Limit Operators, Collective Compactness, and the Spectral Theory of Infinite Matrices

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Dedicated to Professor Bernd Silbermann
on the occasion of his 67th birthday.

Abstract

In the first half of this text we explore the interrelationships between the abstract theory of limit operators (see e.g. the recent monographs of Rabinovich, Roch and Silbermann [74] and Lindner [51]) and the concepts and results of the generalised collectively compact operator theory introduced by Chandler-Wilde and Zhang [23]. We build up to results obtained by applying this generalised collectively compact operator theory to the set of limit operators of an operator A (its operator spectrum). In the second half of this text we study bounded linear operators on the generalised sequence space $^p(\mathrm{Z}^N, U)$, where p $[1, \infty]$ and U is some complex Banach space. We make what seems to be a more complete study than hitherto of the connections between Fredholmness, invertibility, invertibility at infinity, and invertibility or injectivity of the set of limit operators, with some emphasis on the case when the operator A is a locally compact perturbation of the identity. Especially, we obtain stronger results than previously known for the subtle limiting cases of $p = 1$ and ∞. Our tools in this study are the results from the first half of the text and an exploitation of the partial duality between ^{1} and $^\infty$ and its implications for bounded linear operators which are also continuous with respect to the weaker topology (the strict topology) introduced in the first half of the text. Results in this second half of the text include a new proof that injectivity of all limit operators (the classic Favard condition) implies invertibility for a general class of almost periodic operators, and characterisations of invertibility at infinity and Fredholmness for operators in the so-called Wiener algebra. In two final chapters our results are illustrated by and applied to concrete examples. Firstly, we study the spectra and essential spectra of discrete Schrödinger operators (both self-adjoint and non-self-adjoint), including operators with almost periodic and random potentials. In the final chapter we apply our results to integral operators on R^N.

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CHAPTER 1

Introduction

1.1. Overview

This text develops an abstract theory of limit operators and a generalised collectively compact operator theory which can be used separately or together to obtain information on the location in the complex plane of the spectrum, essential spectrum, and pseudospectrum for large classes of linear operators arising in applications. We have in mind here differential, integral, pseudo-differential, difference, and pseudo-difference operators, in particular operators of all these types on unbounded domains. This text also illustrates this general theory by developing, in a more complete form than hitherto, a theory of the limit operator method in one of its most concrete forms, as it applies to bounded linear operators on spaces of sequences, where each component of the sequence takes values in some Banach spectrum of A, i.e. the set of $\lambda \in \mathbb{C}$ for which $\lambda I - A$ is not invertible as a member of the algebra $L(Y)$, is independent of p. One of our main results in Section 6.5 implies that also the essential spectrum of A (by which we mean the set of λ for which $\lambda I - A$ is not a Fredholm operator[1]) is independent of p. Moreover, we prove that the essential spectrum is determined by the behaviour of A at infinity in the following precise sense.

Let $h = (h(j))_{j \in \mathbb{N}} \subset \mathbb{Z}$ be a sequence tending to infinity for which it holds that $a_{m+h(j),n+h(j)}$ approaches a limit $\tilde{a}_{m,n}$ for every $m, n \in \mathbb{Z}$. (The existence of many such sequences is ensured by the theorem of Bolzano-Weierstrass and a diagonal argument.) Then we call the operator A_h, with matrix $[A_h] = [\tilde{a}_{mn}]$, a *limit operator* of the operator A. Moreover, following e.g. [74], we call the set of limit operators of A the *operator spectrum* of A, which we denote by $\sigma^{\mathrm{op}}(A)$. In terms of these definitions our results imply that the essential spectrum of A (which is independent of $p \in [1, \infty]$) is the union of the spectra of the elements A_h of the operator spectrum of A (again, each of these spectra is independent of p). Moreover, this is also precisely the union of the point spectra (sets of eigenvalues) of the limit operators A_h in the case $p = \infty$, in symbols

$$\text{(1.3)} \qquad \mathrm{spec}_{\mathrm{ess}}(A) = \bigcup_{A_h \in \sigma^{\mathrm{op}}(A)} \{\lambda : A_h x = \lambda x \text{ has a bounded solution } x \neq 0\}.$$

This formula and other related results have implications for the spectrum of A. In particular, if it happens that $A \in \sigma^{\mathrm{op}}(A)$ (we call A *self-similar* in that case), then it holds that

$$\text{(1.4)} \quad \mathrm{spec}(A) = \mathrm{spec}_{\mathrm{ess}}(A) = \bigcup_{A_h \in \sigma^{\mathrm{op}}(A)} \{\lambda : A_h x = \lambda x \text{ has a bounded solution}\}.$$

In the case $A \notin \sigma^{\mathrm{op}}(A)$ we do not have such a precise characterisation, but if we construct $B \in L(Y)$ such that $A \in \sigma^{\mathrm{op}}(B)$ (see e.g. [51, §3.8.2] for how to do this), then it holds that

$$\text{(1.5)} \quad \mathrm{spec}(A) \subset \mathrm{spec}_{\mathrm{ess}}(B) = \bigcup_{B_h \in \sigma^{\mathrm{op}}(B)} \{\lambda : B_h x = \lambda x \text{ has a bounded solution}\}.$$

A main aim of this text is to prove results of the above type which apply in the simple setting just outlined, but also in the more general setting where $Y = \ell^p(\mathbb{Z}^N, U)$ is a space of generalised sequences $x = x(m)_{m \in \mathbb{Z}^N}$, for some $N \in \mathbb{N}$, taking values in some Banach space U. In this general setting the definition (1.1) makes sense if we replace $\mathbb{Z}$ by $\mathbb{Z}^N$ and understand each matrix entry a_{mn} as an element of $L(U)$. Such results are the concern of Chapter 6, and are applied to discrete Schrödinger operators and to integral operators on $\mathbb{R}$

Then the spectral properties of an integral operator K on $L^p(\mathbb{R})$, whose action is given by

$$\mathrm{K}f(t) = \int_{\mathbb{R}} k(s,t) f(s)\, ds, \quad t \in \mathbb{R},$$

for some kernel function k, can be studied by considering its *discretisation* $K := G\mathrm{K}G^{-1}$. In turn K is determined by its matrix $[K] = [\kappa_{mn}]_{m,n \in \mathbb{Z}}$, with $\kappa_{mn} \in L(L^p[0,1])$ the integral operator given by

$$\kappa_{mn} g(t) = \int_0^1 k(m+s, n+t) g(s) ds, \quad 0 \le t \le 1.$$

Let us also indicate how the results we will develop are relevant to differential operators (and other non-zero order pseudo-differential operators). Consider the first order linear differential operator L, which we can think of as an operator from $BC^1(\mathbb{R})$ to $BC(\mathbb{R})$, defined by

$$\mathrm{L}y(t) = y'(t) + a(t)y(t), \quad t \in \mathbb{R},$$

for some $a \in BC(\mathbb{R})$. (Here $BC(\mathbb{R}) \subset L^\infty(\mathbb{R})$ denotes the space of bounded continuous functions on $\mathbb{R}$ and $BC^1(\mathbb{R}) := \{x \in BC(\mathbb{R}) : x' \in BC(\mathbb{R})\}$.) In the case when $a(s) \equiv 1$ it is easy to see that L is invertible. Specifically, denoting L by L_1 in this case and defining $\mathrm{C}_1 : BC(\mathbb{R}) \to BC^1(\mathbb{R})$ by

$$\mathrm{C}_1 y(t) = \int_{\mathbb{R}} \kappa(s-t) y(s)\, ds,$$

where

$$\kappa(s) := \begin{cases} e^s, & s < 0, \\ 0, & \text{otherwise}, \end{cases}$$

it is easy to check by explicit calculation that $\mathrm{L}_1\mathrm{C}_1 = \mathrm{C}_1\mathrm{L}_1 = \mathrm{I}$ (the identity operator). Thus the study of spectral properties of the differential operator L is reduced, through the identity

$$\mathrm{L} = \mathrm{L}_1 + \mathrm{M}_{a-1} = \mathrm{L}_1(\mathrm{I} + \mathrm{K}), \tag{1.6}$$

where M_{a-1} denotes the operator of multiplication by the Td[(L)]TJ/1/F11 index set, a family $\{A_i : i \quad I\}$ of linear operators on a Banach space U is said to be *collectively compact* if $\{A_i x : i \quad I,\ x \quad U,\ \ x \ \ \leq 1\}$ is relatively compact in U.) The first half of this text (Chapters 2-5) is devoted to developing an abstract theory of limit operators, in which Y is a general Banach space and in which the role of compactness and collective compactness ideas (in an appropriate weak sense) play specified at the beginning of Chapter 2, the specific translation operators V_n are replaced by a more general discrete group of isometric isomorphisms, and then the definitions (1.8), (1.9), and (1.10) are retained in essentially the same form. The notion of compactness that proves important is with respect to what we term (adapting the definition of Buck [10]) the *strict topology* on Y, a topology in which $\stackrel{s}{\to}$ is the sequential convergence. Moreover, when we study operators of the form $A = I + K$ it is not compactness of K with respect to the strict topology that we require (that K maps a neighbourhood of zero to a relatively compact set), but a weaker notion, that K maps bounded sets to relatively compact sets, operators having this property sometimes denoted *Montel* in the topological vector space and called the *hull* of A. A main result in [33] is the following: if

$$x'(t) + \tilde{A}(t)x(t) = 0, \quad t \in \mathbb{R}, \tag{1.14}$$

has only the trivial solution in $BC^1(\mathbb{R})$, for all $\tilde{A} \in H(A)$, and (1.13) has a solution in $BC^1(\mathbb{R})$, then (1.13) has a solution that is almost periodic. (Since $A \in H(A)$, this is the unique solution in $BC^1(\mathbb{R})$.)

Certain of the ideas and concepts that we use in this text are present already

$H(A) = \mathrm{Lim}\,(A)$, i.e. the hull of A coincides with the set of limit functions of A (cf. Theorem 6.10). Thus this second theorem of Muhamadiev includes his result for the case when A is almost periodic.

The first extension of results of this type to multidimensional problems is the study of systems of partial differential equations in $\mathbb{R}^N$ in [58]. Muhamadiev studies differential operators elliptic in the sense of Petrovskii with bounded uniformly Hölder continuous coefficients, specifically those operators $\underline{L}$ that are what he terms *recurrent*, by which he means that $\sigma^{\mathrm{op}}(\underline{L}) = \sigma^{\mathrm{op}}(\tilde{\underline{L}})$, for all $\tilde{\underline{L}} \in \sigma^{\mathrm{op}}(\underline{L})$. Here $\sigma^{\mathrm{op}}(\underline{L})$ is an appropriate version of the operator spectrum of $\underline{L}$. Precisely, where $A_p(t)$, for $t \in \mathbb{R}^N$ and for multi-indices p with $|p| \le r$, is the family of coefficients of the operator $\underline{L}$ (here r is the order of the operator), the differential operator of the same form $\tilde{\underline{L}}$ with coefficients $\tilde{A}_p(t)$ is a member of $\sigma^{\mathrm{op}}(\underline{L})$ if there exists a sequence $t_k \to \infty$ such that, for every p,

$$A_p(t - t_k) \to \tilde{A}_p(t) \tag{1.15}$$

uniformly on compact subsets of $\mathbb{R}^N$ as $k \to \infty$.

The main result he states is for the case where $\underline{L}$ is recurrent and is also, roughly speaking, almost periodic with respect to the first $N-1$ variables. His result takes the form that if a Favard condition is satisfied ($\tilde{\underline{L}}x = 0$ has no non-trivial bounded solutions for all $\tilde{\underline{L}} \in \sigma^{\mathrm{op}}(\underline{L})$) and if supplementary conditions are satisfied which ensure that approximations to $\underline{L}$ with periodic coefficients have index zero as a mapping between appropriate spaces of periodic functions, then $\underline{L}$ is invertible as an operator between appropriate spaces of bounded Hölder continuous functions.

Muhamadiev's results apply in particular in the case when the coefficients of the differential operator are almost periodic (an almost periodic function is recurrent and its set of limit functions is its hull). Shubin, as part of a review of differential (and pseudo-differential) operators with almost periodic solutions [87], gives a the proof of his Theorem 2.1 show moreover that if $\underline{L}$ is Fredholm then the limit operators of $\underline{L}$ are not only invertible but the inverses are also uniformly bounded, i.e.

$$\sup_{\tilde{\underline{L}} \in \sigma^{\mathrm{op}}(\underline{L})} \left\| \tilde{\underline{L}}^{-1} \right\| < \infty.$$

Extensions of these results to give criteria for normal solvability and Fredholmness of $\underline{L}$ as an operator on Sobolev spaces are made in [60].

In [59] Muhamadiev also, briefly, introduces what we can term a *weak limit operator*. Uniform continuity of the coefficients $A_p(t)$ is required to ensure that every sequence $t_k \to \infty$ has a subsequence, which we denote again by t_k, such that the limits (1.15) exist uniformly on compact subsets (cf. the definition of *richness* in §5.3). The set of all limit operators defined by (1.15) where the convergence is uniform on compact sets we have denoted by $\sigma^{\mathrm{op}}(\underline{L})$. Muhamadiev notes that it is enough to require that the coefficients A_p be bounded (and measurable) for the same *richness* property to hold but with convergence uniformly on compact sets replaced[2] by weak convergence in $L^2(\mathbb{R}^N)$. In the case when the coefficients A_p are bounded, the set of limit operators defined by (1.15) where the convergence is weak convergence in $L^2(\mathbb{R}^N)$ we will term the set of *weak limit operators* of $\underline{L}$. We note that this set coincides with $\sigma^{\mathrm{op}}(\underline{L})$ in the case when each A_p is uniformly continuous. In [60] Muhamadiev gives criteria for Fredholmness of $\underline{L}$ on certain function spaces in terms of invertibility of each of the weak limit operators of $\underline{L}$.

Muhamadiev's work has been a source of inspiration for the decades that followed. For example, similar to his main results in [59] but much more recently, A. and V. Volpert show that, for a rather general class of scalar elliptic partial differential operators L on rather general unbounded domains and also for systems of such, a Favard condition is equivalent to the Φ_+ property of L on appropriate Hölder [93, 94, 95] or Sobolev [92, 94, 95] spaces.

Lange and Rabinovich [44], inspired by and building on Muhamadiev's paper [59], carry the idea of (semi-)Fredholm studies by means of limit operators over to the setting of operators on the discrete domain $\mathbb{Z}^N$. They give sufficient and necessary Fredholm criteria for the class $BDO(Y)$ of band-dominated operators (as defined after (1.12) and studied in more detail below in §6.3) acting on $Y = \ell^p(\mathbb{Z}^N, \mathbb{C})$ spaces. For $1 < p < \infty$, they show that such an operator is Fredholm iff all its limit operators are invertible and if their inverses are uniformly bounded. Their proof combines the limit operator arguments of Muhamadiev [59] with ideas of Simonenko and Kozak [39, 84, 85] for the construction of a Fredholm regulariser of A by a clever assembly of local regularisers. Lange and Rabinovich are thereby the first to completely characterise Fredholmness in terms of invertibility of limit operators for the general class of band-dominated operators on $\ell^p(\mathbb{Z}^N, \mathbb{C})$. Before, Simonenko [84, 85] was able to deal with the subclass of those operators whose coefficients (i.e. matrix diagonals) converge along rays at infinity; later Shteinberg [88] was able to relax this requirement to a condition of slow oscillation at infinity. Lange and Rabinovich require nothing but boundedness of the operator coefficients.

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The final section of [44] studies (semi-)Fredholmness of operators in the so-called *Wiener algebra* W (see our many nonzero entries a_{ij} is in general no longer of finite rank – not even compact. That is why Rabinovich, Roch and Silbermann replace the ideal $K(Y)$ of compact operators by another set, later on denoted by $K(Y, P)$, which is the norm closure of the set of all operators A with finitely many nonzero matrix entries. Also this set is contained in BDO K

Another important thread that should be mentioned here is the determination not only of Fredholmness but also of the Fredholm index by means of limit operators. The key paper in this respect is [71] by Rabinovich, Roch and Roe, where the case $N = 1$, $p = 2$, $U = \mathbb{C}$ has been studied using C

[5] to show the stability in $BC[0,\infty)$ of the finite section method for the classical Wiener-Hopf integral equation

$$y(s) = x(s) + \int_0^\infty \kappa(s-t)y(t)\,dt, \quad s \geq 0, \tag{1.17}$$

with $\kappa \in L^1(\mathbb{R})$. (The finite section method is just the approximation of (1.17) by the equation on the finite interval

$$y(s) = x(s) + \int_0^A \kappa(s-t)y(t)\,dt, \quad 0 \leq s \leq A,$$

and the main issue is to study stability and convergence as $A \to \infty$.) The methods and results of [5] are generalised in Chandler-Wilde [13] to obtain criteria in $BC(\mathbb{R})$ for both stability of the finite section method and solvability for the equation

$$y(s) = x(s) + \int_{-\infty}^\infty \kappa(s-t)z(t)y(t)\,dt, \quad s \in \mathbb{R},$$

in operator form

$$y = x + K_z y, \tag{1.18}$$

where K_z is the integral operator with kernel $\kappa(s-t)z(t)$, and it is assumed that $\kappa \in L^1(\mathbb{R})$ and $z \in L^\infty(\mathbb{R})$.

Limit operators do not appear explicitly in [13], or in generalisations of this work to multidimensional cases [20, 23], to more general classes of kernels [24], to other functions spaces ($L^p(\mathbb{R})$, $1 \leq p \leq \infty$, or weighted spaces) [6, 7], or to general operator equations on Banach spaces [23]. Rather, as we discuss in the paragraphs below, the results of these papers provide criteria for unique solvability of (1.18) expressed in terms of injectivity in $BC(\mathbb{R})$ (or equivalently in $L^\infty(\mathbb{R})$) of the elements of a particular family of operators. The connection to limit operators, explored in Section 5.3 below, is that this family of operators necessarily contains both the operator $I - K_z$ and all the weak limit operators of $I - K_z$. (Here *weak limit operator* has the same meaning as in our discussion of the paper [59] on page 8 above; we call $K_{\tilde{z}}$ a weak limit operator of K_z if, for some unbounded sequence $(t_k) \subset \mathbb{R}$, it holds that $z(\cdot - t_k) \overset{w^*}{\to} \tilde{z}$ as $k \to \infty$, where $\overset{w^*}{\to}$ is weak* convergence in $L^\infty(\mathbb{R})$.)

Collective Compactness. In the mid 1960's Anselone and co-workers (see [3] and the references therein) introduced the concept of collectively compact operators. A family $\mathcal{K}$ of linear operators on a Banach space Y is called *collectively compact* if, for any sequences $(K_m) \subset \mathcal{K}$ and $(x_m) \subset Y$ with $\|x_m\| \leq 1$, there is always a subsequence of $(K_m x_m)$ that converges in the norm of Y. It is immediate that every collectively compact family $\mathcal{K}$ is bounded and that all of its members are compact operators.

There are some important features of collectively compact sets of operators. First, recall that if K is a compact operator on Y and a sequence A_m of operators on Y converges strongly (i.e. pointwise) to 0, then $A_m K$ converges to 0 in the operator norm on Y. But under the same assumption, even $A_m K_m$ converges to 0 in the norm for any sequence $(K_m) \subset \mathcal{K}$ provided $\mathcal{K}$ is collectively compact. This fact was probably the motivation for the introduction of this notion. It was used by

Anselone for the convergence analysis of approximation methods like the Nyström method for second kind integral equations.

Another important feature [3, Theorem 1.6] is that if $\{K_m\}_{m=1}^{\infty}$ is collectively compact and strongly convergent to K, then also K is compact, and the following holds:

$$I - K \text{ is invertible}$$
$$I - K_m \text{ is invertible for large } m, \text{ say } m > m_0, \text{ and } \sup_{m > m_0} (I - K_m)^{-1} < \infty$$

Since K and K_m are compact, the above is equivalent to the following statement

$$\text{(1.19)} \quad I - K \text{ is injective} \qquad m_0 : \inf_{m > m_0} \nu(I - K_m) > 0$$

where $\nu(A) := \inf\{ Ax : x = 1\}$ is the so-called *lower norm* of an operator A.

There are many important examples where K is not compact in the norm topology on Y but does have compactness properties in a weaker topology. To be precise, K, while not compact (mapping a neighbourhood of zero to a relatively compact set) has the property that, in the weaker topology, it maps bounded sets to sets that are relatively compact (such operators are sometimes termed *Montel*)[3]. In particular, this is generically the case when K is an integral operator on an unbounded domain with a continuous or weakly singular kernel; these properties of the kernel make K a 'smoothing' operator, so that K has local compactness properties, but K fails to be compact because the domain is not compact. Anselone and Sloan [5] were the first to extend the arguments of collectively compact operator theory to tackle a case of this type, namely to study the finite section method for classical Wiener-Hopf operators on the half-axis. As mentioned already above, the arguments introduced were developed into a methodology for establishing existence from uniqueness for classes of second kind integral equations on unbounded domains and for analyzing the convergence and stability of approximation methods in a series of papers by the first author and collaborators [13, 63, 20, 24, 56, 18, 23, 6, 7]. A particular motivation for this was the analysis of integral equation methods for problems of scattering of acoustic, elastic and electromagnetic waves by unbounded surfaces [14, 21, 96, 19, 22, 56, 18, 97, 61, 7, 16]. Other applications included the study of multidimensional Wiener-Hopf operators and, related to the Schrödinger operator, a study of Lippmann-Schwinger integral equations [20]. Related developments of the ideas of Anselone and Sloan [5] to the analysis of nonlinear integral equations on unbounded domains are described in [1, 4, 62].

In [23] the first author and Zhang put these ideas into the setting of an abstract Banach space Y, in which a key role is played by the notion of a *generalised collectively compact* family K. Now the sequence $(K_m x_m)$ has a subsequence that converges in a topology that is weaker than the norm topology on additional assumptions hold (see the end of our discussion of limit operators above and Theorem 5.9 below), then

$$(1.20)\quad I - K \text{ is injective for all } K \quad K \qquad\qquad \inf_{K \quad K} \nu(I - K) > 0.$$

If the family K satisfies rather strong additional constraints (see Theorem 5.9 below for details), then also invertibility of $I - K$ for every $K \quad K$ follows from injectivity for all $K \quad K$.

To give a concrete flavour of these results (this was the first concrete application of these ideas made to boundary integral equations in wave scattering [13, 14]), one case where they apply is to the integral equation (1.18), with the family K defined by

$$K := \{K_z : z \quad L^\infty(\mathrm{R}) \text{ and } z(s) \quad Q, \text{ for almost all } s \quad \mathrm{R}\},$$

for some $Q \quad \mathrm{C}$ which is compact and convex. That is, existence and uniform boundedness of $(I - K_z)^{-1}$ (as an operator on $BC(\mathrm{R})$) for all $K_z \quad K$, can be shown to follow from injectivity of $I - K_z$ for all $K_z \quad K$ (see [13, 14 *identity* on Y, satisfying conditions (i) and (ii) at the beginning of the chapter. In the theory of limit operators developed in [74] the following notion of sequential convergence plays a crucial role: we write that $x_n \stackrel{s}{\to} x$ if (x_n) is a bounded sequence and $P_m x_n \to P_m x$ as $n \to \infty$ for every m. In this text, as we have noted already, we call $\stackrel{s}{\to}$ *strict convergence*, by analogy with the strict topology of [10]. Chapter 2 is concerned with study of a topology, which we term the strict topology, in which $\stackrel{s}{\to}$ is the sequential convergence. We recall properties of this topology from [23] (which derive in large part from similar results in [10]) and show further properties, for example characterising the compact and sequentially compact sets in the strict topology, characterising when the strict and norm topologies coincide, and introducing many examples that we build on later.

In Chapter 3 we study a number of subspaces of $L(Y)$, the space of bounded linear operators on the Banach space Y, namely those subspaces that play an important role in the abstract theory of limit operators [74] and in the generalised collectively compact operator theory of [23], and so play an important role in the rest of the text. These subspaces include the classes $L(Y, P)$ and $K(Y, P)$ central to the theory of limit operators [74][5], the class $S(Y)$ of operators that are sequentially continuous on (Y, s) (Y equipped with the strict topology), and the class $SN(Y)$

that a sequence (A_n) $L(Y)$ satisfies $A_n \overset{P}{\to} 0$ if and only if (A_n) is bounded and both $P_m A_n$ □ 0 and $A_n P_m$ □ 0 as $n \to \infty$, for every m, while $A_n \overset{S}{\to} 0$ if and only if (A_n) is bounded and $P_m A_n x \overset{S}{\to} 0$ as $n \to \infty$ for every m and every x

operator A on Y^{∞} and its restriction A_0 to Y^0. The main techniques here are to isometrically embed

Acknowledgements. We would like to dedicate this text to Professor Bernd Silbermann, who has been, and still is, an icon and a very influential figure for both of us. When we started this text he was still 62. But it soon turned out to be a rather long term project, and our plans to dedicate it to his 65th birthday were shattered when we couldn't stop ourselves from continuing to work and write on it. So finally, being an extraordinary man, he is getting the extraordinary honour of a dedication to his 67th birthday. *Happy birthday, Bernd!*

We would also like to acknowledge useful discussions of parts of this work with

CHAPTER 2

The Strict Topology

n $\mathbb{N}_0$,

$$P_n x(m) = \begin{cases} x(m), & |m| \leq n, \\ 0, & |m| > n. \end{cases}$$

Then $P = (P_n)$ satisfies (i) and (ii) with $N(m) =$

This last equation implies that the family of semi-norms, $\{|\cdot|_n : n \quad \mathbb{N}_0\}$, is separating. We will term the metrisable topology generated by this family of semi-norms the *local topology*: with this topology, Y is a separated locally convex topological vector space (TVS). By definition, a sequence (x_n) converges to x

Example 2.10 For $x \quad ^\infty$ and $k \quad \mathrm{Z}$ let $x_k \quad ^\infty$ be defined by $x_k(m) = x(m-k)$, $m \quad \mathrm{Z}$. Call $x \quad ^\infty$ *almost periodic* (e.g. [**51**, Definition 3.58]) if $\{x_k : k \quad \mathrm{Z}\} \quad ^\infty$ is relatively compact. Let $^\infty_{\mathrm{AP}} \quad ^\infty$ denote the set of almost periodic functions. Let $Y = \ ^\infty_{\mathrm{AP}}$ and define P as in Example 2.3, noting that $P_n(\ ^\infty_{\mathrm{AP}}) \quad ^\infty_{\mathrm{AP}}$ for every $n \quad \mathrm{N}_0$. Then $\hat{Y} = \ ^\infty$ and Y_0 is a strict subspace of Y. In particular, if $y(m) = \exp(2\pi iam)$, $m \quad \mathrm{Z}$, with a irrational, then $y \quad Y$ (see e.g. [**51**, Lemma 3.64,Proposition 3.65]) but $\ P_n y - y \ = 2$, for $n \quad \mathrm{N}_0$, so that $y \quad Y_0$ by Lemma 2.7. □

We will also be interested in a third topology on $\hat{Y} \quad Y$, intermediate between the local and norm topologies. Given a positive null-sequence $a : \mathrm{N}_0 \to (0, \infty)$, define

$$|x|_a := \sup_n a(n)|x|_n.$$

Then $\{|\cdot|_a : a$ is a positive null sequence$\}$ is a second separating family on $\hat{Y}$ and generates another separated locally convex topology on $\hat{Y}$ which, by analogy with [**10**], we will term the *strict topology*. For $(x_n) \quad \hat{Y}$, $x \quad \hat{Y}$, we will write $x_n \overset{s}{\to} x$ if x_n converges to x in the strict topology, i.e. if $|x_n - x|_a \to 0$ as $n \to \infty$ for every null sequence a.

The topology we have callJ/F11f-ily on

To see (vi) note that if (a) holds then S is bounded in the strict topology and so in the norm topology by (ii). Also, S is totally bounded in the (coarser) local topology. Thus (b) holds. Conversely, if (b) holds, U is a neighbourhood of zero in the strict topology, $M := \sup_{x \in S} \|x\|$, and $B := \{x : \|x\| \le 2M\}$, then, by (ii), there exists a neighbourhood of zero in the local topology, U', such that $U' \cap B = U \cap B$. Further, there exists a finite set $\{s_1, \dots, s_N\} \subseteq S$ such that $S \subseteq \bigcup_{1 \le j \le N} (s_j + U')$. It follows that

$$S \subseteq \bigcup_{1 \le j \le N} (s_j + U' \cap B) = \bigcup_{1 \le j \le N} (s_j + U \cap B) \subseteq \bigcup_{1 \le j \le N} (s_j + U).$$

Thus also (b) $\Rightarrow$ (a).

To see that (b) and (c) are equivalent note that, as the local topology is metrisable, S is totally bounded in the local topology iff every sequence in S has a subsequence that is Cauchy in the local topology. Further, by (v), a sequence is Cauchy in the strict topology iff it is Cauchy in the local topology and norm-bounded. ■

Note that it follows from (ii) that the linear operators on Y that are bounded with respect to the strict topology (map bounded sets onto bounded sets) are precisely the members of $L(Y)$.

Let E denote one of $\hat{Y}$, Y and Y_0. When it is necessary to make a clear distinction we will denote the TVS consisting of E (considered as a linear space) equipped with the strict topology by (E, s) and will denote the TVS (and Banach space) consisting of E with the norm topology as $(E, \|\cdot\|)$.

Lemma 2.12. *If $P_n = I$ for some n, then the local, strict, and norm topologies coincide on $\hat{Y}$. If $P_n \neq I$ for all n, then:*

(a) on $\hat{Y}$ the local topology is strictly coarser than the strict topology which is strictly coarser than the norm topology; and

(b) $\hat{Y}$, equipped with the local topology, is not complete, while $\hat{Y}$ equipped with the strict topology is complete and non-metrisable.

Proof. It is easy to see that any set open in the local topology is open in the strict topology and that any set open in the strict topology is open in the norm topology. If $P_n = I$ for some n then the converse statements clearly hold, as at least one of the semi-norms defining each topology coincides with the norm. Thus the topologies coincide.

If $P_n \neq I$ for any n then there exists (x_n) such that $\|Q_n x_n\| = 1$ for all n. For all m, $P_m Q_n x_n = 0$ for all sufficiently large n, by (ii). Clearly $Q_n x_n \to 0$, but it follows from (2.6) that $Q_n x_n \not\xrightarrow{s} 0$ as $n \to \infty$. Thus the strict and norm topologies are distinct. To see that the local and strict topologies are distinct, note that $nQ_n x_n$ converges to zero in the local topology but $\|nQ_n x_n\| = n \to \infty$ so that, by (2.6), $nQ_n x_n \not\xrightarrow{s} 0$.

If $\hat{Y}$ equipped with the local topology were complete it would be a Fréchet space and it would follow from the open mapping theorem [82] applied to the identity operator that the local and norm topologies coincide.

Let Y' denote the completion of $\hat{Y}$ in the strict topology. Then $Y' \subseteq X$, since $\hat{Y} \subseteq X$ and X is complete in the coarser local topology. Suppose $Y' \neq \hat{Y}$.

Then there exists $x \in Y$ with $|x|_n \to \infty$ as $n \to \infty$. Let $b_n := 1/2 \max(1, |x|_n)$, $a_n := 1/b_n$, and $a = (a_0, a_1, \dots)$. Then $y \in Y$ and $|x - y|_a < 1$ imply that $|y|_n > |x|_n/2$ for all sufficiently large n, so that $\{y \in \hat{Y} : |x - y|_a < 1\} = \emptyset$. This is a contradiction, for $\hat{Y}$ is dense in its completion. ■

By definition, Y_0 is the completion of $\tilde{Y}$ in the norm topology and we have seen that $Q_n x \to$

CHAPTER 3

Classes of Operators

We have introduced already $L(Y)$ and $K(Y)$, the sets of linear operators that are, respectively, bounded and compact on $(Y, \|\cdot\|)$. We have noted that $L(Y)$ coincides with the set of linear operators that are bounded on (Y, s). Let $C(Y)$ and $S(Y)$ denote the sets of those linear operators that are, respectively, continuous and sequentially continuous on (Y, s). Thus $A \in S(Y)$ if and only if, for every sequence $(x_n) \subset Y$ and $x \in Y$,

$$x_n \xrightarrow{s} x \quad \Rightarrow \quad Ax_n \xrightarrow{s} Ax. \tag{3.1}$$

Let $SN(Y)$ denote the set of those linear operators that are sequentially continuous from (Y, s) to $(Y, \|\cdot\|)$, so that $A \in SN(Y)$ iff

$$x_n \xrightarrow{s} x \quad \Rightarrow \quad Ax_n \to Ax. \tag{3.2}$$

We remark that the operators in $S(Y)$ and $SN(Y)$ are precisely those termed $s-$continuous and $sn-$continuous, respectively, in [6].

From standard properties of topological vector spaces [82, Theorems A6 and 1.30], and Lemma 2.12, it follows that $C(Y) \subset S(Y) \subset L(Y)$. In fact we have the following stronger result.

Lemma 3.1. $C(Y) = S(Y)$.

Proof. Let $C(\hat{Y})$, $S(\hat{Y})$ denote the sets of linear operators on $\hat{Y}$ that are, respectively, continuous and sequentially continuous. For $n \in \mathbb{N}_0$ let Y_n denote the linear subspace of Y,

$$Y_n := \{x \in Y : |x|_n = 0\} = \{x \in Y : P_m x = 0, \; 0 \le m \le n\}. \tag{3.3}$$

Note that, by (ii), for every $m \in \mathbb{N}_0$, $Q_n(\hat{Y}) \subset Y_m$ for all sufficiently large n, and, for all $x \in Y$, $\|x - Q_n x\| = \|P_n x\| \le |x|_n$. Thus Assumption A of [23] holds and it follows from [23, Theorem 3.7] that $C(\hat{Y}) = S(\hat{Y})$.

By Lemma 2.13, the sequential closure of $Y \subset \hat{Y}$ in the strict topology is $\hat{Y}$. In Lemma 3.18 we will show that every $A \in S(Y)$ has an extension $\hat{A} \in S(\hat{Y})$ defined by $\hat{A}x = \lim_{n\to\infty} AP_n x$, where the limit exists in the strict topology. Then $\hat{A} \in C(\hat{Y})$ and $A = \hat{A}|_Y \in C(Y)$. ■

In view of this lemma it holds that

$$SN(Y) \subset C(Y) = S(Y) \subset L(Y). \tag{3.4}$$

As Lemmas 3.3-3.4 below clarify, in general $SN(Y)$ is a strict subset of $S(Y)$. The following example shows that $S(Y) = L(Y)$ in general, indeed that A may be compact on $(Y, \|\cdot\|)$ but not sequentially continuous on (Y, s).

Example 3.2 Let $Y = \;{}^{\infty}$ and P_n be as in Example 2.2. Let c^{+} denote the set of those $x \quad {}^{\infty}$ for which $\lim_{m \to +\infty} x(m)$ exists. By the Hahn-Banach theorem

Proof. Suppose $A \quad S(Y) \cap K(Y)$. Take an arbitrary sequence $(x_n) \quad Y$ with $x_n \overset{s}{\to} 0$ as $n \to \infty$. From $A \quad S(Y)$ we conclude that $Ax_n \overset{s}{\to} 0$ as $n \to \infty$. Since $\{x_n\}$ is bounded and A is compact, we know that $\{Ax_n\}$ is relatively compact, so every subsequence of (Ax_n) has a norm-convergent subsequence, where the latter has limit 0 since $Ax_n \overset{s}{\to} 0$ as $n \to \infty$. Of course, this property ensures that Ax_n itself norm-converges to 0.

To see when equality holds consider that, by (ii), it holds for every m that $P_m Q_n = 0$ for all sufficiently large n. Thus, by Lemma 3.3, $P_m \quad SN(Y)$ for all m. So clearly $SN(Y) \quad K(Y)$ if P_m is not compact for all m. If P_m is compact for all m and $A \quad SN(Y)$ then, by Lemma 3.3 again, A is the norm limit $\lim_{m\to\infty} AP_m$, with AP_m compact, so that A is compact and $SN(Y) \quad K(Y)$. Thus equality holds iff $P_m \quad K(Y)$ for all m. ■

Recall that (e.g. [3]) if $K \quad K(Y)$ and A_n converges strongly to A then, since pointwise convergence is uniform on compact sets, $(A_n - A)K \to 0$. This and Lemma 3.7 have the following implication.

Lemma 3.8. *If P_n converges strongly to I then $Q_n K \to 0$ as $n \to 0$ for all $K \quad K(Y)$, while if P_n converges strongly to $I \quad (P_n$ and I the adjoints of P_n and $I)$ then $KQ_n \to 0$ as $n \to 0$ for all $K \quad K(Y)$, so that $K(Y) \quad SN(Y)$.*

Lemma 3.9. *Let A be a linear operator on Y. Then the following statements* for $x \in U$, $m \in \mathbb{N}$, and $n_m \le j \le n_{m+1}$, it holds that

$$\|AQ_j x\| \le \sum_{i=m+1}^{\infty} 4^{1-i} \|P_{\tilde{n}_{i+1}} x\| \le \sum_{i=m+1}^{\infty} 4^{1-i}/a_{\tilde{n}_{i+1}} \le 2^{3-m},$$

since $a_{\tilde{n}_{i+1}} \ge a_{n_{i+1}}$

In analogy to Lemma 3.7 we have the following result.

Lemma 3.13. $L(Y,P) \cap K(Y) \quad K(Y,P)$

where the limit is understood in the strict topology. It holds that $\hat{A} = A$ *, and if* A $SN(Y)$*,* $L(Y, P)$ *or* $K(Y, P)$*, then* $\hat{A}$ $SN(\hat{Y})$*,* $L(\hat{Y}, P)$ *or* $K(\hat{Y}, P)$*, respectively. Conversely, if* $\hat{A}$ $S(\hat{Y})$*,* $SN(\hat{Y})$*,* $L(\hat{Y}, P)$ *or* $K(\hat{Y}, P)$ *and if* $\hat{A}(Y)$ Y*, then* $A := \hat{A}|_Y$ $S(Y)$*,* $SN(Y)$*,* $L(Y, P)$ *or* $K(Y, P)$*, respectively.*

Proof.

relatively sequentially compact in the strict topology. Operators with this property are termed *sequentially compact with respect to the* (Y, s) *topology* in [23]. The following two lemmas are useful characterisations of $M(Y)$ in the case when Y is sequentially complete (which is the case when $Y = \hat{Y}$, by Lemma 2.13).

Lemma 3.19. *If* $Y = \hat{Y}$ *then* $A \quad M(Y)$ *i* $A \quad L(Y)$ *and* $P_m A \quad K(Y)$ *for every* m*.*

Proof. The lemma follows immediately from the equivalence of (a) and (d) in Lemma 2.14. This implies that $A \quad M(Y)$ iff $A(S)$ is norm-bounded and $P_n A(S)$ is relatively compact in the norm topology, for every n and every norm-bounded set S. ■

Remark 3.20 In the case $Y = L^p(\mathbb{R}^N)$ of Example 2.4 (in which case $Y = \hat{Y}$), an operator which satisfies $P_m A \quad K(Y)$ and also $AP_m \quad K(Y)$ for each m is termed *locally compact* in [16, 51, 70, 74]. In the case $Y = BC(\mathbb{R}^N)$ of Example 2.5 (in which again $Y = \hat{Y}$) an operator $A \quad L(Y)$ is termed *locally compact* in [38] if it holds merely that $P_m A \quad K(Y)$ for every m, i.e. by Lemma 3.19, if $A \quad M(Y)$. □

Lemma 3.21. *If* $A \quad M(Y)$ *then* $AP_n \quad KS(Y)$ *for every* n*. Conversely, if* $Y = \hat{Y}$*,* $A \quad S(Y)$ *and* $AP_n \quad M(Y)$ *for every* n*, then* $A \quad M(Y)$*.*

Proof. By Lemma 3.3, $P_n \quad SN(Y)$, and so, by Lemma 3.9, maps some neighbourhood in (Y, s) to a bounded set in $(Y, \ \cdot\ )$. (In fact every neighbourhood in (Y, s) is mapped to a bounded set.) Thus $AP_n \quad KS(Y)$ if $A \quad M(Y)$.

If $AP_n \quad M(Y)$ for every n then, by Lemma 3.19, $P_m AP_n \quad K(Y)$ for every m and n. If also $A \quad S(Y)$ then, by Corollary 3.5, $P_m A - P_m AP_n \ \to 0$ as $n \to \infty$, so that $P_m A \quad K(Y)$ for every m. Thus $A \quad M(Y)$ by Lemma 3.19. ■

Many of the arguments we make in this text will deal with families of operators that have the following collective compactness property.

Definition 3.22. [23] *We say that a set* K *of linear operators on* Y *is* uniformly Montel on (Y, s) *or is* collectively sequentially compact on (Y, s) *if, for every bounded set* B*,* $\ _{K} \ _{K} K(B)$ *is relatively compact in the strict topology.*

Remark 3.23 Note that, by Lemma 2.14, $\ _{K} \ _{K} K(B)$ is relatively compact in the strict topology iff $\ _{K} \ _{K} K(B)$ is relatively sequentially compact in the strict topology, i.e. iff, for every sequence $(K_n) \quad K$ and $(x_n) \quad B$, $(K_n x_n)$ has a strictly convergent subsequence. □

That being Montel on (Y, s) is significantly weaker than being compact is very clear in the case when P_n is compact for all n. The next two results follow from Corollary 2.17 (the first is also a corollary of Lemma 3.19).

Corollary 3.24. *If* $Y = \hat{Y}$ *and* $P_n \quad K(Y)$ *for every* n *then* $M(Y) = L(Y)$*.*

Corollary 3.25. *If* $Y = \hat{Y}$ *and* $P_n \quad K(Y)$ *for every* n *then a set* K *of linear operators on* Y *is uniformly Montel on* (Y, s) *i* K *is uniformly bounded.*

Remark 3.26 Some of our subsequent results will only apply to operators A of the form $A = I + K$ with $K \quad S(Y) \cap M(Y)$. It follows from Corollary 3.24

that, if $Y = \hat{Y}$ and $P_n \quad K(Y)$ for each n, then $A - I \quad S(Y) \cap M(Y)$ whenever $A \quad S(Y)$, so that every $A \quad S(Y)$ can be written in this form. □

$M(Y)$ is the set of operators which map bounded sets to relatively compact sets in (Y, s), and we have seen in Lemma 3.9 that $SN(Y)$ is precisely the set of those operators that map some neighbourhood in (Y, s) to a bounded set. On the other hand, $KS(Y)$ is the set of those operators that map some neighbourhood to a relatively compact set. Clear94-11.955Td[(a)-330(relati)1(v)28835390955i

Then $K = KP_n$ for all $n > 1$, so that K $KS(Y)$ by Lemma 3.21. But K $K(Y)$ as, defining $x_n(s) = \exp(-ins)$, s R, (Kx_n) has no norm-convergent subsequence since $Kx_n(s) \to 0$ as $s \to \infty$ for every n but $Kx_n(n) = 1$ for each n. □

3.2. Algebraic Properties

We will find the algebraic properties collected in the following lemma useful. These are immediate from the definitions and Lemmas 3.9, 3.21 and 3.27.

Lemma 3.31. *Let A and B*

We will say that $A \in L(Y)$ is *invertible* if it is invertible in the algebra of linear operators on Y, i.e. if it is bijective. Automatically, by the open mapping theorem, it follows that $A^{-1} \in L(Y)$. An interesting question is whether $S(Y) = C(Y)$ is inverse closed, i.e. whether, if $A \in S(Y)$ is invertible, it necessarily holds that $A^{-1} \in S(Y)$. Since (Y, s) is not barrelled [23], this question is not settled by standard generalisations of the open mapping theorem to non-metrisable TVS's [77]. Indeed, it is not clear to us whether $S(Y)$ is inverse closed without further assumptions on Y. But we do have the following result which implies that $S(Y)$ is inverse closed in the case when $Y = \hat{Y}$ and $P_n \in K(Y)$ for each n.

Lemma 3.33. *Suppose $A, B \in S(Y)$ are invertible and that $A^{-1} \in S(Y)$ and $A - B \in M(Y)$. Then $B^{-1} \in S(Y)$.*

Proof. We have that $B^{-1} = D^{-1}A^{-1}$, where $D = I + C$ and $C = A^{-1}(B - A)$. By Lemma 3.31, $C \in S(Y) \cap M(Y)$. To show that $B^{-1} \in S(Y)$ we need only to show that $D^{-1} \in S(Y)$.

Suppose that $(x_n) \subset Y$, $x \in Y$, and $x_n \xrightarrow{s} x$. Let $y_n := D^{-1}x_n$. By (2.6), and since $D^{-1} = B^{-1}A \in L(Y)$, (x_n) and (y_n) are bounded. For each n,

$$y_n + Cy_n = x_n. \tag{3.10}$$

Since $C \in M(Y)$ there exists a subsequence (y_{n_m}) and $y \in Y$ such that $x_{n_m} - Cy_{n_m} \xrightarrow{s} y$. From (3.10) it follows that $y_{n_m} \xrightarrow{s} y$. Since $C \in S(Y)$, it follows that $x_{n_m} - Cy_{n_m} \xrightarrow{s} x - Cy$. Thus $y = x - Cy$, i.e. $y = D^{-1}x$. We have shown that $y_n = D^{-1}x_n$ has a subsequence strictly converging to $y = D^{-1}x$. By the same argument, every subsequence of y_n has a subsequence strictly converging to y. Thus $D^{-1}x_n \xrightarrow{s} D^{-1}x$. So $D^{-1} \in S(Y)$. ∎

Corollary 3.34. *If $Y = \hat{Y}$ and $P_n \in K(Y)$ for all n then $S(Y)$ is inverse closed.*

Proof. If $Y = \hat{Y}$ and $P_n \in K(Y)$ for all n, and $A \in S(Y)$ is invertible, then $I - A \in M(Y)$ by Corollary 3.24, so that $A^{-1} \in S(Y)$ by the above lemma. ∎

CHAPTER 4

Notions of Operator Convergence

A component in the arguments to be developed is that one needs some notion of the convergence of a sequence of operators. For (A_n)

for each m, then, for every $x \in Y$, $(A_n x)$ is bounded and $P_m(A_n x - Ax) \to 0$ for each m, so that $A_n x \stackrel{s}{\to} Ax$ by (2.6). ■

Example 4.6 Let Y, P_n and the multiplication operator M_b be defined as in Example 4.3, and suppose that $(b_n) \subset \ell^\infty(\mathbb{Z}^N, L(U))$. Then, extending the results of Example 4.3, we see that

$$
\begin{aligned}
M_{b_n} \square 0 &\Leftrightarrow \|b_n\|_\infty = \sup_{m \in \mathbb{Z}^N} \|b_n(m)\| \to 0, \\
M_{b_n} \stackrel{P}{\to} 0 &\Leftrightarrow \sup_n \|b_n\|_\infty < \infty \text{ and } \|b_n(m)\| \to 0, \quad \forall m \in \mathbb{Z}^N, \\
M_{b_n} \stackrel{s}{\to} 0 &\Leftrightarrow M_{b_n} \stackrel{S}{\to} 0 \\
&\Leftrightarrow \sup_n \|b_n\|_\infty < \infty \text{ and } \|b_n(m)x(m)\| \to 0, \quad \forall m \in \mathbb{Z}^N,\ x \in Y.
\end{aligned}
$$

Thus $M_{b_n} \stackrel{P}{\to} 0$ requires that each component of b_n converges to zero in norm, while $M_{b_n} \stackrel{s}{\to} 0$ requires that each component of b_n converges strongly to zero. We have (cf. Corollary 4.14 below) that

$$
M_{b_n} \square 0 \Rightarrow M_{b_n} \stackrel{P}{\to} 0 \Rightarrow M_{b_n} \stackrel{s}{\to} 0 \Leftrightarrow M_{b_n} \stackrel{S}{\to} 0 \Rightarrow M_{b_n} \to 0.
$$

If U is finite-dimensional, then $\stackrel{P}{\to}$, $\stackrel{S}{\to}$ and $\stackrel{s}{\to}$ all coincide. If $p = \infty$, then $\to$ is equivalent to $\square$. If $1 < p < \infty$ and U is finite-dimensional, then $\to$ coincides with $\stackrel{P}{\to}$, $\stackrel{S}{\to}$ and $\stackrel{s}{\to}$. □

Lemma 4.7. *Suppose $(A_n) \subset L(Y)$ is bounded, $A \in S(Y)$, and*

$$
\|P_m(A_n - A)\| \to 0 \text{ as } n \to \infty
$$

for each m. Then $A_n \stackrel{s}{\to} A$.

Proof. If the conditions of the lemma hold and $x_n \stackrel{s}{\to} x$ then $Ax_n \stackrel{s}{\to} Ax$ and, by (2.6), $\sup_n \|x_n\| < \infty$, so that $(A_n x_n)$ is bounded, and, for each m,

$$
\|P_m(A_n x_n - Ax)\| \le \|P_m(A_n - A)x_n\| + \|P_m A(x_n - x)\| \to
$$

(A_{n_m}) such that $A_{n_m} \xrightarrow{S} A$. Then Lemma 4.9 and other observations made above imply the following corollary.

Corollary 4.10. *Suppose* $\mathcal{A} \quad L(Y)$*. Then* $\mathcal{A}$ *is s-sequentially compact i* $\mathcal{A} \quad S(Y)$ *and* $\mathcal{A}$ *is s-sequentially equicontinuous and sequentially compact in the strong operator topology on* (Y, s)*.*

In the case that the strict and norm topologies coincide, in which case $C(Y) = S(Y) = L(Y)$, it follows from Lemma 4.5, i.e. from the uniform boundedness theorem in Banach spaces, that if $(A_n) \quad L(Y)$ and $A_n \xrightarrow{S} A$ then $\{A_n : n \quad \mathrm{N}\}$ is s-sequentially equicontinuous. In the case when these topologies do not coincide, in which case, by Lemma 2.12, $(Y, S$

Corollary 4.14. *Suppose that Assumption A holds and that* $(A_n) \quad S(Y)$, $A \quad S(Y)$. *Then*

$$A_n \xrightarrow{P} A \qquad A_n \xrightarrow{s} A \quad \Leftrightarrow \quad A_n \xrightarrow{S} A \qquad A_n \to A.$$

CHAPTER 5

Key Concepts and Results

This chapter introduces the key concepts and develops the key results of the text. We first recall the concepts of invertibility at infinity and Fredholmness and start to explore their inter-relation. Next, we summarise some main results from the abstract generalised collectively compact operator theory developed in [23] and from the abstract theory of limit operators [72, 74, 51]. It then turns out that the collection of all limit operators of an operator A $S(Y)$ is subject to the constraints made in the operator theory of [23

Recall that $A \quad L(Y)$ is called *semi-Fredholm* if it has a closed range, $A(Y)$, and if one of the numbers

$$\alpha(A) := \dim(\ker A) \qquad \text{and} \qquad \beta(A) := \dim(Y/A(Y)) \tag{5.3}$$

is finite, and that A is called *Fredholm* if both $\alpha(A)$ and $\beta(A)$ are finite (in which

$$P\!-\!\lim_{n\to\infty} V_{-h(n)}AV_{h(n)} = M_b$$

where $b := \chi_{\{-\infty,\dots,-4\}}$ (see Figure 5.1). The operator A is rich; this can be seen directly or by applying Lemma 6.21 below. □

The following theorem summarises and extends known results on the operator spectrum $\sigma^{\text{op}}(A)$ and on the relationship between A and its operator spectrum. Statements (i) and (ii) are from [72], (iii) and (iv) are from [74] and statements (v)-(vii) go back to [49, Section 3.3] and can also be found in [74, Section 1.2]. (Note that the proofs of (iii)-(vii) given in [49, 74] work for all A $L(Y)$, although the results state a requirement for A $L(Y,P)$ or make a particular choice of Y, and note also that (iv) is immediate from (ii) and (iii) and that (vii) is immediate from (ii), (v) and (vi), see [17].) Thus we include only a proofo18.371639.01cmq[]0d0J0.398w09.934aq[]0dixre

Taking the limit first as $n \to \infty$ and then as $m \to \infty$, noting (2.2), we get that $\nu(A)\|B^{-1}x\| \leq \|x\|$. We have shown that $\nu(A)\|y\| \leq \|By\|$, for all y Y, as required. ■

Within the subspace $L(Y, P)$ of $L(Y)$, for every fixed sequence h tending to infinity, the mapping $A \to A_h$ is compatible with all of addition, composition, scalar multiplication and passing to norm-limits [72]. That is, the equations

$$
\begin{aligned}
&(A + B)_h = A_h + B_h, \qquad (AB)_h = A_h B_h, \\
&(\lambda A)_h = \lambda A_h, \qquad \left(\lim_{m\to\infty} A^{(m)}\right)_h = \lim_{m\to\infty} A_h^{(m)}
\end{aligned}
\tag{5.13}
$$

hold, in each case provided the limit operators on the right hand side exist. By definition, $L(Y, P)$ is a subalgebra of $L(Y)$. By Lemma 3.10, (5.10) implies that V $L(Y, P)$. Thus, if A $L(Y, P)$ then $T(A)$ $L(Y, P)$, so that $\sigma^{\mathrm{op}}(A)$ $L(Y, P)$ by Lemma 4.4. Similarly, since by Lemma 3.32, $S(Y)$ $L(Y, P)$ is a subalgebra of $L(Y)$, if A $S(Y)$ then $V_{-k}AV_k$ $S(Y)$ for 011.495Td[(AkV)]TJ/F1000 TJ/. 5550

One case in which the connection between the properties (5.14) is evident is the case in which A is self-similar. Here, following [**17**] call A $L(Y)$ *self-similar* if A $\sigma^{\mathrm{op}}(A)$ and, generalising [**58**], call A $L(Y)$ *recurrent* if, for every B $\sigma^{\mathrm{op}}(A)$, it holds that $\sigma^{\mathrm{op}}(B) = \sigma^{\mathrm{op}}(A)$. It is immediate from these definitions and Theorem 5.12 (iv) that if A is recurrent then all limit operators of A are self-similar and recurr CONCsell298(w2)1(e)-299(c3s)-1(e)-414(w2)wh-43/F119.963Tf175.2620Td[(r)5TJ/F89.963Tf10.445383T

for each k N, so that $(K_n - V_{-h(n)}K_nV_{h(n)})x_n \xrightarrow{s} 0$ by (2.6). Thus K_nx_n has a strictly convergent subsequence, so that, by Remark 3.23, $\sigma^{op}(K)$ is uniformly Montel.

Conversely, suppose that K is rich and $\sigma^{op}(K)$ is uniformly Montel. Take an arbitrary sequence $h = (h(n))_{n=1}^{\infty}$ Z^N which tends to infinity and an arbitrary bounded sequence (x_n) Y. Since K is rich, $(h(n))$ and (x_n) have subsequences, denoted again by $(h(n))$ and (x_n), such that $V_{-h(n)}KV_{h(n)} \xrightarrow{P} K_h$ $\sigma^{op}(K)$. Thus

$$P_k(V_{-h(n)}KV_{h(n)} - K_h)x_n \to 0, \quad n \to \infty,$$

for each k N, so that $(V_{-h(n)}KV_{h(n)} - K_h)x_n \xrightarrow{s} 0$. On the other hand, K_hx_n has a strictly convergent subsequence since K_h is Montel. Thus $V_{-h(n)}KV_{h(n)}x_n$ has a strictly convergent subsequence. ■

Remark 5.18 Note that, clearly, $(V_{-k}KV_k)_{k\ Z^N}$ is asymptotically Mo[illegible] $(V_{-k}K)_{k\ Z^N}$ is asymptotically Montel since $V = \{V_k\}_{k\ Z^N}$ $iso(Y)$. □

An extension of Theorem 5.16 can be derived by applying Theor[illegible]

$$K := \sigma^{op}(K) \quad T(K),$$

with $T(K)$ defined by (5.11), so that $I - K = \sigma^{op}(A)$ $T(A$[illegible] (iii) of Theorem 5.9 can be checked in a similar way as [illegible] Theorem 5.9, that K is uniformly Montel on (Y, s), is eq[illegible] any of the properties (i)-(iii) of Lemma 6.23 below, [illegible] arbitrary K $L(Y)$. Note that, for a rich operator [illegible]e properties is moreover equivalent to $\sigma^{op}(K)$ bein[illegible]and K $M(Y)$. Then we get the following slightly [illegible]art of Theorem 5.16, which in addition allows to co[illegible]e to the closedness of the range of A.

Theorem 5.19. *Suppose $Y = \hat{Y}$, $A =$ [illegible]su cient, K is subject to any of (i)-(iii) of Lemma 6*[illegible]*mit operators are injective. Then A is bounded below*[illegible]

Note that Theorems 5.16 and 5.1[illegible] just a special case of Theorem 5.8. W[illegible]

Theorem 5.20. *Suppose that Y* [illegible] $A_n = I$. Y $L(Y)$

Proof. Let $K_n := T(K_n)$, $n \in \mathbb{N}$, and set $K := I - B$, $S := V$. Then (c) and (d) imply that conditions (i)–(iv) of Theorem 5.8 are satisfied, and (a) and (b) imply that the condition in Theorem 5.8 b) is satisfied. Thus, applying Theorem 5.8, the result follows. ■

Later on, in Chapter 6, this result will be used to derive Theorem 6.37 on the

Remark 5.23 We note, by Remark 5.21, that, necessarily, $A \quad B$ and $\sigma^{\mathrm{op}}(A)$ B. □

More concrete statements than Theorems 5.16, 5.19 and 5.20 can be given when we pass to a more concrete class of spaces Y. This is what we do in Chapter 6.

CHAPTER 6

Operators on $\ell^p(\mathbb{Z}^N, U)$

In this chapter we focus on the case, introduced already briefly in Example 4.3, when $Y = {}^p(\mathbb{Z}^N, U)$, where $1 \le p \le \infty$, N N and U is an arbitrary complex Banach space. The elements of Y are of the form $x = (x(m))_{m \ \mathbb{Z}^N}$ with $x(m)$ U for every $m = (m_1, ..., m_N)$ $\mathbb{Z}^N$. We equip Y with the usual ^{p} norm of the scalar sequence ($x(m)$ $_U$). We also consider the case when $Y = c_0(\mathbb{Z}^N, U)$, the Banach subspace of ${}^\infty(\mathbb{Z}^N, U)$ consisting of the elements that vanish at infinity, i.e. $x(m)$ $_U \to 0$ as $m \to \infty$.

Since the parameter N N is of no big importance in almost all of what follows, we will use the abbreviations $Y^0(U) := c_0(\mathbb{Z}^N, U)$ and $Y^p(U) := {}^p(\mathbb{Z}^N, U)$ for $1 \le p \le \infty$. If there is no danger of confusion about what U is, we will even write Y^0 and Y^p. Some of our following statements hold for all the spaces under consideration. In this case we will simply write Y, which then can be replaced by any of Y^0 and Y^p with $1 \le p \le \infty$.

In terms of dual spaces, we have $(Y^0(U)) = Y^1(U\ )$, $(Y^1(U)) = Y^\infty(U\ )$, and $(Y^p(U)) = Y^q(U\ )$ for $1 < p < \infty$ and $1/p + 1/q = 1$ (see e.g. [81]). To give t9.963363Tf40Ybrominen= 1 (see e.

$L(Y)$ (this is precisely the definition of an almost periodic operator in Kurbatov [42]). Call A $L(Y)$ *absolutely rich* or *periodic* if every sequence in $T(A)$ has a constant subsequence, i.e. i $T(A)$ is a finite set. It is easy to establish the following characterisation.

Lemma 6.5. *An operator A $L(Y)$ is absolutely rich/periodic i there exist $m_1, ..., m_N$ N such that*

$$VA = AV \qquad \text{for all} \qquad V \quad \tilde{V}_A := \{V_{m_j e^{(j)}}\}_{j=1}^N$$

with $e^{(1)}, ..., e^{(N)}$ denoting the standard unit vectors in R^N, i.e. $e^{(j)}(i) = 1$ if i =

Then, clearly, for each n, $\tilde{P}_n Q_j = 0$ for all su□ciently large j, so that $\tilde{P}_n \quad SN(Y^\infty)$ by Lemma 3.3. (Note however that $\tilde{P}_n \quad L(Y^\infty, P)$.)

The last part of the following result and its proof can be seen as a generalisation of Theorem 2.10 in [20]

Theorem 6.7. *If $A \quad L(Y^\infty)$ is absolutely rich/periodic then $A(Y_n^\infty) \quad Y_n^\infty$ for each n, and*

$$\sigma^{\mathrm{op}}(A) = \{V_{-i} A V_i : i \quad \mathbb{Z}^N\} = \{V_{-i} A V_i : i \quad C_1\}. \tag{6.6}$$

If also $A = I + K$ with $K \quad S(Y^\infty) \cap M(Y^\infty)$ and A is injective then A is invertible.

Proof. If $x \quad Y_n^\infty$ then $Ax \quad Y_n^\infty$ since V_2

Lemma 6.9. *$L^{n\$}(Y)$ is an inverse closed Banach subalgebra of $L(Y)$.*

Proof. Let $A, B \in L^{n\$}(Y)$ and take an arbitrary sequence $h = (h(1), h(2), ...) \subset \mathbb{Z}^N$. Pick a subsequence g of h such that both $V_{-g(n)}AV_{g(n)}$ and $V_{-g(n)}BV_{g(n)}$ converge in norm. Then, clearly, also $V_{-g(n)}(A + B)V_{g(n)}$ and

$$V_{-g(n)}(AB)V_{g(n)} = (V_{-g(n)}AV_{g(n)})(V_{-g(n)}BV_{g(n)})$$

converge in norm. To see that $L^{n\$}(Y)$ is closed in the operator norm take $A_1, A_2, ... \in L^{n\$}(Y)$ with $A_k \rightrightarrows A \in L(Y)$ and an arbitrary sequence $h = (h(n))_{n=1}^{\infty} \subset \mathbb{Z}^N$. Pick subsequences $\cdots \subset h^{(2)} \subset h^{(1)} \subset h$ such that, for every $k \in \mathbb{N}$,

$$\|V_{-h^{(k)}(m)}A_kV_{h^{(k)}(m)} - V_{-h^{(k)}(n)}A_kV_{h^{(k)}(n)}\| < 1/k, \qquad m, n > k, \tag{6.7}$$

and put $g(n) := h^{(n)}(n)$ for all $n \in \mathbb{N}$. Then, for all $k \in \mathbb{N}$ and all $m, n > k$, noting that $g(n) = h^{(k)}(n')$ for some $n' \geq n > k$,

$$\begin{aligned}\|V_{-g(m)}AV_{g(m)} - V_{-g(n)}AV_{g(n)}\| &\leq \|V_{-g(m)}A_kV_{g(m)} - V_{-g(n)}A_kV_{g(n)}\| \\ &\qquad + 2\|A_k - A\| \\ &\leq 1/k + 2\|A_k - A\| \to 0\end{aligned}$$

as $k \to \infty$. This shows that the sequence $(V_{-g(n)}AV_{g(n)})$ is Cauchy and therefore convergent in $L(Y)$. Since $g \subset h$, we get that $A \in L^{n\$}(Y)$.

To see the inverse closedness suppose $A \in L^{n\$}(Y)$ is invertible in $L(Y)$ and take an arbitrary sequence $h = (h(1), h(2), ...) \subset \mathbb{Z}^N$. Since $A \in L^{n\$}(Y)$, there is a subsequence g of h such that $A_n := V_{-g(n)}AV_{g(n)} \rightrightarrows B$ for some $B \in L(Y)$. Since $\|A_n^{-1}\| = \|V_{-g(n)}A^{-1}V_{g(n)}\| = \|A^{-1}\|$ is bounded independently of n, it follows from a basic result on Banach algebras (see, e.g. Lemma 1.3 of [51]) that B is invertible and

$$A_n^{-1} = V_{-g(n)}A^{-1}V_{g(n)} \rightrightarrows B^{-1},$$

showing that $A^{-1} \in L^{n\$}(Y)$. ■

Theorem 6.10. *For $A \in L^{n\$}(Y)$, the following holds.*

(i) *If, for some sequence $h = (h(1), h(2), ...) \subset \mathbb{Z}^N$ and $B \in L(Y)$,*

$$V_{-h(n)}AV_{h(n)} \xrightarrow{P} B \quad \text{holds, then} \quad V_{-h(n)}AV_{h(n)} \rightrightarrows B.$$

(ii) *$A \in \sigma^{op}(A)$ (i.e. A is self-similar).*

(iii) *$\sigma^{op}(A) = \mathrm{clos}_{L(Y)}T(A)$ is a compact subset of $L^{n\$}(Y)$.*

(iv) *A is invertible iff any one of its limit operators is invertible.*

(v) *$\nu(A) = \nu(B)$ for all $B \in \sigma^{op}(A)$, so that A is bounded below iff $\sigma^{op}(A)$ is uniformly bounded below.*

(vi) *If x is almost periodic, then Ax is almost periodic.*

(vii) *If A is invertible on Y^{∞}, then it is invertible on Y^{∞}_{AP}.*

Proof. (i) Since $A \in L^{n\$}(Y$

as $n \to \infty$, showing that $A = A_f \quad \sigma^{\mathrm{op}}(A)$ with $f(n) = g(n+1) - g(n) \to \infty$.

(iii) The inclusion $\sigma^{\mathrm{op}}(A) \quad \mathrm{clos}_{L(Y)} T(A)$ follows from (i). The reverse inclusion follows from (ii), from Theorem 5.12 (ii) and the closedness of $\sigma^{\mathrm{op}}(A)$ (see Theorem 5.12 (iii) above or [51

Proof. If the conditions of the lemma are satisfied then, by (6.11), there exists $c > 0$ such that $Cz \geq c \inf_{y \quad \ker C} z - y$, $z \quad Z$. But, since $Z_0 \quad Z$ and $\ker C = \ker C_0$, this implies that $C_0 z \geq c \inf_{y \quad \ker C_0} z - y$, $z \quad Z_0$, so that the range of C_0 is closed. ■

Corollary 6.14. *If $A \quad S(Y) \cap L_0(Y)$ and A_0 is semi-Fredholm with $\alpha(A_0) < \infty$, then A is semi-Fredholm and* $\ker A = \ker A_0$.

Proof. If the conditions of the lemma are satisfied then, from standard results on Fredholm operators (e.g. [38]), we have that A_0 and A_0

Lemma 6.16. *If* A $S(Y) \cap L_0(Y)$, *then* A $(\check{Y}_1)$ $\check{Y}_1$ *and* $\check{A}_0 = A$ $|_{\check{Y}_1}$, *so that* $\ker \check{A}_0$ $\ker A$ *.*

Proof. For x $\check{Y}_1$, y Y, where $z = J$

characterisation of $BO(Y)$ is the following [51]: that

Proof. It is clear from the definitions that (i) (ii), (ii) (iii) and that (iv) (v). By Lemma 5.17, (ii) implies (i).

Suppose now that (iii) holds and that $h = (h(n))_{n=1}^{\infty}$ $\mathbb{Z}^{N}$ and that (x_n) Y is bounded. If h does not have a subsequence that tends to infinity, then h is bounded, and hence it has a subsequence that is constant. In the case that h has a subsequence that tends to infinity, $(V_{-h(n)}KV_{h(n)}x_n)$ has a strictly convergent subsequence since $(V_{-k}KV_k)$ is asymptotically Montel. In the case that h has a constant subsequence, $(V_{-h(n)}KV_{h(n)}x_n)$ has a strictly convergent s051

The purpose of the following two lemmas is to prove that, for every A $L(Y^{1}(U))$, the operator spectra $\sigma^{\mathrm{op}}(A)$ $L(Y^{1}(U))$ and $\sigma^{\mathrm{op}}(A\ )$ $L((Y^{1}(U))\ ) = L(Y^{\infty}(U\ ))$ correspond elementwise in terms of adjoints.

Lemma 6.25. *If* A $L(Y^{1}(U))$, *then*

$$\sigma^{\mathrm{op}}(A\ ) = \{B\ : B \quad \sigma^{\mathrm{op}}(A)\}.$$

Proof. It is a standard result that $B = A_h$ $\sigma^{\mathrm{op}}(A)$ implies $B\ = (A_h)\ = (A\ )_h$ $\sigma^{\mathrm{op}}(A\ )$ (see, e.g. [**51**, Proposition 3.4 e]).

For the reverse implication, suppose C $\sigma^{\mathrm{op}}(A\ )$ $L(Y^{\infty}(U\ ))$. Then

$$(V_{-h(m)}AV_{h(m)})\ = V_{-h(m)}A\ V_{h(m)} \xrightarrow{P} C$$

as $m \to \infty$

by (6.19), (6.18) and (6.20). Consequently, $(B_m x)$ is a Cauchy sequence in $Y^1(U)$. Let us denote its limit in $Y^1(U)$ by Bx, thereby defining an operator $B \quad L(Y^1(U))$. Passing to the strong limit as $r \to \infty$ in (6.19), we get

$$BP_k = B_k, \qquad k \quad \mathrm{N}. \tag{6.21}$$

Summing up, we have $A_m P_k \ \square\ B_k = BP_k$, and hence $(A_m - B)P_k \ \square\ 0$ as $m \to \infty$, for all $k \quad$ N. Passing to adjoints in the latter gives $P_k(A_m - B\ ) \ \square\ 0$ in $L(Y^\infty(U\ ))$ as $m \to \infty$. If we subtract this from (6.17) we get $P_k(B \quad - C) = 0$ for all $k \quad$ N, and consequently $C =$

(i) *If A is Fredholm and $p = \infty$ then A is invertible at infinity;*
(ii) *If A is invertible at infinity and either U is finite-dimensional or $A = C + K$ with $C \in BDO(Y)$ invertible and $K \in M(Y)$ then A is Fredholm;*
(iii) *A is invertible at infinity if and only if all limit operators of A are invertible and their inverses are uniformly bounded;*
(iv) *The condition on uniform boundedness in (iii) is redundant if $p \in \{0, 1, \infty\}$;*
(v) *It holds that* $\mathrm{spec}(B) \subset \mathrm{spec}(A)$ *for all* $B \in \sigma^{\mathrm{op}}(A)$, *indeed* $\mathrm{spec}(B) \subset \mathrm{spec}_{\mathrm{ess}}(A)$, *for $p = \infty$;*
(vi) *It holds that* $\mathrm{spec}_\varepsilon(B) \subset \mathrm{spec}_\varepsilon(A)$, *for all* $B \in \sigma^{\mathrm{op}}(A)$ *and* $\varepsilon > 0$.

b) *In the case $p = \infty$ if, in addition, it holds that U has a predual U_*, and A has a preadjoint, $A_* \in L(Y_*)$, where $Y_* = Y^1(U_*)$, then (i) and (v) also apply for $p = \infty$; that is, A being Fredholm implies A being invertible at infinity, so that* $\mathrm{spec}(B) \subset \mathrm{spec}_{\mathrm{ess}}(A) \subset \mathrm{spec}(A)$ *for all* $B \in \sigma^{\mathrm{op}}(A)$.

consequence of this and Lemma 6.25. Indeed, if all $B \quad \sigma^{\mathrm{op}}(A)$ are invertible on $Y^1(U)$ then also all their adjoints $C = B$ are invertible on $Y^\infty(U\ )$, which, by Lemma 6.25, are all elements of $\sigma^{\mathrm{op}}(A\ )$. Since $A \quad BDO(Y^\infty(U\ ))$ is rich as well, we know from the results about $p = \infty$ that

$$\sup_{B \quad \sigma^{\mathrm{op}}(A)} B^{-1} = \sup_{B \quad \sigma^{\mathrm{op}}(A)} (B^{-1}) = \sup_{B \quad \sigma^{\mathrm{op}}(A)} (B\ )^{-1}$$

$$= \sup_{C = B \quad \sigma^{\mathrm{op}}(A\ )} C^{-1} < \infty$$

since $B \quad \sigma^{\mathrm{op}}(A)$ i□ $C = B \quad \sigma^{\mathrm{op}}(A$

(b) *all limit operators of A are injective ($\alpha(A_h) = 0$ for all $A_h \quad \sigma^{\mathrm{op}}(A)$) and there is an $S-$dense subset, σ, of $\sigma^{\mathrm{op}}(A)$ such that $\alpha(A_h) = 0$ for all $A_h \quad \sigma$;*

(c) *A is invertible at infinity;*

Proof. Note first that, since $K \quad \sigma^{\mathrm{op}}(K)$ if K is norm rich (Theorem 6.10 (ii)), that $\sigma^{\mathrm{op}}(K)$ uniformly Montel implies $K \quad M(Y^\infty)$.

We shall establish the theorem by proving, by induction, that, for $r = 0, 1, \ldots, N$,

(6.30) if A satisfies the conditions of the theorem and, additionally, A is r-partially periodic, then A is invertible.

Statement (6.30) for $r = 0$ is precisely the theorem that we wish to prove.

That (6.30) holds for $r = N$ follows from Theorem 6.7. Now suppose that (6.30) holds for $r = s$, for some $s \quad \{1, \ldots, N\}$, and that A satisfies the conditions the sth component of). Then $V = V\hat{\ } V\tilde{\ }$ and, for j N

Defining

$$\hat{K} := \sum_{k \in D} M_{\hat{b}_k} V_k,$$

we have that $\varepsilon_j \to 0$ as $j \to \infty$ for each $k \in D$, so that

$$V_{-\check{}(j)} K V_{\check{}(j)} \text{ [illegible] } \sum_{k \in D} M_{\hat{b}_k} V_k \text{ [illegible] } \hat{K}$$

Since also $I + K_n \square I + K = A$, we get, by [51, Lemma 1.3], that A is invertible. From Theorem 6.10 (iv) and (v) it now follows that each limit operator A_h of A is invertible and

$$(A_h)^{-1} = \quad 1$$

Theorem 6.40. *If A W is rich then the following statements are equivalent.*

(i) *A is invertible at infinity on **one** of the spaces Y.*
(ii) *A is invertible at infinity on **all** the spaces Y.*
(iii) *All limit operators of A are invertible on **one** of the spaces Y.*
(iv) *All limit operators of A are invertible on **all** the spaces Y and*

$$\sup_{p\ \{0\}\ [1,\infty]}\ \sup_{A_h\ \sigma^{\mathrm{op}}(A)} A_h^{-1}{}_{L(Y^p)} < \infty. \tag{6.34}$$

Remark 6.41 This theorem is a significant strengthening and simplification of Theorem 2.5.7 in [74]. Theorem 2.5.7 requires that U is reflexive, and, in the case that U is reflexive, it implies only a reduced version of our Theorem 6.40 with the value of Y restricted to Y^p, p $\{0\}$ $(1, \infty)$, in (i)–(iii). □

Proof of Theorem 6.40. (i) (iii) follows from Theorem 6.28 (iii).

(iii) (iv): Suppose (iii) holds. We have observed already that $\sigma^{\mathrm{op}}(A)$ W is independent of the space Y by Lemma 6.39 (ii). Applying Lemma 6.39 (i) to the limit operators of A, it follows that these limit operators are invertible on all the spaces Y. By Theorem 6.28 (iv),

$$s_p := \sup_{A_h\ \sigma^{\mathrm{op}}(A)} A_h^{-1}{}_{L(Y^p)}$$

is finite for p $\{0, 1, \infty\}$. Now, by Riesz-Thorin interpolation (as demonstrated in the proof of [74, Theorem 2.5.7]), we get that $s_p \leq s_1^{1/p}\, s_\infty^{1-1/p} < \infty$ for all p $(1, \infty)$, which proves (iv).

(iv) (ii) follows from Theorem 6.28 (iii).

Finally, (ii) (i) is evident. ■

From the above result and the relationship between invertibility at infinity and Fredholmness, (see Remark 6.2 and Theorem 6.28 a)(i) + (ii) and b)), we can deduce Corollary 6.43 below, which relates Fredholmness to invertibility of limit operators.
I(3[this(4(6.)is-Thor1(ite)-956Td[e-956T-285(qu15)1(e1(e)-enc04(6.)i11.9555i)1(on)-266(b-(is [redacted]

(d) *A is Fredholm on **all** the spaces Y.*

In the case that U has a predual U_* and A, considered as an operator on $Y^\infty(U)$, has a preadjoint $A_* \in Y^1(U_*)$, then (a)–(d) are equivalent to

(e) *A is Fredholm on **one** of the spaces Y.*

Proof. For the clarity of our argument we introduce two more statements:

(f) All limit operators of A are invertible on Y^∞;
(g) A is invertible at infinity on Y^2.

Each of these will turn out to be equivalent to (a)–(d).

By Theorem 6.3, statement (a) is equivalent to (f), which, by Theorem 6.40, is equivalent to each of (b), (c) and (g).

Since $K \in M(Y)$, the implication (b) $\Rightarrow$ (d) follows from Theorem 6.28 (ii) (applied with $C = -I$).

Since, obviously, (d) implies Fredholmness of A on Y^2, it also implies (g), by Theorem 6.28 (i). Another obvious consequence of (d) is (e).

Finally, suppose U_* and A_* exist and (e) holds for $Y = Y^p$. If $p = \infty$, then (c) follows by Theorem 6.28 b), and otherwise, if $p < \infty$, then (c) follows by Theorem 6.28 (i). ■

The above corollary implies, for rich operators in the Wiener algebra which are of the form $A = I - K \in W$ with $K \in U\!M(Y)$, i.e. K is subject to the (equivalent) properties (i)-(v) in Lemma 6.23, and which possess a preadjoint, that Fredholmness on one of the spaces Y implies Fredholmness on all spaces Y. The argument to show this is indirect: it depends on the connection between Fredholmness and invertibility at infinity and on the equivalence of (i) and (ii) in Theorem 6.40.

Recently, Lindner [52] has studied directly the invariance of the Fredholm property across the spaces Y for general operators in the Wiener algebra, but with a slight restriction on the Banach space U, that it is either finite-dimensional or possesses the *hyperplane property*, meaning that it is isomorphic to a subspace of U of co-dimension 1. An equivalent characterisation of the hyperplane property [52] is that there exists a $B \in L(U)$ which is Fredholm of index 1. This characterisation suggests that infinite-dimensional Banach spaces which do not have the hyperplane property are unusual. In [53] Lindner lists many sets of conditions on U which ensure that U has the hyperplane property, and recalls that it was a long-standing open problem due to Banach – the so-called *hyperplane problem* – whether there exist any infinite-dimensional Banach spaces which do not have the hyperplane property. An example was finally constructed by Gowers in [35], for which work, and the resolution of other long-standing open questions posed by Banach, he received the Fields medal in 1998.

The main result proved by Lindner [52] is the following:

Theorem 6.44. *Suppose that U is finite-dimensional or has the hyperplane property and that $A \in W(U)$. Then:*

(a) *If A is Fredholm on one of the spaces Y^p, with $p \in \{0\} \cup [1, \infty)$, then A is Fredholm on all spaces Y.*

b) For simplicity, [78, Lemma 2.1] was only stated for band operators but its proof actually applies to all operators $A \in W$. We make use of this fact in the proof of our Corollary 6.46.

c) Also, for simplicity, [78, Lemma 2.1] was only stated for operators on $Y^p(\mathbb{C})$ instead of $Y^p(U)$ with $n := \dim U < \infty$. This is not a loss of generality since these two spaces are isomorphic and the discussed operator properties are preserved under this isomorphism. Indeed, fix a basis in U and let $\phi : U \to \mathbb{C}^n$ refer to the isomorphism that maps $u \in U$ to its coordinate vector $\phi(u) =: (\phi_1(u), ..., \phi_n(u)) \in \mathbb{C}^n$ with respect to this basis. Then $\Phi : Y^p(U) \to Y^p(\mathbb{C})$ with

$$(\Phi x)(n \cdot m_1 + k,\, m_2,\, \dots,\, m_N) \;=\; \phi_k\big(x(m_1, ..., m_N)\big) \in \mathbb{C}, \qquad k \in \{1, ..., n\},\; m_1, ..., m_N \in \mathbb{Z}$$

for every $x \in Y^p(U)$ is an isomorphism. On the operator side, roughly speaking, the matrix representation of an operator on $Y^p(U)$ is an infinite matrix the entries of which are $n \times n$ matrices (operators on $U = \mathbb{C}^n$, via ϕ). Via Φ this matrix is identified, in a natural way, with an infinite matrix with scalar entries, and this is the setting in which [78, Lemma 2.1] applies. Note that this identification preserves membership of the Wiener algebra, Fredholmness, and the index of the operator. $\square$

Finally, we note that in the one-dimensional case $N = 1$ we have the following refinement of Corollaries 6.43, 6.45 and 6.46, as a consequence of Theorem 6.31.

Corollary 6.48. *Suppose $N = 1$ and that $A = I - K \in W$ is rich and $K \in UM(Y)$. Then the following statements are equivalent:*

(a) *All limit operators of A are injective on Y^∞;*
(b) *All limit operators of A are invertible on* **one** *of the spaces Y;*
(c) *All limit operators of A are invertible on* **all** *the spaces Y and (6.34) holds;*
(d) *A is invertible at infinity on* **all** *the spaces Y;*
(e) *A is invertible at infinity on* **one** *of the spaces Y;*
(f) *A is Fredholm on* **all** *the spaces Y.*

In the case that U has a predual U_* and A, considered as an operator on $Y^\infty(U) = \ell^\infty(\mathbb{Z}, U)$, has a preadjoint A_* on $Y^1(U_*) = \ell^1(\mathbb{Z}, U_*)$, then (a)–(f) are equivalent to

(g) *A is Fredholm on* **one** *of the spaces Y;*

and on every space Y it holds that

$$\text{spec}_{\text{ess}}(A) \;=\; \bigcup_{B \in \sigma^{\text{op}}(A)} \text{spec}(B) \;=\; \bigcup_{B \in \sigma^{\text{op}}(A)} \text{spec}^\infty_{\text{point}}(B). \tag{6.36}$$

Here we denote by $\text{spec}^\infty_{\text{point}}(B)$ the point spectrum (set of eigenvalues) of B, considered as an operator on Y^∞.

Corollary 6.49. *Suppose $N = 1$, $A \in W$ and U is finite-dimensional. Then statements (a)–(g) of Corollary 6.48 are equivalent6.46, as of.ary 6.48 are equivalen]T/F7428(of)-40f bt*

CHAPTER 7

Discrete Schrödinger Operators

In this chapter we illustrate the results of Chapter 6, in particular the results of Section 6.5, in the relatively simple but practically relevant setting of $Y = Y^p = Y^p(U) = \ell^p(\mathbb{Z}^N, U)$ with $p \in \{0\} \cup [1, \infty]$ and a finite-dimensional space U. For applications to a class of operators on $Y^p(U)$ with U infinite-dimensional, see Chapter 8.

In this chapter we suppose that our operator A is a *discrete Schrödinger operator* on Y in the sense e.g. of [27]. By this we mean that A is of the form

$$A = L + M_b$$

with a translation invariant operator L, i.e. $V_{-\alpha} L V_\alpha = L$ for all $\alpha \in \mathbb{Z}^N$, and with a multiplication operator M_b, given by (4.4), with $b \in Y^\infty(L(U))$. A translation invariant operator L on Y is often referred to as a *Laurent operator*, and the sequence b is typically called the *potential* of A. The matrix representation of L is a *Laurent matrix* $[L] = [\lambda_{i-j}]_{i,j \in \mathbb{Z}^N}$ with $\lambda_k \in L(U)$ for all $k \in \mathbb{Z}^N$. To be able to apply the results of the previous subsections we will suppose that $A = L + M_b \in L(Y^p)$, for $1 \le p \le \infty$, which is the case if $L \in W$, i.e. if

$$\|L\|_W = \sum_{k \in \mathbb{Z}^N} \|\lambda_k\| < \infty.$$

Discrete (or lattice) Schrödinger operators are widely studied in mathematical physics (see e.g. [76, § [illegible]

We will say that A is *symmetric* if $A = A$. For example, this is the case when L is the classical operator (7.1).

For $b \quad Y^{\infty}(L(U))$ let $\mathrm{Lim}\,(b)$ denote the set of *limit functions* of b, by which we mean the set of all functions $b_h \quad Y^{\infty}(L(U))$ for which there exists a sequence $h : \mathrm{N} \to \mathrm{Z}^N$ tending to infinity such that

$$b_h(m) = \lim_{n \to \infty} b(m + h(n)), \quad m \quad \mathrm{Z}^N. \tag{7.2}$$

It follows from (4.5) that

$$\sigma^{\mathrm{op}}(A) = \{L + M_c : c \quad \mathrm{Lim}\,(b)\}.$$

Noting that Corollary 6.46 applies to $A = L + M_b$, we have the following result. In this result, for an operator $B \quad W$ we denote by $\mathrm{spec}^p(B)$, $\mathrm{spec}^p_{\mathrm{ess}}(B)$, and $\mathrm{spec}^p_{\mathrm{point}}(B$

Remark 7.2 We note that main parts of the above result, namely equality (7.3) and that the spectrum and essential spectrum do not depend on $p \in [1, \infty]$, are well known (see e.g. [74, Theorem 5.8.1]). The characterisation of the essential spectrum by (7.4) appears to be new. □

Clearly, equations (7.3) – (7.6) simplify when L is symmetric, for example if L is given by (7.1), since we then have that $\mathrm{spec}^1_{\mathrm{point}}(A^*) = \mathrm{spec}^1_{\mathrm{point}}(A)$ and $\mathrm{spec}^1_{\mathrm{point}}(L + M_c) \subset \mathrm{spec}^\infty_{\mathrm{point}}(L + M_c)$ for all $c \in \mathrm{Lim}\,(b)$. Simplifications also occur when the potential b is almost periodic, $b \in Y^\infty_{\mathrm{AP}}(L(U))$, in which case $\mathrm{Lim}\,(b)$ is precisely what is often called the *hull* of b, the set $\mathrm{clos}_{Y^\infty(L(U))}\{V_k b : k \in \mathbb{Z}^N\}$, the closure of the set of translates of b.

Theorem 7.3. *If b is almost periodic then, for all p and all $\tilde{b} \in \mathrm{Lim}\,(b)$,*

$$\mathrm{spec}^p_{\mathrm{ess}}(A) = \mathrm{spec}^p(A) = \mathrm{spec}^p(L + M_{\tilde{b}}) = \bigcup_{c \in \mathrm{Lim}\,(b)} \mathrm{spec}^\infty_{\mathrm{point}}(L + M_c). \tag{7.7}$$

Proof. L is absolutely rich/periodic and so norm rich. Since b is almost periodic, M_b is norm rich by Lemma 6.35. Thus A is norm rich. Further, $\sigma^{\mathrm{op}}(A - I)$ is uniformly bounded by Theorem 5.12 (i) and so uniformly Montel on Y^∞ by Corollary 3.25, since $\dim U < \infty$. The result thus follows from Theorem 6.38, Theorem 6.10 (iv), and the equivalence of statements (b) and (d) in Theorem 7.1. ■

Remark 7.4 That $\mathrm{spec}^p_{\mathrm{ess}}(A) = \mathrm{spec}^p(A) = \mathrm{spec}^p(L + M_{\tilde{b}})$ for all $\tilde{b} \in \mathrm{Lim}\,(b)$, the hull of b, is a classical result, see e.g. [83, 89, 74]. The result that $\mathrm{spec}^p(A) = \bigcup_{c \in \mathrm{Lim}\,(b)} \mathrm{spec}^\infty_{\mathrm{point}}(L + M_c)$ appears to be new in this discrete setting, although analogous results for uniformly elliptic differential operators on $\mathbb{R}^N$ with almost periodic coefficients date back to Shubin [87].

Moreover, note that this result is well-known, as a part of Floquet-Bloch theory [40, 41, 28], in the case when b is periodic; in fact one has the stronger result in that case, at least when L is given by (7.1), that λ is in the spectrum of A iff there exists a solution $x \in Y^\infty(U)$ of $\lambda x = Ax$ which is quasi-periodic in the sense of [40]. The latter means that $x(m) = \exp(ik \cdot m)y(m)$ for all $m \in \mathbb{Z}^N$, where $y \in Y^\infty(U)$ is periodic and $k \in \mathbb{R}^N$ is fixed, so that if x is quasi-periodic then it is certainly almost periodic. Thus, if b is periodic then λ is in the spectrum of A iff there exists a solution $x \in Y^\infty(U)$ of $\lambda x = Ax$ which is almost periodic.

Natural questions are whether this statement still holds for the case when b is almost periodic, at least for L given by (7.1), or whether the weaker statement holds that λ is in the spectrum of A iff, for some $c \in \mathrm{Lim}\,(b)$, there exists a solution $x \in Y^\infty(U)$ of $\lambda x = (L + M_c)x$ which is almost periodic. The answer is 'no' on both counts. In particular, in the case $N = 1$ and L given by (7.1), see [26, 65] and [64, p. 454], there exist almost periodic potentials b for which the spectrum of A as an operator on $Y^2(U)$ has the property that every solution $x \in Y^\infty(U)$ of $\lambda x = (L + M_c)x$, for some $c \in \mathrm{Lim}\,(b)$ decays exponentially at infinity. □

To illustrate the application of the above theorem in the 1D case ($N = 1$) we consider a widely studied class of almost periodic operators obtained by the following construction. For some $d \in \mathbb{N}$ let $B : \mathbb{R}^d \to L(U)$ be a continuous function satisfying

$$B(s + m) = B(s), \quad s \in \mathbb{R}^d,\ m \in \mathbb{Z}^d.$$

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ and, for $s \in \mathbb{R}^d$ let $b_s : \mathbb{Z} \to L(U)$ be given by

$$b_s(n) = B(\alpha n + s), \quad n \in \mathbb{Z}. \tag{7.8}$$

If $\alpha_1, \dots, \alpha_d$ are all rational, then b_s is periodic. Whatever the choice of $\alpha_1, \dots, \alpha_d$, b_s is almost periodic ($b_s \in Y^\infty_{\mathrm{AP}}(L(U))$).

For $s \in \mathbb{R}^d$ let $[s]$ denote the coset $[s] = s + \mathbb{Z}^d$ in $\mathbb{R}^d/\mathbb{Z}^d$. An interesting case is that in which $1, \alpha_1, \alpha_2, \dots, \alpha_d$ are rationally independent, in which case $\{[\alpha m] : m \in \mathbb{Z}\}$ is dense in $\mathbb{R}^d/\mathbb{Z}^d$. Then it is a straightforward calculation to see that

$$\mathrm{Lim}\,(b_s) = \{b_t : t \in \mathbb{R}^d\}. \tag{7.9}$$

Thus, for this case, (7.7) reads as

$$\mathrm{spec}^p_{\mathrm{ess}}(L + M_{b_s}) = \mathrm{spec}^p(L + M_{b_s}) = \bigcup_{t \in \mathbb{R}^d} \mathrm{spec}^\infty_{\mathrm{point}}(L + M_{b_t}). \tag{7.10}$$

As a particular instance, this formula holds in the case when $U = \mathbb{C}$, $d = 1$, and $B(s) = \lambda \cos(2\pi s)$, $s \in \mathbb{R}$, for some $\lambda \in \mathbb{C}$. Then

$$b_s(n) = \lambda \cos(2\pi(\alpha n + s)), \quad n \in \mathbb{Z}, \tag{7.11}$$

and (7.10) holds if α is irrational, in which case b_s is the so-called *almost Mathieu* potential.

We next modify the above example to illustrate the application of Theorem 7.1 in a particular 1D ($N = 1$) case.

Example 7.5 Define $b_s \in Y^\infty_{\mathrm{AP}}(L(U))$ by (7.8) and suppose that $1, \alpha_1, \alpha_2, \dots, \alpha_d$ are rationally independent. Suppose that $f : \mathbb{Z} \to \mathbb{R}^d$ satisfies

$$\lim_{|n| \to \infty} |f(n+1) - f(n)| \to 0.$$

Define $b \in Y^\infty(L(U))$ by

$$b(n) = B(\alpha n + f(n)), \quad n \in \mathbb{Z}.$$

Then it is straightforward to see that $\mathrm{Lim}\,(b) \subset \{b_s : s \in \mathbb{R}^d\}$. Since [51, Corollary 3.97], $b_s \in \mathrm{Lim}\,(b)$ implies that $\mathrm{Lim}\,(b_s) \subset \mathrm{Lim}\,(b)$, we have, by (7.9), that

$$\mathrm{Lim}\,(b) = \{b_s : s \in \mathbb{R}^d\}.$$

Thus, applying Theorem 7.1 and (7.10), we see that, for every $s \in \mathbb{R}^d$ and every $p \in \{0\} \cup [1, \infty]$,

$$\mathrm{spec}^p_{\mathrm{ess}}(L + M_b) = \mathrm{spec}^p_{\mathrm{ess}}(L + M_{b_s}) = \mathrm{spec}^p(L + M_{b_s}) = \bigcup_{t \in \mathbb{R}^d} \mathrm{spec}^\infty_{\mathrm{point}}(L + M_{b_t}). \tag{7.12}$$

We note that, in the special case that L is given by (7.1) (with $N = 1$), $U = \mathbb{C}$, and B is real-valued, the statement that

$$\mathrm{spec}^2_{\mathrm{ess}}(L + M_b) = \mathrm{spec}^2(L + M_{b_s})$$

for all $s \in \mathbb{R}^d$ is Theorem 5.2 of Last and Simon [47] (established by limit operator type arguments). As a specific instance where (7.12) holds, let us take $U = \mathbb{C}$, $d = 1$, and $B(s) = \lambda \cos(2\pi s)$, $s \in \mathbb{R}$, for some $\lambda \in \mathbb{C}$. Then b_s is given by (7.11) and, taking (as one possible choice), $f(n) = |n|^{1/2}$, one has

$$b(n) = \lambda \cos(2\pi(\alpha n + |n|^{1/2}))$$

(cf. [47, Theorem 1.3]). □

As a further example we consider the case when b is pseudo-ergodic in the sense of Davies [27]. Following [27], we call the function $b \in Y^\infty(\mathbb{Z}^N, U)$ *pseudo-ergodic*, if, for every $\varepsilon > 0$, every finite set $S \subset \mathbb{Z}^N$ and every function $f : S \to \Sigma := \mathrm{clos}_U\, b(\mathbb{Z}^N)$, there is a $z \in \mathbb{Z}^N$ such that

$$\|f(s) - b(z + s)\|_U < \varepsilon, \qquad \forall s \in S.$$

One can show [51, Corollary 3.70] that b is pseudo-ergodic iff $\mathrm{Lim}\,(b)$ is the set $\Sigma^{\mathbb{Z}^N}$ of all functions $c : \mathbb{Z}^N \to \Sigma$. In particular, $b \in \mathrm{Lim}\,(b)$ if b is pseudo-ergodic.

Theorem 7.6. *If b is pseudo-ergodic then, for all p,*

$$\mathrm{spec}^p_{\mathrm{ess}}(A) = \mathrm{spec}^p(A) = \bigcup_{c \in \Sigma^{\mathbb{Z}^N}} \mathrm{spec}^p(L + M_c) = \bigcup_{c \in \Sigma^{\mathbb{Z}^N}} \mathrm{spec}^\infty_{\mathrm{point}}(L + M_c).$$

Proof. The first two '=' signs follow from (7.3) and the fact that $b \in \mathrm{Lim}\,(b) = \Sigma^{\mathbb{Z}^N}$. For the proof of the remaining equality, we refer to the following s-dense subset of $\mathrm{Lim}\,(b) = \Sigma^{\mathbb{Z}^N}$: Let $m_1 = m_2 = \ldots = m_N = 1$, and let ς stand for the set of all periodic functions $x : \mathbb{Z}^N \to \Sigma$, that is

$$\varsigma := \bigcup_{n \in \mathbb{N}} Y^\infty_n(\Sigma)$$

with $Y^\infty_n(\Sigma)$ defined as in (6.4) (with the slight abuse of notation by writing $Y^\infty(\Sigma)$ for $\Sigma^{\mathbb{Z}^N}$, i.e. the set of all functions $x : \mathbb{Z}^N \to \Sigma$). Then ς is s-dense in $\Sigma^{\mathbb{Z}^N}$ as every $x \in \Sigma^{\mathbb{Z}^N}$ can be strictly approximated by the sequence $(\tilde{P}_n x) \subset \varsigma$ with $\tilde{P}_n$ as defined in (6.5). If $\lambda \in \mathbb{C}$ and all limit operators $\lambda I - (L + M_c)$ of $\lambda I - A = \lambda I - (L + M_b)$, including those with $c \in \varsigma$, are injective, then, by Theorem 6.7, we have that $\lambda I - (L + M_c)$ is surjective for every $c \in \varsigma$. By the equivalence between (a) and (d) in Theorem 7.1, this shows that $\lambda I - A = \lambda I - (L + M_b)$ is Fredholm. ■

Remark 7.7 It is shown that

$$\mathrm{spec}^2_{\mathrm{ess}}(A) = \mathrm{spec}^2(A) = \bigcup_{c \in \Sigma^{\mathbb{Z}^N}} \mathrm{spec}^2(L + M_c)$$

in [27]. The result that $\mathrm{spec}^p(A) = \bigcup_{c \in \Sigma^{\mathbb{Z}^N}} \mathrm{spec}^\infty_{\mathrm{point}}(L + M_c)$ appears to be new. □

The above theorems show that, in each of the cases L symmetric, b almost periodic, and b pseudo-ergodic, it holds that

$$\mathrm{spec}^p_{\mathrm{ess}}(A) = \bigcup_{c \in \mathrm{Lim}\,(c)} \mathrm{spec}^\infty_{\mathrm{point}}(L + M_c). \tag{7.13}$$

We conjecture that, in fact, this equation holds for all $c \in Y^\infty(L(U))$. For $N = 1$ this is no longer a conjecture, as we showed in Corollary 6.49 (which follows from our more general results in [17], also see Theorem 6.31 above). For $N \geq 2$ however, this is an open problem.

We finish this chapter with an example demonstrating how Theorem 7.6 can be used to compute spectra of Schrödinger operators with random potential b.

Example 7.8 Let $N = 1$, $p \in [1, \infty]$, $U = \mathbb{C}$ and take a compact set Σ in the complex plane. We compute the spectrum of $A = L + M_b$ as an operator on

$Y = Y^p(U) = \ell^p(\mathbb{Z})$ where $L = V_{-1}$ is the backward shift and the function values $b(k)$, $k \in \mathbb{Z}$, of the random potential b are chosen independently of each other from the set Σ. We assume that, for every $\sigma \in \Sigma$ and $\varepsilon > 0$, $\mathrm{P}(|b(k) - \sigma| < \varepsilon) > 0$. Then it is easy to see (the argument is sometimes called 'the Infinite Monkey Theorem' and it follows from the Second Borel Cantelli Lemma, see [**12**, Theorem 8.16] or [**25**, Theorem 4.2.4]) that, with probability 1, b is pseudo-ergodic.

For the calculation of the point spectra in Theorem 7.6, let $c \in \Sigma^{\mathbb{Z}}$ and take $\lambda \in \mathbb{C}$. If $x : \mathbb{Z} \to \mathbb{C}$ is a nontrivial solution of $(L + M_c)x = \lambda x$ then $x(n_0) \neq 0$ for some $n_0 \in \mathbb{Z}$ and $x(k+1) = (\lambda - c(k))\,x(k)$ for all $k \in \mathbb{Z}$. Note that $\lambda \neq c(k)$ for all $k < n_0$ since otherwise $x(n_0) = 0$ and, w.l.o.g., suppose that $x(n_0) = 1$. As a consequence we get that

$$x(n) = \begin{cases} \displaystyle\prod_{k=n_0}^{n-1} (\lambda - c(k)), & n \geq n_0, \\ \displaystyle\prod_{k=n}^{n_0-1} (\lambda - c(k))^{-1}, & n < n_0 \end{cases} \tag{7.14}$$

for every $n \in \mathbb{Z}$. Now put, for $r > 0$,

$$\Sigma^r := \bigcup_{\sigma \in \Sigma} (\sigma + r\overline{\mathbb{D}}) \qquad \text{and} \qquad \Sigma^r_\circ := \bigcup_{\sigma \in \Sigma} (\sigma + r\mathbb{D})$$

with $\mathbb{D}$ denoting the open unit disk in $\mathbb{C}$ and $\overline{\mathbb{D}}$ its closure.

D)x

Clearly, if $\lambda \notin \Sigma^1$ then $|\lambda - \sigma| > 1$ for all $\sigma \in \Sigma$ and hence25.4286Tf41.5670Td[(ITf8.8170Td[(2))]TJ/F89.96

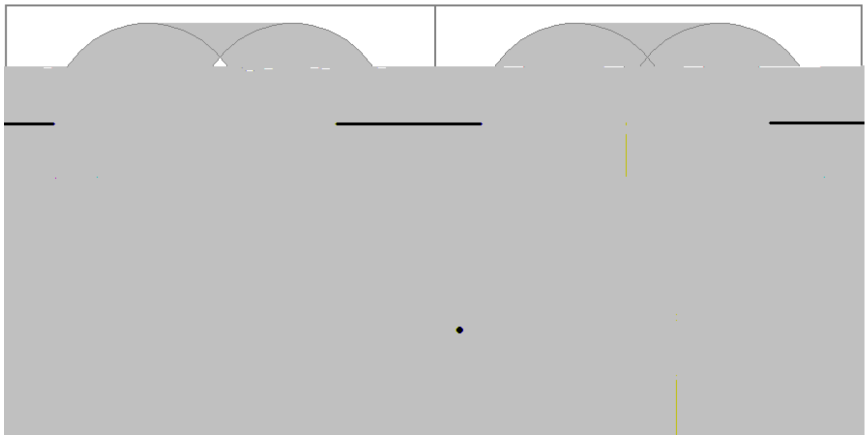

Figure 7.1. The left image shows, as a gray shaded area, $\mathrm{spec}^p A$ when Σ is the black straight line of length 1.5. In the right image, one more point (the centre of the lower circle) has been added to Σ which results in $\Sigma^1_\cap = \square$.

L on Y, where every entry λ_k of $[L] = [\lambda_{i-j}]_{i,j \in \mathbb{Z}^N}$ is the operator of convolution by $\kappa(\cdot + k)$ on U. By Young's inequality, we get that

$$\|\lambda_k\| \leq \|\kappa|_{k+[-1,1]^N}\|_1$$

for every $k \in \mathbb{Z}^N$, and by $\kappa \in L^1(\mathbb{R}^N)$ it follows that $L \in W = W(U)$.

We denote by A^o the smallest algebra in $L(L^{p,q})$ containing all operators of these two types; that is the set of all finite sum-products of operators of the form M_b and C_κ with $b \in L^\infty_\$$ and $\kappa \in L^1$. From the above considerations it follows that every operator $A \in A^o$, if identified with an operator on Y, is contained in the Wiener algebra W. By A we denote the closure of A^o in the norm $\|\cdot\|_W$. Note that, by (6.33), the closure of a set $S \subset W$ in the W-norm is always contained in the closure of S in the usual operator norm.

Lemma 8.1. *The predual space U' exists and, if $p = \infty$, then every $A \in A$ has a preadjoint operator A' on $L^{1,q'}$ with $1/q + 1/q' = 1$.*

Proof. By the choice $q \in (1, \infty]$, it is clear that the predual space U' of $U = L^q([0,1]^N)$ exists and can be identified with $L^{q'}([0,1]^N)$, where $1/q + 1/q' = 1$, including the case $q' = 1$ if $q = \infty$. Now suppose $p = \infty$. Then the predual $Y^1(U')$ of

Proof. If A A is invertible at infinity and b from its representation (8.2) is invert[redacted]
ible in L^∞, then A is Fredholm by Theorem 6.28 (ii) and Lemma 8.2. Conversel[redacted]
let A A be Fredholm. By Theorem 6.28 (i) and b), together with Lemma 8.1, w[redacted]
get that A is invertible at infinity. It remains to show that b from (8.2) is invertib[redacted]
in L^∞. To see this, take B $L(L^{p,q})$ and S, T $K(L^{p,q})$ such that $AB = I +$ [redacted]
and $BA = I + T$. Then, for every k

CHAPTER 9

Some Open Problems

We conclude with a small list of open problems the solutions of which, we believe, could be crucial in extending the picture that we have tried to draw in this text. We would like to see this list as both a future agenda for ourselves as well as an invitation for the interested reader.

1. Is a version of the limit operator theory possible with $L(Y,P)$ and $K(Y,P)$ replaced by $S(Y)$ and $SN(Y)$? We know from Lemma 3.3, Corollary 3.5, (3.5) and Lemma 3.10 that $S(Y)$ and $SN(Y)$ are "one-sided versions" of $L(Y,P)$ and $K(Y,P)$. Moreover, $SN(Y)$ is a closed two-sided ideal in $S(Y)$ – just like $K(Y,P)$ is in $L(Y,P)$. The ideal $K(Y,P)$ shapes the theory presented here in two ways: It defines the notion of invertibility at infinity (see Definition 5.1) and that of P-convergence (see Definition 4.1). How do these properties change if one works with the ideal $SN(Y)$ instead and what is the connection between the new notions of 'invertibility at infinity' and 'limit operator'?

2.

6.43, 6.45 and 6.48? Recall that we use the existence of predual and preadjoint to conclude Fredholmness of $A_0 = A|_{Y_0}$ from that of A on Y^∞, including preservation of the index.

5. *Is the hyperplane condition redundant in Theorem 6.44?*

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