+1} := T(\Gamma_n)$, $n \in \mathbb{N}$, then $d(\Gamma_n, S) \to 0$ as $n \to \infty$. In particular, $d(S_n,S) \to 0$ as $n \to \infty$. Since also (by an easy induction argument) $S_1 \subset S_2 \subset \ldots$, it follows that $S = \cup_{n\in\mathbb{N}} S_n$.

Define v_{-i} for $i = 0,1,\ldots$ by $v_{-i} := d_i u_i$, which implies that $v_0 = 0$, and set $v_1 = 1$. Then, since u_i is defined uniquely for $i \geq 0$ by the requirement that it satisfy (6) for $i \geq 0$ with the initial conditions that $u_0 = 0$ and $u_1 = 1$, to show (vii) it is enough to check that the sequence v_i satisfies (6) for $i \leq 0$, i.e. that

$$v_{-i+1} = v_{-i} - c_{-i}v_{-i-1}, \quad i = 0,1,\ldots.$$

But $v_1 - v_0 + c_0 v_{-1} = 1 + c_0 d_1 u_1 = 0$, so the equation holds for $i = 0$, and, for $i \in \mathbb{N}$,

$$\begin{aligned} v_{-i+1} - v_{-i} + c_{-i}v_{-i-1} &= d_{i-1}u_{i-1} - d_i u_i + c_{i+1}d_{i+1}u_{i+1} \\ &= (d_{i-1} - c_i c_{i+1}d_{i+1})u_{i-1} - (d_i - c_{i+1}d_{i+1})u_i, \end{aligned}$$

since $u_{i+1} = u_i - c_i u_{i-1}$. Since $u_0 = 0$, the right hand side of this last equation is zero for $i \in \mathbb{N}$ provided that $d_i = c_{i+1}d_{i+1}$ for $i \in \mathbb{N}$. But this follows from the definitions of the sequences c and d. ■

Remark 2.4 The standard infinite discrete Sierpinski triangle (e.g. [24]) is the set $\mathcal{S} \subset \mathbb{N}^2$ defined by $\mathcal{S} := \cup_{n\in\mathbb{N}}\mathcal{S}_n$, where $\mathcal{S}_1 := \{(1,1)\}$ and the sets $\mathcal{S}_n$, $n = 2,3,\ldots$, are defined recursively by $\mathcal{S}_{n+1} := 2\mathcal{S}_n + \nabla$, where $\nabla := \{(0,0),(-1,-1),(0,-1)\}$. One instance where $\mathcal{S}$ arises is as the pattern of odd coefficients in Pascal's triangle: for $i \in \mathbb{N}$ and $j = 1,\ldots,i$ the coefficient of x^{j-1} in $(1+x)^{i-1}$ is odd iff $(i,j) \in \mathcal{S}$, so that the discrete Sierpinski triangle is often referred to as Pascal's triangle modulo 2 (e.g. [13]). Proposition 2.1(vi) (cf. Remark 2.2) makes clear that the pattern $S \subset \mathbb{N}^2$ of the non-zero coefficients in table (7) is essentially that of the standard discrete Sierpinski triangle $\mathcal{S}$; indeed, the sets $\mathcal{S}$ and S are connected by a linear mapping: $(i,j) \in \mathcal{S}$ iff $(2i-j,j) \in S$, for $i,j \in \mathbb{N}$. □

Remark 2.5 Note that the sequence c from Proposition 2.1 is not pseudo-ergodic since, by $c_{2i+1} = c_{2i}$, the patterns "+ + +" and " " can never occur as consecutive entries in the sequence c. □

Based on Theorems 1.1 and 2.3 and the numerical results displayed in Figures 1.1 and 1.2, we make the following conjecture.

Conjecture We conjecture that clos($_1$) = clos($_1$) is spec[illegible]