An applicati n f the re-iterated Galerkin appr ximati n in -dimensi ns

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I confirm that this is my own work
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C a ter 1

An acoustics roblem

Consider the following idealised acoustics problem.

In a region that will be known as a *wavegu de*, of height b, and length and breadth of far greater dimension, a continual disturbance at one of the ends causes a disruption to the equilibrium state within the waveguide. The sound waves that are produced by the disturbance travel through the waveguide towards the opposite end. ithin the waveguide, the relationship between pressure and density, and hence wave speed, is constant as in air) in all but a fixed region. This region is referred to as the *obstacle*. The effect of the obstacle is to scatter the incident waves. At some non-specific point in time after the disturbance first occurs, the behaviour within the waveguide settles to a periodic steady state.

In addition, it is known that the geometry of the obstacle is that it resembles a uniform tube, lying in the breadth direction. That is, it has a constant height by length cross-section. The walls i.e. boundaries of the waveguide) are highly absorbent, or so-called 'sound-soft'. This will be important later.

From this information, we aim to construct a mathematical model that will determine the structure of this periodic steady state. By concentrating our attention on the steady state problem, we are absolved from any need for an initial state, that could lead to an initial value problem. The steady state problem is of use in determining the energy transfer that results from the wave scattering. This issue will become better defined in due course.

The inherent symmetry in the problem means that, under some simplifying assumptions, one of the three spatial dimensions is redundant. Specifically, if it is assumed that the motion of the sound waves caused by the disturbance is perpendicular to the generators of the obstacle, that is, entirely without breadth direction, then the problem is unsensitive to breadth variation. This means it may be solved by working only over a length by height cross-sectional portion. This will be U)D˜˜˜ :7V7:fi6,i?zxB77

For the present, the subdomain on which the wave speed varies will be undefined, suffice to say that it is connected and bounded. It will be denoted D', and obviously $D' \subset D$.

It is assumed that the disturbance is small in amplitude, in which case we may solve for the $l\ \ ear$ wave profile $\Phi = \Phi\ x, y, t)$. It is well known see [6] for example) that Φ satisfies the *wave equat o*

$$\nabla^2 \Phi = \frac{1}{c^2} \Phi_{tt} \qquad \text{in } D, \forall t, \tag{1.1}$$

where $c = c\ x, y)$ denotes the wave speed over D. As the wave equation is linear, Φ is the linear combination of solution waves. In particular, Φ may be constructed as the superposition of harmonic waves, as is manifest in a *Four er ser es* representation

$$\Phi\ x, y, t) = \sum_{n=-}\ \phi_n\ x, y) e^{-\ \omega_n t}. \tag{1.2}$$

Eac

1.1 Non-di ension lis tion

By suitably redefining the independent variables in such a way that one dimension is set to a non-dimensional scale, we gain the practical advantage of removing a parameter from matters. Consistency then dictates that all dimensions are lost, and the problem is said to be *d me s o less.*

The choice of re-scaling is ours. So, let

$$x' = ^{\pi}$$

For large positive or negative x, that is x away from D', $\phi \sim \tilde{\phi}$ such that

$$\nabla^2 \tilde{\phi} + k_0^2 \tilde{\phi} = 0,$$

and $\tilde{\phi}$ satisfies condition 1.5). The separation solutions of this equation may easily be determined, but first let us make a further simplifying assumption. Although generalip

is a wave travelling in the negative x direction. Therefore, from

to be solved for the A_n

Consider applying the appropriate conditions to the cases A_1 and A_n $n \geq 2$) separately.

In the limit $x \to -\infty$, the inequality $x_0 > x$ holds, hence

$$A_1 = c_1 e^{\beta_0 x} + c_2 e^{-\beta_0 x}, \qquad \text{as } x \to -\infty$$

and the first part of condition 1.12) implies that $c_1 = 0$.

Now, taking the limit $x \to \infty$, so that $x_0 < x$,

$$\begin{aligned} A_1(x \mid x_0, y_0) &= c_2 e^{-\beta_0 x} - \frac{2 \sin y_0) \sin\left({}_0 x - x_0)\right)}{{}_0\pi} \\ &= c_2 e^{-\beta_0 x} + i\frac{\sin y_0)}{{}_0\pi} e^{\beta_0(x-x_0)} - e^{-\beta_0(x-x_0)}) \\ &= c_2 - i\frac{\sin y_0)}{{}_0\pi} e^{\beta_0 x_0})e^{-\beta_0 x} + i\frac{\sin y_0)}{{}_0\pi} e^{\beta_0(x-x_0)} \end{aligned}$$

and to satisfy the second part of condition 1.12) we must set

$$c_2 = i\frac{\sin y_0)}{{}_0\pi} e^{\beta_0 x_0}.$$

This leaves

$$A_1(x \mid x_0, y_0) = \begin{cases} i\frac{\sin(y_0)}{\beta_0 \pi} e^{-\beta_0(x-x_0)} & x_0 > x) \\ i\frac{\sin(y_0)}{\beta_0 \pi} e^{-\beta_0(x-x_0)} - \frac{2 \sin y_0) \sin\left({}_0 x - x_0)\right)}{{}_0\pi} & x_0 < x). \end{cases}$$

It is easily shown that G satisfies the symmetry property

$$G(x, y \mid x_0, y_0) = G(x_0, y_0 \mid x, y), \qquad \forall\, x, y),\ x_0, y_0) \quad D : x, y) \neq x_0, y_0).$$

This enables us to write A_1 as a single expression or, it at least saves much calculation in doing so)

$$A_1(x \mid x_0, y_0) = i\frac{\sin y_0)}{{}_0\pi} e^{\beta_0 |x-x_0|} \qquad x, y),\ x_0, y_0) \quad D.$$

Similarly, as $x \to -\infty$

$$A_n(x \mid x_0, y_0) = p\ e\ y\ i\ y\ \partial \quad y\ N \quad i\ i > q\ q\ I\ \chi\ \chi\ \chi\ P > i\ E > \quad y\ i\ L\ \partial \quad y\ N\ r$$

and we must set the constant

$$c_3 = \frac{\sin\ ny_0)}{\gamma_n \pi} e^{-\gamma_n x_0}$$

for boundedness.

Thus

$$\begin{aligned} A_n\ x \mid x_0, y_0) &= \begin{cases} \frac{\sin(ny_0)}{\gamma_n \pi} e^{\gamma_n (x - x_0)} & x_0 > x) \\ \frac{\sin(ny_0)}{\gamma_n \pi} e^{\gamma_n (x - x_0)} - \frac{2\sin(ny_0)}{\gamma_n \pi} \sinh\big(\gamma_n\ x - x_0)\big) & x_0 < x) \end{cases} \\ &= \frac{\sin\ ny_0)}{\gamma_n \pi} e^{-\gamma_n |x - x_0|} \qquad x, y),\ x_0, y_0) \quad D. \end{aligned}$$

The required Green's function is now defined as

$$\begin{aligned} G\ x, y \mid x_0, y_0) &= \frac{i}{{}_0\pi} \sin\ y_0) \sin\ y) e^{\ \beta_0 |x - x_0|} \\ &+ \sum_{n=2} \frac{1}{\gamma_n \pi} \sin\ ny_0) \sin\ ny) e^{-\gamma_n |x - x_0|}. \end{aligned} \tag{1.13}$$

This series expression is convergent everywhere except the point $x, y) = x_0, y_0)$, where G has the singularity spoken of. Now

$$\frac{1}{\gamma_n} \sim \frac{1}{n}$$

for large n implies that the singularity is logarithmic in nature this will be made explicit in §6.3). Such functions are measurable and hence expressions containing the Green's function beneath an integral sign are well defined.

e proceed by applying Green's theorem in the plane

$$\int_D \big(\ \nabla^2 v - v\nabla^2\ \big) \mathrm{d}x\mathrm{d}y = \int_{\delta D} \big(\ \frac{\partial v}{\partial n} - v\frac{\partial}{\partial n}\big) \mathrm{d}c \tag{1.14}$$

to our problem, where $\frac{\partial}{\partial n}$ denotes the outward unit normal derivative.

Substituting $= \phi$ and $v = G$ in 1.14) gives

$$\int_D \big(\phi \nabla^2 G - G\nabla^2 \phi\big) \mathrm{d}x\mathrm{d}y = \int_{\delta D} \big(\phi \frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\big) \mathrm{d}c.$$

Let us analyse each side of this equality in turn, beginning with the left hand side

$$\int_D \big(\phi \nabla^2 G - G\nabla^2 \phi\big) \mathrm{d}x\mathrm{d}y$$

$$= \int_D \big[\phi\ x, y)\big(-k_0^2 G\ x, y \mid x_0, y_0) - \delta\ x - x_0)\delta\ y - y_0)\big) - G\ x, y \mid x_0, y_0)\big(-k^2\ x, y)\phi\ x, y)\big)\big] \mathrm{d}x\mathrm{d}y$$

$$= -\phi\ x_0, y_0) + \int_{D'} k^2\ x, y) - k_0^2)\phi\ x, y) G\ x, y \mid x_0, y_0) \mathrm{d}x\mathrm{d}y.$$

In order to implement the conditions as $|x| \to \infty$, it is necessary to evaluate the right hand side using the finite domain $D_X = \{x, y : |x| < X, 0 < y < \pi$, with boundary $\delta D_X = \{x, y : x \leq |X|$ on $y = 0, \pi$ and $0 \leq y \leq \pi$ on $x = \pm X$. For large X, recall that $\phi = 0$ on $y = 0, \pi)$

$$\int_{\delta D_X} \Big(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n} \; dc$$

$$= - \int_0^{\pi} \big(\phi \frac{\partial G}{\partial x} - G \frac{\partial \phi}{\partial x}\big)_{x=-X} \mathrm{d}y + \Big\} \Big[$$

$$\pi$$

$$+ \int_{\pi}$$

$$+ \int_{\pi}$$

both of which depend on the solution ϕ in D'.

Therefore, producing approximations to the reflected and transmitted energies may be regarded as our final goal, with the approximations to ϕ in D' that must be made en route to achieve this final goal, coming almost as a by-product.

C a ter 2

Integral equations

In §1 we found that an attempt to solve a differential equation, that models a realistic situation, using a Green's function, results in an implicit expression for the solution, in the form of an integral equation. This example will serve as both motivation and a suitable illustration for what follows.

Background reading on integral equations can be found in Porter and Stirling [1].

The term *tegral equat o* refers to an unknown function appearing beneath an integral sign. Integral equations can often appear as reformulations of initial and boundary value problems. Although the relative merits of using either formulation over the other will not be pursued, it is worth highlighting that integral equations have the advantage that they carry no extra conditions: all boundary and initial values are encompassed by the equation. It has also been noted that the

2.1 Inte r l e u tions of the second kind

The one-dimensional, scalar integral equation of the *seco d k d* takes the form

$$\phi\ x) = f\ x) + \lambda \int_a^b k\ x,t)\phi\ t)dt \qquad a \leq x \leq b).$$

Definition The *ra k* of an operator is the dimension of its range.

If the operator K

the required assumptions can not be expected to hold in general. This point is being emphasised as methods that will be investigated in subsequent chapters seek to approximate the solution of 2.5) rather than 2.1). In particular, it will be shown that the two-dimensional example of the first chapter th

C a ter 3

alerkin Met ods

Our main problem in finding an analytic solution to 2.4) is in our inability to deal with infinities. It is then natural to constuct our approximations in finite dimensional spaces where we may easily produce a general framework to solve equations. e seek to strike a balance between letting the finite dimension get large, as it will obviously play a role in increasing the accuracy of the approximation, and keeping the dimension small enough to simplify calculations.

3.1 The G lerkin ethod

This is a well-established and widely used form of *project o method.* e may attempt to make a Galerkin approximation in any finite dimensional subspace of $\mathscr{H}$. Let us fix this subspace to be E_N, where N denotes the dimension. As a subspace of a Hilbert space it possesses an orthonormal basis $\{\chi_1, \ldots, \chi_N$, so that

$$E_N = \text{span}\{\chi_1, \ldots, \chi_N \ ,$$

where

$$\chi \ , \chi_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

This basis may be used to define an *orthogo al project o* P_N from $\mathscr{H}$ into E_N

$$\begin{aligned} P_N : \mathscr{H} &\to E_N \\ \varphi &\mapsto \sum_{=1}^{N} \varphi, \chi \ )\chi \ . \end{aligned} \tag{3.1}$$

Now, note that

$$\varphi = \sum_{=1} \varphi, \xi \ )\xi$$

for any orthonormal basis $\{\xi_1, \xi_2, \ldots$ of $\mathscr{H}$. So P_N reduces each element in the Hilbert space into its components that lie in the chosen subspace, and hence

$$P_N p = p \qquad \forall p \quad E_N. \tag{3.2}$$

The projection P_N has the property that it maps each element in $\mathscr{H}$ onto the element that best approximates it

e solve equation 2.4) for its components within the subspace. This is achieved by using the projected forms of the terms on the right hand side of 2.4):

$$f \mapsto P_N f \qquad K \mapsto P_N K.$$

The solution p_N of this projected equation , which is obviously in the subspace, is the Galerkin approximation

$$\begin{aligned} p_N &= P_N f + P_N K p_N \\ \Rightarrow p_N &= \ I - P_N K)^{-1} P_N f. \end{aligned} \tag{3.4}$$

It has been assumed that this solution exists, ie. the operator $I - P_N K)^{-1}$ exists. e shall see that, this assumption is guarteed by another assumption made in subsequent approximations.

Definition The functions that correspond to the basis elements are known as *tr al fu ct o s*, and the space of functions spanned by these functions the *tr al space*.

Some properties of the Galerkin solution will now be highlighted.

Only in rare circumstances does the Galerkin approximation p_N coincide with the projection of the solution of 2.4) ϕ. To see this consider the difference between the two elements

$$\begin{aligned} P_N\phi - p_N &= P_N \ I - K)^{-1} f - \ I - P_N K)^{-1} P_N f \\ &= P_N \ I - K)^{-1} - \ I - P_N K)^{-1} P_N) f \\ &= I - P_N K)^{-1} \ I - P_N) P_N - P_N \ I - K)) \ I - K)^{-1} f \\ &= I - P_N K)^{-1} P_N \ K - K P_N)\phi, \end{aligned} \tag{3.5}$$

if $\phi = \sum_{=1} \alpha \chi$, then 3.5) vanishes only if, for all $j > N$, either

$$a_j = 0$$

or

$$K\chi_j, \chi \ ) = 0 \qquad i = 1, \ldots, N.$$

Otherwise $p_N \neq P_N\phi$, and

$$\exists q \quad E_N : \|q - \phi\| < \|p_N - \phi\|, \tag{3.6}$$

so it is not the optimal element in the subspace with respect to the pointwise error). The choice $q = P_N\phi$ will satisfy 3.6).

Nor does the Galerkin approximation minimise the residual error. The approximation q_N that does this satisfies the projected equation

$$Aq_N = Q_N f$$

where Q_N is the orthogonal projection onto the space AE_N.

The orthonormal basis $\{\chi_1, \ldots, \chi_N$ may be extended indefinitely to an orthonormal basis of $\mathscr{H}$. This is what is implied when $N \to \infty$ is written. Using 3.3), as $N \to \infty$

$$P_N\varphi \to \varphi, \qquad \forall\varphi \quad \mathscr{H},$$

and in particular

$$P_N f \to f \qquad \text{as } N \to \infty,$$
$$P_N K \to K \qquad \text{as } N \to \infty.$$

Therefore, despite 3.6), the Galerkin approximation satisfies 2.4) increasingly more closely as $N \to \infty$, and since

$$p_N = \ I - P_N K)^{-1} P_N f \to \ I - K)^{-1} f = \phi \qquad \text{as } N \to \infty,$$

the Galerkin approximation tends towards the exact solution.

To improve our approximation we need only extend our subspace. Unfortunately, as will be shown, in practice this approach on its own is flawed .

The dimension of the subspace is not the only factor that dictates the accuracy of the Galerkin approximation, the choice of basis relative to the problem in hand has a major role. In fact, a priori

T

This says that the null space of P_N is orthogonal to the basis $\{\chi_1, \ldots, \chi_N$ and hence to the space E_N). e conclude that the Galerkin approximation must have a residual error $Ap_N - f$ that satisfies the N equalities

$$\begin{aligned} & Ap_N - f, \chi\ ) = 0 \qquad i = 1, \ldots, N \\ \iff \quad & Ap_N, \chi\ ) = \ f, \chi\ ) \qquad i = 1, \ldots, N. \end{aligned} \tag{3.9}$$

As p_N E_N it may be expressed in terms of the basis $\{\chi_1, \ldots, \chi_N$

$$p_N = \sum_{=1}^{N} a\ \chi \tag{3.10}$$

where the scalar coefficients a are currently unknown as p_N is currently unknown). Substituting 3.10) into 3.9) yields an $N \times N$ system of equations

$$\sum_{j=1}^{N} a_j\ A\chi_j, \chi\ ) = \ f, \chi\ ), \qquad i = 1, \ldots, N \tag{3.11}$$

that may in theory be solved by the earlier assumption that $I - P_N K$ is invertible) for the unknown coefficients a_j. This fixes our Galerkin approximation p_N via 3.10).

The basis $\{\chi_1, \ldots, \chi_N$ was chosen in equations 3.9) to 3.11) for continuity of notation, although any basis of E_N would suffice.

Thus, every time we wish to find a Galerkin approximation we must solve an $N \times N$ system 3.11). Although achievable, this is computationally very expensive and sensitive to the value of N. Another drawback is that information is non-transferable, that is, work done to calculate p_N will not aid calculation of p_{N+1}. For these reasons we wish to keep N small, and so disregard the afore-mentioned approach, of expanding the subspace, for finding a sequence of approximations that tends to the exact solution as, in this approach, N is unrestricted.

Is it possible to improve on the Galerkin approximation p_N without increasing the dimension N? e address this in what remains of this chapter.

Definition If T is a bounded, linear operator on a Hilbert space, then the *adjo t* of T is the unique bounded linear operator T^* with the property

$$T\ , v) = \ , T^* v)$$

for all $, v$ $\mathscr{H}$.

If $T = T^*$, then T is said to be *self-adjo t*.

3.2 Iter ted G lerkin

The structure of 2.4) lends itself to an iteration process. Let us take the Galerkin approximation p_N as our 'initial guess', and iterate using 2.4) to produce the approximation

$$\hat{p}_N = f + Kp_N. \tag{3.12}$$

This intuitively shares more of the exact solution's structure.

Before investigating the effects of this iteration, 3.12) is introduced independent of the Galerkin approximation.

3.2.1 Degenerate kernels

A kernel $k\ x, t)$ is known

If the approximation is written

$$\phi_N = v_0 + \sum_{=1} a\ v\ ,$$

where $Kv_0 = K_N v_0 = 0$, then by 3.15)

$$\begin{aligned} K_N \phi_N &= \sum_{=1} a\ K_N v \\ &= \sum_{=1}^{N} a\ K_N v\ . \end{aligned} \qquad 3.16)$$

Substituting 3.16) into 3.13), we obtain the expression

$$\phi_N = f + \sum_{=1}^{N} a\ Kv \qquad 3.17)$$

and all that m

3.2.2 Analysis of $\hat{p}_N$

Can it be shown that the iterate $\hat{p}_N$ is an improvement over p_N as an approximation to ϕ?

To answer this we seek an expression for the new error $\hat{p}_N - \phi$ in terms of the old error $p_N - \phi$. Firstly, we need to derive an expression for $\hat{p}_N$ that will simplify this task. To do this, an auxiliary operator equality is required, namely

$$\begin{aligned} P_N\ I - KP_N) = P_N - P_NKP_N = \ I - P_NK)P_N \\ \Rightarrow P_N\ I - KP_N)^{-1} = \ I - P_NK)^{-1}P_N. \end{aligned} \tag{3.19}$$

Note that, this equality also tells us that the existence of the inverses of $I - KP_N$ and $I - P_NK$ are co-dependent.

Combining 3.4) and 3.12) gives

$$\hat{p}_N = f + K\ I - P_NK)P_Nf,$$

then rearranging the second term on the right hand side using 3.19) we obtain

$$\begin{aligned} \hat{p}_N &= f + KP_N\ I - KP_N)^{-1}f \\ &= \big(\ I - KP_N) + KP_N\big)\ I - KP_N)^{-1}f \\ &= \ I - KP_N)^{-1}f. \end{aligned} \tag{3.20}$$

From here on 3.20) will be used to define the iterate.

Now, analysing the error using 3.20) and 2.3) gives

$$\begin{aligned} \hat{p}_N - \phi &= \ I - KP_N)^{-1}f - \ I - K)^{-1}f \\ &= \ I - KP_N)^{-1}\big(\ I - K) - \ I - KP_N)\big)\ I - K)^{-1}f \\ &= -\ I - KP_N)^{-1}K\ I - P_N)\phi. \end{aligned} \tag{3.21}$$

This is not quite what is required. It is an expression for the new error in terms of the exact solution only, but may be overcome by noting that from 3.8)

$$\begin{aligned} I - P_N)p_N &= 0 \\ \Rightarrow\ I - KP_N)^{-1}K\ I - P_N)p_N &= 0. \end{aligned} \tag{3.22}$$

Thus, adding the left hand side of 3.22) to the right hand side of 3.21) we obtain

$$\begin{aligned} \hat{p}_N - \phi &= \ I - KP_N)^{-1}K\ I - P_N)p_N - \ I - KP_N)^{-1}K\ I - P_N)\phi \\ &= \ I - KP_N)^{-1}K\ I - P_N)\ p_N - \phi), \end{aligned} \tag{3.23}$$

as required, expressing the new error in terms of the old. For brevity, 3.23) will be written

$$\hat{p}_N - \phi = S\ p_N - \phi), \tag{3.24}$$

where

$$S = \ I - KP_N)^{-1}K\ I - P_N) \tag{3.25}$$

may be called the *Sloa operator*.

Note that, by 3.19), the existence of the Sloan operator implicitly requires the existence of the Galerkin approximation.

hat can this expression tell us?

Taking norms on 3.24) and applying the Schwarz inequality

$$\|\hat{p}_N - \phi\| \leq \|S\|\|p_N - \phi\|. \tag{3.26}$$

ecall that a good approximation is one that minimises the norm of the error, then 3.26) tells us that $\hat{p}_N$ is an improvement to p_N, i.e.

$$\|\hat{p}_N - \phi\| < \|p_N - \phi\|$$

f the Sloan operator satisfies the inequality

$$\|S\| < 1. \tag{3.27}$$

Sadly, this is not always the case.

Fortunately though, it is a condition that can be controlled through our choice of subspace E_N. This is s

3.2.3 Does the improvement repeat?

A natural question to ask is whether a further improvement will result if a second iteration of the form 3.12) is performed. Let the second iterate be

$$\begin{aligned} \hat{\hat{p}}_N &= f + K\hat{p}_N \\ &= f + K\ I - KP_N)^{-1} f \end{aligned} \qquad 3.30)$$

by 3.20).

Again we seek an expression for the new error $\hat{\hat{p}}_N - \phi$ in terms of the old error $\hat{p}_N - \phi$. Using 3.30) and 2.3)

$$\hat{\hat{p}}_N - \phi \mathrm{i}$$

How should this approximation be made? Firstly, a useable expression for e_0 must be found. The above equality 3.31) is not suitable, as it involves the unknown ϕ. However, 3.31) may be manipulated as follows. e note that

$$\phi \quad + \hat{}:+\hat{}:+\hat{}:,$$

bound on the improvement of $\hat{p}_0$ over p_0, hence our new approximation is an improvement over the original Galerkin approximation with upper bound $\|S\|^2$.

The algebraic manipulations leading to 3.34) do not involve any properties of the original approximation $\hat{p}_0$, rather that we add on $\hat{p}_1$ to the original approximation, where $\hat{p}_1$ is the iterated Galerkin approximation to the pointwise error, made using 3.33). As such, it is clear that we are able to repeat the above procedure indefinitely, substituting in our latest approximation for $\hat{p}_0$, and the improvement will hold. This may be shown formally using induction.

Let the n^{th} approximation to ϕ be

$$\hat{}_n = \sum_{=0}^{n} \hat{p} \ ,$$

where p_0 and $\hat{p}_0$ are as previously defined, and

$$\begin{aligned} \hat{p}_j &= \hat{r}_{j-1} + Kp_j \\ &= I - KP_N)^{-1}\hat{r}_{j-1} \end{aligned}$$

is the iterate of

$$p_j = I - KP_N)P_N\hat{r}_{j-1}, \qquad \dot{A} = 1, \ldots,)$$

the Galerkin approximation to the current error $e_{j-1} = \phi - \hat{}_{j-1}$ in E_N, obtained from the equation

$$Ae_{j-1} = \hat{r}_{j-1},$$

which says that the current pointwise error is the solution of an operator equation, identical to the original equation 2.4), excepting the free term that has been replaced by the current residual error.

Suppose that the current approximation is $\hat{}_n$, and up until this point each new approximation has been an improvement over the previous by a factor bounded above by $\|S\|$, i.e.

$$\|\hat{}_j\| \leq \|S\|\|\hat{}_{j-1}\|, \qquad \dot{A} = 1$$

$y\ t\ a\ p\ e\ x\ M\ .\ \ t\ y\ y\ y\ i \quad Q\ N\ \ r > \dot{A} > \quad \varphi\ r$

the method can be repeated as required.

It is important to emphasise the computational efficiency of the re-iterated Galerkin method. The expense of an approximation method is largely dictated by the number of and size of systems of equations that arise in their evaluation. In re-iterated Galerkin there are two types of approximation that must be made, the Galerkin approximation and the iterated Galerkin approximation. Only in the ordinary Galerkin approximation are we required to solve a system of equations. As the subspace E_N has been fixed, this will always be an $N \times N$ system of the form 3.11). In addition, as the operator A is the same in each step of the re-iterated Galerkin method, the co-efficients on the left hand side of 3.11) are the same. This means we need only calculate the inverse of the resulting matrix once and re-apply it to vectors resulting from the right hand side of 3.11) with the free term changed to the appropriate residual error. This constitutes a substantial computational saving.

This computational saving relies on only one subspace being used. Our ability to only use one subspace yet still incur improvements, is a consequence of the intermediate iterations. To see this, suppose we were to attempt the re-iteration process without using the iterations $\hat{p}_j$, then

$$r_0 = f - Ap_0$$

and

$$\begin{aligned} P_N r_0 &= P_N f - P_N A p_0 \\ &= \underbrace{P_N f + P_N K p_0}_{=p_0 \text{ by } (3.7)} - P_N p_0 \\ &= p_0 - p_0 \\ &= 0. \end{aligned}$$

Our Galerkin approximation p_1 in E_N to the current error satisfies

$$\begin{aligned} I - P_N K)p_1 = P_N r_0 = 0 \\ \Rightarrow p_1 = 0 \end{aligned}$$

and the new approximation

$$p_0 + p_1 = p_0$$

is equal to the previous approximation. Thus, the re-iteration process breaks down at the first step without the use of the $\hat{p}_j$.

Another practical advantage of the re-iterated Galerkin method is that inaccuracies are not propagated. This is because improvements are calculated relative to the error in our current approximation.

3.3.1 A more accurate improvement factor

By using the Schwartz inequality to deduce condition 3.27), which if satisfied ensures iterations will improve our approximation, a certain amount of information is lost. For this reason, 3.27) is only a sufficient condition. e are only able to say that f the Sloan operator has a norm

value less than one, *the* an iteration will improve matters. However, this does not preclude the possibility that iteration could improve the approximation although $\|S\| \geq 1$.

To remedy this situation, consider our approximation $\hat{}_n = \sum_{=0}^{n} \hat{p}$. ather than ask whether each successive approximation is an improvement, we seek to determine if the series representation of the approximation converges and if it does is its limit ϕ?

Analysis of the convergence is eased if $\hat{}_n$ is written as a geometric progression. First, note that

$$\begin{aligned} & \hat{p} &=& \hat{r}_{-1} + Kp \\ \Rightarrow \quad & P_N\hat{p} &=& P_N\hat{r}_{-1} + P_NKp \\ \Rightarrow \quad & P_N\hat{p} &=& p\ , \end{aligned}$$

so that the projection of the iterate is the Galerkin approximation, a result of independent interest. Now,

$$\begin{aligned} S\hat{p} &= I - KP_N)^{-1}K\ I - P_N)\hat{p} \\ &= I - KP_N)^{-1}K\ \hat{p}\ - p\ ) \\ &= I - KP_N)^{-1}\ \hat{r}_{-1} - \hat{p}\ + Kp\ ) \\ &= I - KP_N)^{-1}\hat{r} \\ &= \hat{p}_{+1}. \end{aligned}$$

Therefore, by induction

$$\hat{p}\ = S\ \hat{p}_0 \tag{3.35}$$

and the approximation

$$\hat{}_n = \sum_{=0}^{n} S\ \hat{p}_0. \tag{3.36}$$

Call $\mathscr{H}'$ the smallest Hilbert subspace of $\mathscr{H}$ possibly $\mathscr{H}$ itself) containing all of the terms in the series, $S\ e_0$, on which S may be considered as an operator. By application of the root test, the condition

$$\lim_{n\to}\ \|S^n\|^{\frac{1}{n}} < 1 \tag{3.37}$$

can be deduced for convergence of the series 3.36) as $n \to \infty$. It is well known see [3] for example) that the limit to the left hand side of 3.37) is equal to and sometimes used to define) $\rho_{\mathscr{H}'}\ S) = \max\{|\nu| : \exists\varphi \quad \mathscr{H}'$ such that $S\varphi = \nu\varphi$ the *spectral rad us* of S over $\mathscr{H}'$. Hence, the condition that 3.36) converges is

$$\rho_{\mathscr{H}'}\ S) < 1$$

This is a weaker condition than $\|S\| < 1$, as $\rho_{\mathscr{H}'}\ S) \leq \|S\|$. In fact

$$\rho_{\mathscr{H}'}\ S) \leq \rho_{\mathscr{H}}\ S) \leq \|S\|,$$

and $\rho_{\mathscr{H}}\ S) = \|S\|$ only if S is self-adjoint.

If 3.36) converges, necessarily $\hat{p}_n \to 0$ as $n \to \infty$. From the definition of $\hat{p}_n$, $\hat{p}_n \to 0$ is equivalent to $\hat{r}_{n-1} \to 0$. The residual error tending to zero is enough to guarantee that the limit of the series is ϕ, and the approximations $\hat{}_n$ converges to the exact solution.

In §3.3 we saw that $e_{n+1} = Se_n$. Using this equality, a similar expression to 3.36) is obtainable for the sum of the error terms

$$\sum_{=0}^{n} e \ = \sum_{=0}^{n} S \ e_0 .$$

Assuming convergence, ϕ is the limit of the re-iterated Galerkin approximations, all of which belong to $\mathscr{H}'$ by definition, and the closed property of Hilbert spaces then ensures

$$\phi \quad \mathscr{H}'.$$

Hence, for the same reasons, this series converges as $n \to \infty$ if and only if $\rho_{\mathscr{H}'} \ S) < 1$. This is equivalent to saying that the series

$$\sum \|e_n\|$$

converges if and only if $\rho_{\mathscr{H}'} \ S) < 1$. e may now deduce that the ratio of successive terms

$$\frac{\|e_{n+1}\|}{\|e_n\|} \to \rho_{\mathscr{H}'} \ S) \qquad \text{as } n \to \infty.$$

At worst, we can expect

$$\frac{\|e_{n+1}\|}{\|e_n\|} \to \rho_{\mathscr{H}} \ S) \qquad \text{as } n \to \infty.$$

Thus, the *theoret cal mproveme t factor* is $\rho_{\mathscr{H}'} \ S)$, the spectral radius of the Sloan operator over $\mathscr{H}'$. This result has been verified in practice see [2]), although for reasons that will be discussed in §6.4 it is shown in terms of the ratio of the norm of successive residual errors. However, we must be careful to note that the theoretical improvement factor is an 'eventual trend'. Early iterations may not bear any relation to this value, with the ratio of successive error terms being bounded only by $\|S\|$. This means, if $\|S\| > 1$ and $\rho_{\mathscr{H}} \ S) < 1$, the re-iterated Galerkin approximation could get of*i*χ*i*Pca

$$DS$$

C a ter

Quantities of interest

The links between deriving an approximation to the unknown function ϕ and approximating a quantity that involves this function, as one may expect,

with f the free term of 2.4) and g as in 4.1), then

$$\begin{aligned} L : \mathscr{H} \oplus \mathscr{H} &\to \mathbb{C} \\ p, q) &\mapsto f, q) + p, g) - Ap, q). \end{aligned}$$

Although L is now only dependent on p and q, f and g are still arbitrary. To emphasise this, the operator will be written

$$L = L\ p, q, f, g).$$

Consider the stationary point $\phi, \psi)$ the reason for the clash in notation 0˜:7:

to $\psi = g + K^*\psi$ resulting from the Galerkin approximation $q_0 \quad F_M$. Let P_N and S be as in 3.1) and 3.25) respectively, and Q_M and T be the corrosponding orthogonal projection and Sloan operator for the subspace F_M, i.e.

$$T = \; I - K^*Q_M)^{-1}K^* \; I - Q_M).$$

ecall that, it has been assumed that $\|S\| < 1$. By the same argument, it can also be assumed that $\|T\| < 1$. Following the theory of §3.3 we have

$$\|\phi - \hat{\ }_n\| \le \|S\|^{n+1}\|p_0 - \phi\| \quad {}^*\psi \text{ resulting from}$$

It would be reasonable to assume that as $n \to \infty$ our approximation $L\ \hat{}_n, \hat{\tau}_n, f, g)$ will tend to the stationary value $\phi, g)$. Can this be shown, and if so, can the improvements be described?

In §4.1, it was shown that

$$L\ \phi + \delta\phi, \psi + \delta\psi, f, g) = \ \phi, g) - \ A\delta\phi, \delta\psi) \tag{4.9}$$

for any variations $\delta\phi, \delta\psi \quad \mathscr{H}$. As, for any p and q in $\mathscr{H}$, there exist $\delta\phi$ and $\delta\psi$ in $\mathscr{H}$, such that

$$p = \phi + \delta\phi; \qquad q = \psi + \delta\psi,$$

4.9) may be written

$$\phi, g) - L\ p, q, f, g) = \ A\ p - \phi), q - \psi) \qquad \forall p, q \quad \mathscr{H}.$$

Substituting $\hat{}_n$ for p and $\hat{\tau}_n$ for q, and using the Schwartz inequality

$$|\ \phi, g) - L\ \hat{}_n, \hat{\tau}_n, f, g)| \leq \|A\|\|\phi - \hat{}\ \ |p, g) - \hat{}_n, \hat{\tau}_n, f, g)$$

C a ter 5

T e acoustics roblem revisited

Having covered the theory of how to approximate the solution of certain operator equations in §3, we wish to use this theory to solve the problem set out in §1. In order to do this, it is necessary to classify the integral equation 1.15) in an appropriate setting. This process will lead neatly into making explicit how to use the theory of §4 to approximate the reflection and transmission amplitudes, R and T.

5.1 The Hilbert sp ce $L_2(D')$

ecall that the integral equation in question is

$$\phi\ x_0, y_0) = e^{\ \beta_0 x_0} \sin\ y_0) + \int_{D'} k^2\ x, y) - k_0^2) G\ x, y \mid x_0, y_0) \phi\ x, y) \mathrm{d}x\mathrm{d}y \qquad 5.1)$$

for $x_0, y_0)$ D', where D' is a bounded subset of $D = \{x_0, y_0 : x_0 \quad \mathbb{R}, 0 < y_0 < \pi$. Let us suppose that we wish to recast 5.1) in the Hilbert space of Lebesgue square integrable functions, over the domain D'. This space will be denoted $L_2\ D')$. More generally, any space of two-dimensional Lebesgue integrable functions will be referred to as a $L_2 \oplus L_2$ space, i.e. the direct sum of one-dimensional Lebesgue spaces.

The space $L_2\ D')$ is defined with the inner product

$$, v) = \int_{D'} \bar{v}, \qquad \forall\ , v \quad L_2\ D') \qquad 5.2)$$

and norm

$$\|\ \| = \quad , \ )^{\frac{1}{2}}.$$

Note that, other Hilbert spaces could have been considered. However, the $L_2\ D')$ space is the most natural c V m
$\mathrm{A}_2 D$

$'$

of 5.1). In addition, we define the multiplication operator M as

$$M\varphi)\ x,y) = \big(k^2\ x,y) - k_0^2\big)\varphi\ x,y)$$

and the integral operator $\mathcal{G}$ as

$$\mathcal{G}\varphi)\ x_0,y_0) = \int_{D'} G\ x,y \mid x_0,y_0)\varphi\ x,y)\mathrm{d}x\mathrm{d}y,$$

where $G\ x,y \mid x_0,y_0)$ is the Green's function defined in 1.13). The integral equation 5.1) may now be solved by solving the equivalent operator equation

$$\phi = f_0 + \mathcal{G}M\phi \tag{5.3}$$

in the Hilbert space $L_2\ D')$. This may be achieved via application of the re-iterated Galerkin method.

It is also required that the reflection and transmission amplitudes, R and T, be determined. In §1, it was shown that these unknowns may be written as

$$R = \frac{i}{{}_0\pi}\int_{D'} k^2\ x,y) - k_0^2)e^{\ \beta_0 x}\phi\ x,y)\mathrm{d}x\mathrm{d}y$$

and

$$T = 1 + \frac{i}{{}_0\pi}\int_{D'} k^2\ x,y) - k_0^2)e^{-\ \beta_0 x}\phi\ x,y)\mathrm{d}x\mathrm{d}y.$$

To access the theory of §4, these definitions must be expressed in terms of inner-products on $L_2\ D')$. Our choice of function space makes this task particularly simple, with

$$R = \frac{i}{{}_0\pi}\ \phi, Mg_1) : g_1\ x,y) = e^{-\ \beta_0 x}\sin\ y)$$

and

$$T = 1 + \frac{i}{{}_0\pi}\ \phi, Mg_2) : g_2\ x,y) = e^{\ \beta_0 x}\sin\ y).$$

It is apparent that g_1 and g_2 are equal to the conjugate of f_0 and f_0 respectively. This point will be made use of later.

There is a purely theoretical issue that arises from the choice of $L_2\ D')$ as our Hilbert space, that should be briefly addressed.

Elements of $L_2 \oplus L_2$ spaces define equivalence classes of two-dimensional square integrable functions, rather than individual functions. A set of suitable functions that differ only on a set of measure zero are all represented by the same $L_2 \oplus L_2$ element. It may be helpful to think of this as the correspondence between the set of $L_2 \oplus L_2$ functions and the set of $L_2 \oplus L_2$ elements being non-injective. It is, therefore, not unreasonable to foresee formal difficulties in re-mapping our Hilbert space approximation onto a function that approximates the exact solution of 5.1) in a pointwise sense. ecall though, from 1.15), that the full approximation, $\phi_{app}\ x_0,y_0)$ on D, is defined as

$$\phi_{app}\ x_0,y_0) = e^{\ \beta_0 x_0}\sin\ y_0) + \int_{D'} k^2\ x,y) - k_0^2)G\ x,y \mid x_0,y_0)\hat{}\ x,y)\mathrm{d}x\mathrm{d}y,$$

where $\hat{}\ x, y)$ is any function that belongs to the equivalence class defined by the re-iterated Galerkin approximation $\hat{}$. By definition of the equivalence class, any $\hat{}\ x, y)$ will provide the same approximation, ϕ_{app}. This may be described as one of the 'smoothing' properties of the integral.

In practice, when making a Hilbert space approximation, we must reintroduce the independent variables to define $L_2 \oplus L_2$ elements. In doing so, we will exclusively consider ourselves to be working with continuous, or, at worst, piecewise continuous functions. The question of re-mapping elements onto functions does not arise. For this reason, from now on we shall speak, in the main, of functions rather than elements and assume an implicit understanding of the analogous Hilbert space situation.

ith the problem now fully defined, we may make some physical interpretations of previously abstract quantities. To begin with, let us consider the error.

5.1.1 Error

The re-iterated Galerkin method seeks to reduce the norm of the pointwise error

$$\|e\| = \|\phi - \hat{}_n\| = \Big(\int_{D'} |\phi - \hat{}_n|^2 \mathrm{d}x\mathrm{d}y \Big)^{\frac{1}{2}}. \qquad 5.4)$$

Again, the smoothing property of the integral removes any contention of definition of pointwise error.

The norm of the error is a measure of the error in the approximation, across the domain D'. A decrease in this value does not imply that the error is decreasing at all points in the domain, rather that the error is decreasing in an 'overall' sense. e can draw a link between the norm error and pointwise error as follows. It can be shown that

$$| \mathcal{G}M\varphi)\ x_0, y_0)| \leq C\|\varphi\| \qquad 5.5)$$

where $C^2 = \int\int_{D'} |\ k^2\ x, y) - k_0^2) G\ x, y\ |\ x_0, y_0)|^2 \mathrm{d}x\mathrm{d}y$ is a finite constant. I

Therefore, as the norm of the error tends to zero, so does the pointwise error at all points in the domain. If we had the time to estimate C, we could estimate the magnitude of the point

for some finite integer N. This property may be established by noting the non-separability of the Green's function G.

There is however another form of degeneracy that we must be aware of. This degeneracy would not effect the infinite rank of $\mathcal{G}M$, rather it would mak

where the Green's function, G, is fully defined as

$$G\ x, y \mid x_0, y_0) = \frac{i}{{}_0\pi} \sin\ y_0) \sin\ y) e^{\ \beta_0 |x - x_0|} + \sum_{n=2} \frac{1}{\gamma_n \pi} \sin\ n y_0) \sin\ n y) e^{-\gamma_n |x - x_0|}$$

and D' is a bounded sub-domain of D.

As in §1.4, let

$$D_X = \{x, y : |x| < X, 0 < y < \pi\ \ ,$$

but will now be considered only as a *fi te* domain, and let $D' \subset D_X$. Obviously, if a function is a $L_2 \oplus L_2$-kernel on D_X, then it is a $L_2 \oplus L_2$-kernel on any subset of D_X. Thus, we may deduce 5.10) by showing that

$$\int_{D_X} \int_{D_X} |G\ x, y \mid x_0, y_0)|^2 \mathrm{d}x \mathrm{d}y\, \mathrm{d}x_0 \mathrm{d}y_0 < \infty. \qquad 5.1\mathrm{fA}$$

C a ter 6

Nu erical Considerations

This chapter deals with the factors that separate the task of actually carrying out the re-iterated Galerkin method from the theoretical issues of §3-4.

6.1 The discrete proble nd nu eric l u dr ture

In performing calculations on a computer, our continuous problem is necessarily reduced to a discrete problem. This means that instead of solving for elements that were earlier described as 'infinite v

Let $k^2\ x, y) - k_0^2$ be a real-valued function. This is physically valid as the function $k\ x, y)$ and constant k_0 are produced by the ratio of the wave frequency and wave speed. Only in extremely rare circumstances would either of these measures involve an imaginary part.

6.2 A re l v lued kernel

In practice, it is of significant advantage to work with operators that map the real-valued elements of a space onto real-valued elements. Operators with this property allow the equation to be split into its real and imaginary parts, and solved independently. In terms of integral equations, such operators arise from real-valued kernels. Suppose

$$\phi = f + K\phi,$$

where $f = f_{re} + if_{\ m}$ and the kernel of K is a real-valued function, then

$$\phi = \phi_{re} + i\phi_{\ m},$$

such that $\phi_j \quad \mathbb{R}$ and

$$\phi_j = f_j + K\phi_j, \qquad {}_{\dot{\iota}}j = re, im).$$

Thus, any program written to perform re-iterated Galerkin on a problem involving a real-valued kernel need only deal with real numbers.

Our kernel $MG)\ x, y \mid x_0, y_0) = k^2\ x, y) - k_0)G\ x, y \mid x_0, y_0)$, unfortunately does not possess this property due to the appearance of the complex term $\frac{}{\beta_0\pi}\sin\ y_0)\sin\ y)e^{\ \beta_0|x-x_0|}$ in the Green's function G. However, this problem can be amended. Note that, by De-Moivre's theorem

$$\frac{i}{{}_0\pi}\sin\ y_0)\sin\ y)e^{\ \beta_0|x-x_0|} = \frac{i}{{}_0\pi}\sin\ y_0)\sin\ y)\big[\cos\big(\ {}_0|x-x_0|\big) + i\sin\big(\ {}_0|x-x_0|\big)\big]. \qquad 6.1)$$

Using the even property of the cosine function 6.1) gives

$$\frac{i}{{}_0\pi}\sin\ y_0)\sin\ y)\big[\cos\big(\ {}_0\ x - x_0)\big) + i\sin\big(\ {}_0|x-x_0|\big)\big]$$

$$= \frac{1}{{}_0\pi}\sin\ y_0)\sin\ y)\big[i\cos\quad {}_0x)\cos\quad {}_0x_0) + i\sin\quad {}_0x)\sin\quad {}_0\chi$$

and the integral operator

$$\tilde{\mathcal{G}}\phi)\ x_0, y_0) = \int_{D'} \tilde{G}\ x, y \mid x_0, y_0)\phi\ x, y)\mathrm{d}x\mathrm{d}y,$$

is such that

$$\tilde{G}\ x, y \mid x_0, y_0) = \frac{1}{{}_0\pi}\sin\ y_0)\sin\ y)\sin\big(\ {}_0|x - x_0|\big) + \sum_{n=2}\frac{1}{\gamma_n\pi}\sin\ ny_0)\sin\ ny)e^{-\gamma_n|x-x_0|}. \quad 6.3)$$

It is easily seen from our assumptions on the function $k^2\ x, y) - k_0^2$, that the operator $\tilde{\mathcal{G}}M$ is an integral operator arising from the real-valued kernel $M\tilde{G})\ x, y \mid x_0, y_0)$.

The arguments made in §5.2, which showed that the operator $\mathcal{G}M$ satisfies the relevant assumptions needed to apply the re-iterated Galerkin method, hold for the modified operator $\tilde{\mathcal{G}}M$.

The move made by redefining our problem as 6.2) is to remove the terms in the kernel that prevent it from being real-valued, and adding their contribution to the free term. In doing this, the original kernel, $MG)\ x, y \mid x_0, y_0)$, has been split into a degenerate kernel of rank 2

$$\frac{i}{{}_0\pi}\big(k^2\ x, y) - k_0^2\big)\sin\ y_0)\sin\ y)\cos\ \ {}_0|x - x_0|),$$

and a real kernel of infinite rank, $M\tilde{G})\ x, y \mid x_0, y_0)$. The gain in creating a operator with a real-valued kernel is made at the cost of a less straightforward sum of free terms, as the kernel of rank 2 has been moved into the free term. This however does not prove to be problematic.

First though, let

$$\phi = \phi_{re} + i\phi\ _m,$$

and

$$\begin{aligned} c_j &= \ \phi_{re}, Mf_j) + i\ \phi\ _m, Mf_j) \\ &= \ c_{j_{re}} + ic_{j_{im}}, \qquad\qquad j = 1, 2), \end{aligned}$$

then the equations

$$\begin{aligned} \phi_{re} &= \Re\mathrm{e}\ f_0) - \frac{1}{{}_0\pi}c_{1_{im}}f_1 - \frac{1}{{}_0\pi}c_{2_{im}}f_2 + \tilde{\mathcal{G}}M\phi_{re}, \\ \phi\ _m &= \Im\mathrm{m}\ f_0) + \frac{1}{{}_0\pi}c_{1_{re}}f_1 + \frac{1}{{}_0\pi}c_{2_{re}}f_2 + \tilde{\mathcal{G}}M\phi\ _m \end{aligned}, \quad 6.4)$$

are both real-valued. Before

To deal with the unknown constants that appear in these equations, define $\phi \quad i = 1, 2)$ to be the solution of

$$\phi \ = f \ + \tilde{\mathcal{G}} M \phi \ ,$$

so that

$$\phi_{re} = \big(1 - \frac{c_{1_{im}}}{{}_0\pi}\big)\phi_1 - \frac{c_{2_{im}}}{{}_0\pi}\phi_2$$

and

$$\phi \ _m = \frac{c_{1_{re}}}{{}_0\pi}\phi_1 + \big(1 + \frac{c_{2_{re}}}{{}_0\pi}\big)\phi_2.$$

Substituting these solutions into the definitions of the $c \ _j$

$$c_{1_{re}} = \big(1 - \frac{c_{1_{im}}}{{}_0\pi}\big) \ \phi_1, Mf_1) - \frac{c_{2_{im}}}{{}_0\pi} \ \phi_2, Mf_1),$$

$$c_{2_{re}} = \big(1 - \frac{c_{1_{im}}}{{}_0\pi}\big) \ \phi_1, Mf_2) - \frac{c_{2_{im}}}{{}_0\pi} \ \phi_2, Mf_2),$$

$$c_{1_{im}} = \frac{c_{1_{re}}}{{}_0\pi} \ \phi_1, Mf_1) + \big(1 + \frac{c_{2_{re}}}{{}_0\pi}\big) \ \phi_2, Mf_1),$$

$$c_{2_{im}} = \frac{c_{1_{re}}}{{}_0\pi} \ \phi_1, Mf_2) + \big(1 + \frac{c_{2_{re}}}{{}_0\pi}\big) \ \phi_2, Mf_2),$$

a 4×4 system to be solved for the unknown coefficients. It is unsurprising that, in separating the kernel into a degenerate part and a part of infinite rank, we are left to approximate the solutions involving the kernel of infinite rank, and then solve an equation of finite rank. The size of the system to be solved reflects that we are solving for the real and imaginary parts of a system arising from a kernel of rank 2. The inner products involved in this system may be approximated using the theory of §4.1.2. In turn, the solution of the system of equations, the coefficients, will be approximations.

Moreover, the structure of the operator $\tilde{\mathcal{G}} M$ allows these inner products to be approximated at aAnbducts to be a7oˆˆDV6t?ˆ6o?ps seallˆ:Φr6)0Hr)ˆˆˆ6?AvAnrV˜H:ˆ:ΦrˆDˆˆ::Φr6thnΦr6in?lsbT

and the auxiliary equation

$$A^*\psi = g.$$

Our

Note that

$$\begin{aligned} g_1\ x,y) &= \cos\ \ {}_0x) - i\sin\ \ {}_0\big)\sin\ y) \\ &= f_1\ x,y) - if_2\ x,y) \end{aligned}$$

and

$$\begin{aligned} g_2\ x,y) &= \cos\ \ {}_0x) + i\sin\ \ {}_0\big)\sin\ y) \\ &= f_1\ x,y) + if_2\ x,y), \end{aligned}$$

so R and T may be written

$$\begin{aligned} R &= -\frac{1}{{}_0\pi}\ \phi, Mf_2) + \frac{i}{{}_0\pi}\ \phi, Mf_1) \\ &= -\frac{1}{{}_0\pi}\big(c_2 - ic_1\big) \\ &= -\frac{1}{{}_0\pi}\big(\ c_{1_{im}} + c_{2_{re}}) + i\ c_{2_{im}} - c_{1_{re}})\big), \\ T &= 1 + \frac{1}{{}_0\pi}\ \phi, Mf_2) + \frac{i}{{}_0\pi}\ \phi, Mf_1) \\ &= 1 + \frac{1}{{}_0\pi}\big(c_2 + ic_1\big) \\ &= 1 + \frac{1}{{}_0\pi}\big(\ c_{2_{re}} - c_{1_{im}}) + i\ c_{1_{re}} + c_{2_{im}})\big). \end{aligned}$$

Thus, we may deduce approximations to R and T by com

with

$$a_0 = a_0\ x, y \mid x_0, y_0) = \frac{1}{{}_0\pi} \sin\ y_0) \sin\ y) \sin\left(\ {}_0|x - x_0|\right)$$

and

$$s_n = s_n\ x, y \mid x_0, y_0) = \frac{1}{\gamma_n \pi} \sin\ ny_0) \sin\ ny) e^{-\gamma_n |x - x_0|}.$$

As already stated, a

In

where

$$i_{ka} \equiv i_k \mod X_p), \qquad i_{kb} = \frac{i_k - i_{ka}}{X_p}$$

and

$$X = \frac{b-a}{X_p}, \qquad Y = \frac{d-c}{Y_p}.$$

Now

$$\begin{aligned} A_{X,Y} &= \int_{y_{j_b}-Y}^{y_{j_b}+Y} \int_{x_{j_a}-X}^{x_{j_a}+X} \mathcal{L}_3\ \boldsymbol{x}_j \mid \boldsymbol{x}_j)\mathrm{d}x\mathrm{d}y = \int_{-Y}^{Y} \int_{-X}^{X} \ln\ ^2 + v^2)\mathrm{d}\ \mathrm{d}v \\ &= 4\Big[XY \ln\ X^2 + Y^2) - 3XY + X^2 \arctan\ \frac{Y}{X}) + Y^2 \arctan\ \frac{X}{Y})\Big] \end{aligned}$$

an analytic expression that holds for all $\boldsymbol{x}_j$.

Thus, the integral 6.9) may be approximated by the modified rectangle midpoint rule

$$\int_{D'} \mathcal{L}_3\ x, y \mid x_0, y_0)\mathrm{d}x\mathrm{d}y \approx \begin{cases} \mathscr{L}\ x_0, y_0) & x_0, y_0) \neq \boldsymbol{x}\ , \\ \mathscr{L}\ x_0, y_0) & x_0, y_0) = \boldsymbol{x}\ , \end{cases}$$

where

$$\mathscr{L}\ x_0, y_0) = Q \sum_{=1}^{X_p Y_p} \mathcal{L}_3\ \boldsymbol{x}\ \mid x_0, y_0)$$

and

$$\mathscr{L}_j\ x_0, y_0) = Q \sum_{\substack{=1 \\ \neq j}}^{X_p Y_p} \mathcal{L}_3\ \boldsymbol{x}\ \mid\ x_0, y_0)) + A_{X,Y}.$$

The entire approximation is

$$\begin{aligned} \tilde{\mathcal{G}} M\varphi)\ x_0, y_0) \ \approx\ & Q \sum_{=1}^{X_p Y_p} \left(k^2\ \boldsymbol{x}\ ) - k_0^2\right)\Big[- a_0\ \boldsymbol{x}\ \mid x_0, y_0) + \mathcal{S}_N\ \boldsymbol{x}\ \mid x_0, y_0) + \ldots \\ & \ldots \frac{1}{4\pi}\mathcal{L}_1\ \boldsymbol{x}\ \mid x_0, y_0) - \frac{1}{4\pi}\mathcal{L}_4\ \boldsymbol{x}\ \mid x_0, y_0)\Big]\varphi\ \boldsymbol{x}\ ) \\ & - \frac{Q}{4\pi} \sum_{=1}^{X_p Y_p} \mathcal{L}_3\ \boldsymbol{x}\ \mid\ x_0, y_0)\big(\ M\varphi)\ \boldsymbol{x}\ ) -\ M\varphi)\ x_0, y_0)\big) \\ & - \frac{1}{4\pi}\ M\varphi)\ x_0, y_0) \begin{cases} \mathscr{L}\ x_0, y_0) & x_0, y_0) \end{cases} \end{aligned}$$

6.3.1 The series approximation

For simplicity, let

$$\mathcal{S}_N = \sum_{n=1}^{N} \big(s_n - t_n\big) \approx \sum_{n=1} \big(s_n - t_n\big),$$

where N is some fixed positive integer. Increasing N improves the accuracy of $\mathcal{S}_N$, although it is expected that N should not have to be chosen particularly large, as the terms in the series are exponentially small accross the domain, except at the points $x_0, y_0) = x, y)$.

As an analytic solution is not at our disposal it is of importance to have as many ways of verifying the numerical solution as possible.

Check i. ecall that the functions that will be approximated by the re-iterated Galerkin method satisfy the real-valued operator equations

$$\phi = f + \tilde{\mathcal{G}} M\phi \,, \qquad i = 1, 2),$$

which implies that

$$Mf = M - M\tilde{\mathcal{G}}M)\phi \,, \qquad i = 1, 2). \tag{6.10}$$

From the known self-adjoint property of the operators $\tilde{\mathcal{G}}$ and M, it is easily deduced that the operator

$$M - M\tilde{\mathcal{G}}M$$

is also self-adjoint.

Now, consider the quantity

$$\phi \,, Mf_j), \qquad i, j = 1, 2),$$

that must be approximated to find R and T. Using the above information

$$\begin{aligned} \phi \,, Mf_j) &= \phi \,, \; M - M\tilde{\mathcal{G}}M)\phi_j) \\ &= M - M\tilde{\mathcal{G}}M)\phi \,, \phi_j) \\ &= Mf \,, \phi_j) \\ &= \phi_j, Mf \,). \end{aligned}$$

This equality can be used to check the validity of the operator equations 6.10) used iner7U UH7ions

Check ii Conservation of energy see §1.3)

$$|R|^2 + |T|^2 = 1.$$

e shall come accross more checks as we progress.

In the preceeding theory, approximation methods were concerned with the pointwise error $e = \phi -$

Thus

$$\begin{aligned}
\hat{r}_{n+1} &= f - A\hat{}_{n+1} \\
&= f - A\hat{}_n - A\hat{p}_{n+1} \\
&= \hat{r}_n - A\ I - KP_N)^{-1}\hat{r}_n \\
&= K\ I - P_N)\ I - KP_N)^{-1}\hat{r}_n \\
&= \ I - KP_N)^{-1}K\ I - P_N)\hat{r}_n \\
&= S\hat{r}_n. \qquad (6.11)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{r}_{n+1} &= S\hat{r}_n \\
&= SAe_n
\end{aligned}$$

and

$$\begin{aligned}
\hat{r}_{n+1} &= Ae_{n+1} \\
&= ASe_n,
\end{aligned}$$

which imply that

$$SAe_n = ASe_n.$$

As e_n

6.5 Accur cy

Let us summarise the errors that will effect our approximations:

- *The error hera t a approx mat o .*
- *The rou d g error the*

C a ter 7

esults

All calculations are made with MatLabv6, using the code given in appendices A-F.

It is now time to settle on a final problem, so that we may produce concrete results. This entails choosing the constant k_0, function $k\ x, y)$ and domain D'. This choice is essentially our own, excepting the constraints previously outlined. These are

$$1 < k_0 < 2,$$

$$k\ x, y) \to k_0 \qquad \text{as} \quad x, y) \to \delta D',$$

and the domain D' is rectangular. It should

any choice that may have been made of the function $k\ x, y)$ and rectangular domain D'.

The final influence that we must make over the problem is in constructing an appropriate subspace, in which the Galerkin approximation is made. The choice of this subspace will determine the ultimate success or failure of the application of the re-iterated Galerkin method.

7.1 Tri l functions

The basis elements that form the subspace, are defined by their associated trial functions. As successive approximations will be built around these functions, it is logical that they share common traits with the exact solution. So, let us consider the solutions of

$$\phi\ \ x_0, y_0) = F\ \ x_0) \sin\ y_0) + \int_0^{\pi} \int_{-\frac{\pi}{m}}^{\frac{\pi}{m}} \mu\ x, y) \tilde{G}\ x, y \mid x_0, y_0) \phi\ \ x, y) \mathrm{d}x \mathrm{d}y$$

Turning

This shows that $\phi_1 = \phi_{1,ev_{x_0}}$ is even in x_0, and $\phi_2 = \phi_{2,odd_{x_0}}$ is odd in x_0. This can and will be reflected in our choice of trial space. That is, the trial space corresponding to the solution ϕ_1 shall be even in x_0, and the trial space corresponding to ϕ_2 odd in x_0.

ecall that the Galerkin approximation is defined as

$$p = \ I - P_N K)^{-1} P_N f.$$

Expanding the operator $\ I - P_N K)^{-1}$ into series form gives

$$p = \sum_{j=0} \ P_N K)^j P_N f.$$

ith $K = \tilde{\mathcal{G}} M$ and $f = f \quad i = 1, 2)$, the Galerkin approximation becomes

$$p_{\ ,0} = \sum_{j=0} \ P_N \tilde{\mathcal{G}} M)^j P_N f \ .$$

As any orthogonal projection, onto a purely odd or even trial space, preserves the odd or even property of a function, and from what has been noted of the odd and even properties of the operator and free terms, it is clear that the Galerkin approximations to $\phi_1 \ x_0, y_0)$ and $\phi_2 \ x_0, y_0)$ will be even and odd in x_0, respectively.

In turn, the iterate

$$Qq\dot{\iota}yNr{>}\dot{\iota}_{\dot{K}}{>}y\xi QIPrQI\chi yiSiy\dot{\iota}_{\dot{K}}\xi\dot{\iota}_{\dot{K}}{>}\chi{>}yiMDe\partial QQ{>}\chi{>}yiM\partial S\xi bk{,}q{>}y\chi{>}$$

Check iii. The re-iterated Galerkin approximations must satisfy the even and odd properties spoken of.

This property could be used to quarter the computational domain. However, it is more useful to us as a device to help verify results.

Check iv. The inner-products that make up the matrix used in the re-iterated Galerkin method, are of the form

$$A\chi\ ,\chi_j),$$

where χ is a trial function. ith $A = I - \tilde{\mathcal{G}}M$, and the specified trial functions, i.e.

$$\chi\ \ x_0, y_0) = T_{_p}\ x_0)\sin\ i_p y_0),$$

where $T_{_p}\ x_0)$ is equal to either $\cos\ \frac{_q \pi x_0}{m})$ or $\sin\ \frac{_q \pi x_0}{m})$, and i_p, i_q $\mathbb{N}$, this inner product becomes

$$\chi\ ,\chi_j) -\ \tilde{\mathcal{G}}M\chi\ ,\chi_j).$$

Concentrating our attentions to the term $\tilde{\mathcal{G}}M\chi\ ,\chi_j)$, we have

$$\begin{aligned}
\tilde{\mathcal{G}}M\chi\ ,\chi_j) &= \int_{0}^{\pi}\int_{-\frac{\pi}{m}}^{\frac{\pi}{m}}\int_{0}^{\pi}\int_{-\frac{\pi}{m}}^{\frac{\pi}{m}} \not{\mu}\ x,y)\Big[-\frac{1}{_0\pi}\sin\big(\ _0|x-x_0|\big)\sin\ y)\sin\ y_0) \\
&+ \sum_{n=2}\frac{1}{\gamma_n\pi}e^{-\gamma_n|x-x_0|}\sin\ ny)\sin\ ny_0)\Big]T_{_p}\ x)\sin\ i_p y)\mathrm{d}x\mathrm{d}y\,T_{_p}\ x_0)\sin\ i_p y_0)\mathrm{d}x_0\mathrm{d}y_0 \\
&= \int_{-\frac{\pi}{m}}^{\frac{\pi}{m}}\int_{0}^{\pi}\int_{-\frac{\pi}{m}}^{\frac{\pi}{m}} \not{\mu}\ x,y)s_{j_p}\ x,y\mid x_0)T_{_p}\ x)T_{j_p}\ x_0)\sin\ i_p y)\mathrm{d}x\mathrm{d}y\,\mathrm{d}x_0,
\end{aligned}$$

thus removing the infinite series. This allows us to check the inner-products calculated with an approximation to the infinite series, with an integral in which we may use the exact integrand.

Check v. The approximations to ϕ_1 and ϕ_2 will be used in the functional L see §6.2). In particular, we m

to be the residual error in the n^{th} Sloan iterate to $\phi \quad i = 1, 2$).

The Sloan iteration should converge or diverge independently of the trial space see §3.2.3). So, so let

$$\text{Trial Space 1: } 1 \quad E \; D') \tag{7.7}$$

and

$$\text{Trial Space 2: } x_0 \quad O \; D'). \tag{7.8}$$

Case $\alpha = 1$: esults of the application of Sloan iteration are recorded in table 7.1. Problem 1 displays evident divergence, whereas problem 2 is converging. In both series, the ratio of normed residuals settles within 10 iterations. Although the divergence in problem 1 is very slow, settling to 1.064, it is nevertheless divergence, and requires the re-iterated Galerkin method. As problem 2 does not diverge, and does in fact converge reasonably fast, it does not require the re-iterated Galerkin method. This does, however, provide us with another opportunity: to observe how the re-iterated Galerkin method affects the speed of convergence of an already convergent series.

Table 7.1: Sloan iteration

$\alpha = 1$; $\mathcal{S}_N = \mathcal{S}_3$; refinement: 40×40
Trial space 1 : 7.7); Trial space 2: 7.8)

Iterate	$\|\check{r}_{1,n}\|$	$\|\check{r}_{2,n}\|$	$\frac{\|\check{r}_{1,n}\|}{\|\check{r}_{1,n-1}\|}$	$\frac{\|\check{r}_{2,n}\|}{\|\check{r}_{2,n-1}\|}$
0	0.9119	0.8802		
1	1.177	0.2728	1.291	0.3099
2	0.9753	0.1274	0.828	0.4670
3	1.116	5.970×10^{-2}	1.144	0.4686
4	1.162	2.800×10^{-2}	1.041	0.4689
5	1.245	1.310×10^{-2}	1.079	0.4690
6	1.322	6.200×10^{-3}	1.062	0.4690
7	1.407	2.900×10^{-3}	1.065	0.4690

and

$$\rho_{O(D')}\ \tilde{\mathcal{G}}M) \approx 0.4690,$$

for $\alpha = 1$.

Although, there is no guarantee that, if the operator $\tilde{\mathcal{G}}M$ has small eigenvalues, a relatively small trial space will ensure S also has small eigenv

Table 7.2: e-iterated Galerkin

$\alpha = 1$; $\mathcal{S}_N = \mathcal{S}_3$; refinement: 40×40
Trial space 1 : 7.9); Trial space 2: 7.10)

However, this is at the cost of a higher computational expense to achieve each new approximation. By the twenty-fifth re-iterate, both approximations are tending towards a settled divergence.

Table 7.3: e-iterated Galerkin

$\alpha = 1$; $\mathcal{S}_N = \mathcal{S}_3$; refinement: 40×40
Trial space 1 : 7.11); Trial space 2: 7.12)

e-iterate	$\|\hat{r}_{1,n}\|$	$\|\hat{r}_{2,n}\|$	$\frac{\|\hat{r}_{1,n}\|}{}$

Table 7.4: Sloan iteration

$\alpha = 2$; $\mathcal{S}_N = \mathcal{S}_3$; refinement: 40×40
Trial space 1 : 7.7); Trial space 2: 7.8)

Iterate	$\|\check{r}_{1,n}\|$	$\|\check{r}_{2,n}\|$	$\frac{\|\check{r}_{1,n}\|}{\|\check{r}_{1,n-1}\|}$	$\frac{\|\check{r}_{2,n}\|}{\|\check{r}_{2,n-1}\|}$
0	0.9528	1.244		
1	2.847	1.054	2.988	0.8476
2	5.318	1.158	1.868	1.099
3	14.23	1.273	2.675	1.099
4	34.18	1.399	2.403	1.099
5	84.84	1.5378	2.482	1.099
6	208.52	1.691	2.458	1.099
7	514.0	1.858	2.465	1.099
8	1.266×10^3	2.043	2.463	1.099
9	3.119×10^3	2.246	2.464	1.099
10	7.682×10^3	2.269	2.463	1.099

show divergence, although not in the predicted problem. In addition, table 7.5 provides the approximation ρ_O $_4$)

and

$$\text{Trial Space 2: } \begin{cases} \sin\ 2x_0)\sin\ y_0), \\ \sin\ 4x_0)\sin\ y_0), \end{cases}$$

improves matters see table 7.6) and provides convergence in both problems. However, the extremely slow convergence in problem 2 is impractical.

Table 7.6: e-iterated Galerkin

$\alpha = 2$; $\mathcal{S}_N = \mathcal{S}_3$; refinement: 40×40
Trial space 1 : 7.11); Trial space 2: 7.12)

Iterate	$\|\hat{r}_{1,n}\|$	$\|\hat{r}_{2,n}\|$	$\frac{\|\hat{r}_{1,n}\|}{\|\hat{r}_{1,n-1}\|}$	$\frac{\|\hat{r}_{2,n}\|}{\|\hat{r}_{2,n-1}\|}$	$\|R\|^2$	$\|T\|^2$
Galerkin	0.2795	2.557				
0	0.4454	1.179			7.020×10^{-2}	0.9298
1	0.2367	1.131	0.5314	0.9598	5.670×10^{-2}	0.9433
2	0.1311	1.118	0.5540	0.9882	5.190×10^{-2}	0.9481
3	7.300×10^{-2}	1.102	0.5564	0.9857	4.790×10^{-2}	0.9521
			$\vdots$			
9	2.100×10^{-3}	1.013	0.5548	0.9860	2.310×10^{-2}	0.9769
10	1.200×10^{-3}	0.9983	0.5547	0.9860	1.840×10^{-2}	0.9816

The results that come from further expanding the trial spaces to

$$\text{Trial Space 1: } \begin{cases} \sin\ y_0), \\ \cos\ 2x_0)\sin\ y_0), \\ \sin\ 3y_0) \end{cases} \tag{7.14}$$

and

$$\text{Trial Space 2: } \begin{cases} \sin\ 2x_0)\sin\ y_0), \\ \sin\ 4x_0)\sin\ y_0), \\ \sin\ 2x_0)\sin\ 3y_0), \end{cases} \tag{7.15}$$

are recorded in table 7.7. Observe the sharp increase in the rate of convergence in problem 2, that has been created by adding the trial function $\sin\ 2x_0)\sin\ 3y_0)$. Also note that, for the current and previous trial space 1, the norm of the residual error deteriorates between the Galerkin approximation and the iterated Galekin approximation. This is not unexpected, and does not indicate that the norm of the pointwise error is deteriorating see §6.4).

esults of an even further expansion of the trial spaces to

$$\text{Trial Space 1: } \begin{cases} \sin\ y_0), \\ \cos\ 2x_0)\sin\ y_0), \\ \sin\ 3y_0), \\ \cos\ 4x_0)\sin\ y_0), \\ \cos\ 2x_0)\sin\ 3y_0), \\ \sin\ 5y_0), \end{cases} \tag{7.16}$$

Table 7.7: e-iterated Galerkin

$\alpha = 2$; $\mathcal{S}_N = \mathcal{S}_3$; refinement: 40×40
Trial space 1 : 7.14); Trial space 2: 7.15)

Iterate	$\|\hat{r}_{1,n}\|$	$\|\hat{r}_{2,n}\|$	$\frac{\|\hat{r}_{1,n}\|}{\|\hat{r}_{1,n-1}\|}$	$\frac{\|\hat{r}_{2,n}\|}{\|\hat{r}_{2,n-1}\|}$	$\|R\|^2$	$\|T\|^2$
Galerkin	0.2606	0.7044				
0	0.3026	0.1048			0.2127	0.7873
1	0.1403	2.400×10^{-3}	0.4636	2.260×10^{-2}	0.2011	

esults with varying trial spaces are recorded in tables 7.10-7.14, with the largest trial spaces being

$$\text{Trial Space 1: } \begin{cases} \sin\ y_0), \\ \cos\ 2x_0)\sin\ y_0), \\ \sin\ 3y_0), \\ \cos\ 4x_0)\sin\ y_0), \\ \cos\ 2x_0)\sin\ 3y_0), \\ \sin\ 5y_0), \\ \cos\ 6x_0)\sin\ y_0), \\ \cos\ 4x_0x_0)\sin\ 3y_0), \\ \cos\ 2x_0)\sin\ 5y_0), \\ \sin\ 7y_0), \end{cases} \tag{7.18}$$

and

$$\text{Trial Space 2: } \begin{cases} \sin\ 2x_0)\sin\ y_0), \\ \sin\ 4x_0)\sin\ y_0), \\ \sin\ 2x_0)\sin\ 3y_0), \\ \sin\ 6x_0)\sin\ y_0), \\ \sin\ 4x_0)\sin\ 3y_0), \\ \sin\ 2x_0)\sin\ 5y_0), \\ \sin\ 8x_0)\sin\ y_0), \\ \sin\ 6x_0)\sin\ 5y_0), \\ \sin\ 4x_0)\sin\ 5y_0), \\ \sin\ 2x_0)\sin\ 7y_0). \end{cases} \tag{7.19}$$

Table 7.10: e-iterated Galerkin

$\alpha = 3$; $\mathcal{S}_N = \mathcal{S}_3$; refinement: 40×40
Trial space 1 : 7.9); Trial space 2: 7.10)

e-iterate	$\|\hat{r}_{1,n}\|$	$\|\hat{r}_{2,n}\|$	$\frac{\|\hat{r}_{1,n}\|}{\|\hat{r}_{1,n-1}\|}$	$\frac{\|\hat{r}_{2,n}\|}{\|\hat{r}_{2,n-1}\|}$	$\|R\|^2$	$\|T\|^2$
Galerkin	0.9969	1.199				
0	4.690	1.265			0.5816	0.4184
1	4.409	0.1003	0.9402	7.930×10^{-2}	0.6328	0.3672
2	5.262	2.360×10^{-2}	1.194	0.2352	0.4962	0.5038
3	6.515	2.100×10^{-3}	1.2380	8.840×10^{-2}	0.4334	0.5666
4	8.128	1.100×10^{-3}	1.2476	0.5499	0.3988	0.6012
			$\vdots$			
12	48.56	2.018×10^{-7}	1.250	0.7009	0.3469	0.6531531

Table 7.11: e-iterated Galerkin

$\alpha = 3$; $\mathcal{S}_N = \mathcal{S}_3$; refinement: 40×40
Trial space 1 : 7.11); Trial space 2: 7.12)

e-iterate	$\|\hat{r}_{1,n}\|$	$\|\hat{r}_{2,n}\|$	$\frac{\|\hat{r}_{1,n}\|}{\|\hat{r}_{1,n-1}\|}$	$\frac{\|\hat{r}_{2,n}\|}{\|\hat{r}_{2,n-1}\|}$	$\|R\|^2$	$\|T\|^2$
Galerkin	2.867	0.6383				
0	6.961	0.1789			0.3743	0.6257
1	24.96	4.150×10^{-2}	3.590	0.2322	0.3474	0.6526
2	89.16	1.100×10^{-2}	3.572	0.2646	0.3457	0.6543
			$\vdots$			
12	2.948×10^{6}	1.196×10^{-7}	3.564	0.5112	0.3455	0.6545
13	1.051×10^{8}	4.005×10^{-7}	3.564	3.348	0.3455	0.6545
14	3.744×10^{8}	1.677×10^{-6}	3.564	4.188	0.3455	0.6545
15	1.334×10^{9}	7.035×10^{-6}	3.564	4.194	0.3455	0.6555

Table 7.12: e -iterated Galerkin

$\alpha = 3$; $\mathcal{S}_N = \mathcal{S}_3$; refinement: 40×40
Trial space 1 : 7.14); Trial space 2: 7.15)

e-iterate	$\|\hat{r}_{1,n}\|$	$\|\hat{r}_{2,n}\|$	$\frac{\|\hat{r}_{1,n}\|}{\|\hat{r}_{1,n-1}\|}$	$\frac{\|\hat{r}_{2,n}\|}{\|\hat{r}_{2,n-1}\|}$	$\|R\|^2$	$\|T\|^2$
Galerkin	3.336	0.6059				
0	6.809	4.570×10^{-2}			0.3523	0.6477
1	27.74	6.200×10^{-3}	4.074	0.1364	0.3460	0.6540
2	113.8	1.400×10^{-3}	4.101	0.2296	0.3455	0.6545
			$\vdots$			
11	3.731×10^{7}	2.519×10^{-8}	4.101	0.4743	0.3455	0.6545
12	1.530×10^{8}	8.332×10^{-8}	4.101	3.308	0.3455	0.6545
			$\vdots$			
15	1.055×10^{10}	6.138×10^{-6}	4.101	4.194	0.3455	0.6555

Table 7.13: e-iterated Galerkin

$\alpha = 3$; $\mathcal{S}_N = \mathcal{S}_3$; refinement: 40×40
Trial space 1 : 7.16); Trial space 2: 7.17)

e-iterate	$\|\hat{r}_{1,n}\|$	$\|\hat{r}_{2,n}\|$	$\frac{\|\hat{r}_{1,n}\|}{\|\hat{r}_{1,n-1}\|}$	$\frac{\|\hat{r}_{2,n}\|}{\|\hat{r}_{2,n-1}\|}$	$\|R\|^2$	$\|T\|^2$
Galerkin	0.3045	0.5268				
0	0.1985	2.800×10^{-2}			6.200×10^{-3}	0.9938
1	8.000×10^{-3}	4.000×10^{-3}	4.010×10^{-3}	0.1436	1.996×10^{-4}	0.9998
2	3.300×10^{-3}	7.442×10^{-4}	0.4193	0.1854	1.976×10^{-4}	0.9998
3	7.336×10^{-4}	1.472×10^{-4}	0.2196	0.1978	1.978×10^{-4}	0.9998
4	1.998×10^{-4}	2.985×10^{-5}	0.2723	0.2028	1.978×10^{-4}	0.9998
5	5.376×10^{-5}	6.121×10^{-6}	0.2691	0.2051	1.978×10^{-4}	0.9998
6	1.483×10^{-5}	1.262×10^{-6}	0.2758	0.2062	1.978×10^{-4}	0.9998
			$\vdots$			
9	3.261×10^{-7}	1.121×10^{-8}	0.2818	0.2077	1.978×10^{-4}	0.9998
10	9.221×10^{-8}	4.487×10^{-6}	0.2827	0.4001	1.978×10^{-4}	0.9998
11	2.613×10^{-8}	1.613×10^{-6}	0.2834	3.595	1.978×10^{-4}	0.9998
12	7.419×10^{-9}	6.762×10^{-8}	0.2839	4.192	1.978×10^{-4}	0.9998
			$\vdots$			
15	1.769×10^{-10}	4.989×10^{-6}	0.2947	4.194	1.978×10^{-4}	0.9998
16	1.031×10^{-10}	2.092×10^{-5}	0.5829	4.194	1.978×10^{-4}	0.9998
17	1.796×10^{-10}	8.7758×10^{-5}	1.741	4.194	1.978×10^{-4}	0.9998
			$\vdots$			
20	1.370×10^{-9}	6.500×10^{-3}	1.972	4.194	1.978×10^{-4}	0.9998

Table 7.14: e-iterated Galerkin

$\alpha = 3$; $\mathcal{S}_N = \mathcal{S}_3$; refinement: 40×40
Trial space 1 : 7.18); Trial space 2: 7.19)

e-iterate	$\|\hat{r}_{1,n}\|$	$\|\hat{r}_{2,n}\|$	$\frac{\|\hat{r}_{1,n}\|}{\|\hat{r}_{1,n-1}\|}$	$\frac{\|\hat{r}_{2,n}\|}{\|\hat{r}_{2,n-1}\|}$	$\|R\|^2$	$\|T\|^2$
Galerkin	8.220×10^{-2}	0.4597				
0	1.860×10^{-2}	1.070×10^{-2}			6.200×10^{-3}	0.9938
1	3.000×10^{-3}	7.571×10^{-4}	0.1596	7.100×10^{-2}	1.977×10^{-4}	0.9998
2	5.491×10^{-4}	7.826×10^{-5}	0.1849	0.1034	1.978×10^{-4}	0.9998
			$\vdots$			
8	3.257×10^{-8}	4.532×10				

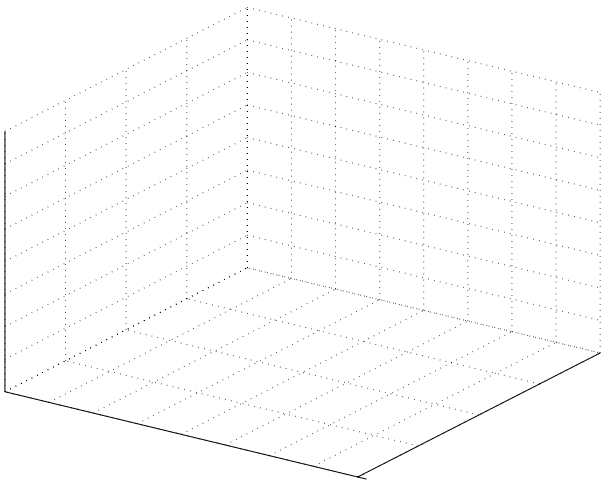
7.4 Eventu l diver ence

This problem is highlighted in tables 7.2, 7.3, 7.7, 7.8, 7.10, 7.11, 7.12, 7.13, 7.14, occurring in one or both of the problems. The problem appears to arise from a small residual error. e have already raised the point that an approximation may only be as good as the order of computational rounding error. However, this is not the answer as the rounding error is far smaller than the size of the residual error, at which the approximations go wrong. Also, re-iterating after rounding error has been reached, would cause a 're-shuffling' of errors, rather than the consistent divergence that we are finding.

Consider the following explanation.

The behaviour we are witnessing, seems to suggest that the re-iterated Galerkin method finds a new eigenvalue of the Sloan operator when the residual gets small – but why does this eigenvalue arise only for a small residual error, and why can it not be eliminated by expansion of the trial space? In §7.1, we discussed the properties of the Hilbert spaces in which each problem is set, and chose trial spaces accordingly. These Hilbert spaces we denoted $E\ D'$) and $O\ D'$), and are subspaces of the 'global' Hilbert space $L_2\ D'$). The appearance of the subspaces was essentially a product of the various symmetries in the problems. By choosing trial functions from these subspaces, we are attempting to minimize the spectral radius of the Sloan operatorattlVˆ6wh?;rˆ:Φr6op?,)oyUHˆ:ˆ:Φr9ˆDn˜ n

v



7.5.1 Quadrature re nement

To make an accurate comparison, all other variables must be fixed. So, making use of prior results, we shall work with $\alpha = 1, 2$, and 3, choosing a pair of trial spaces, for each α, that produced convergence in $|R|^2$ and $|T|^2$ to four decimal places within 5 re-iterations. All other variables, excepting the refinement, are unchanged.

e-iterated Galerkin approximations, where varying quadrature refinements are used, are contained in tables 7.15-7.17. Values of $|R|^2$ and $|T|^2$ are recorded after 5 re-iterations, by which time they had converged to at least 4 decimal places.

Table 7.15: esults of varying quadrature refinement after 5 re-iterations

$\alpha = 1$; $\mathcal{S}_N = \mathcal{S}_3$;
Trial space 1 : 7.11); Trial space 2: 7.10)

efinement	$\|R\|^2$	$\|T\|^2$
10×10	3.200×10^{-3}	0.9968
20×20	3.000×10^{-3}	0.9700
30×30	3.000×10^{-3}	0.9700
40×40	3.000×10^{-3}	0.9700
50×50	3.000×10^{-3}	0.9700

Table 7.16: esults of varying quadrature refinement after 5 re-iterations

$\alpha = 2$; $\mathcal{S}_N = \mathcal{S}_2$

Table 7.17: esults of varying quadrature refinement after 5 re-iterations

$\alpha = 3$; $\mathcal{S}_N = \mathcal{S}_3$;
Trial space 1 : 7.16); Trial space 2: 7.15)

efinement	$\|R\|^2$	$\|T\|^2$
10×10	6.000×10^{-3}	0.9940
20×20	8.748×10^{-4}	0.9991
30×30	3.323×10^{-4}	0.9997
40×40	1.978×10^{-3}	0.9998
50×50	1.978×10^{-4}	0.9998
60×60	1.216×10^{-4}	0.9999

Table 7.18: esults of varying series truncation after 5 re-iterations

$\alpha = 1$; 40×40
trial space 1 = 2; trial space 2 = 1

Terms	$\|R\|^2$	$\|T\|^2$
1	2.900×10^{-3}	0.9971
2	2.900×10^{-3}	0.9971
3	3.000×10^{-3}	0.9970
4	3.000×10^{-3}	0.9970
5	3.000×10^{-3}	0.9970
10	3.000×10^{-3}	0.9970

Table 7.19: esults of varying series truncation after 5 re-iterations

$\alpha = 2$; 40×40
trial space 1 = 2; trial space 2 = 1

Terms	$\|R\|^2$	$\|T\|^2$
1	0.2009	0.7991
2	0.2009	0.7991
3	0.1976	0.8024
4	0.1976	0.8024
5	0.1974	0.8026
10	0.1974	0.8026

Table 7.20: esults of varying series truncation after 5 re-iterations

$\alpha = 3$; 40×40
trial space 1 = 6; trial space 2 = 3

Terms	$\|R\|^2$	$\|T\|^2$
1	6.100×10^{-3}	0.9939
2	6.100×10^{-3}	0.9939
3	1.978×10^{-3}	0.9998
4	1.978×10^{-3}	0.9998
5	8.082×10^{-5}	0.9999
10	7.877×10^{-3}	0.9999

As terms in the infinite series are ordered in decreasing size, it seems a fair assumption that an increase of five to ten terms, having little relative effect, is a good indication that a further increase in terms used would have even less effect.

It is easily seen that, a low number of terms provides an approximation to the infinite series that produces 'good' approximations to the required quantities. This was predicted in §6.3.1,

Figure 7.5 plots the energies $|R|^2$ and $|T|^2$ against α, where values of α have been taken in steps of 0.05. Values of $|R|^2$ and $|T|^2$ were allowed to converge to 3 decimal place, and recorded at this accuracy. The trial spaces used in the re-iterated Galerkin method, were varied as required, although this point is immaterial to figure 7.5.

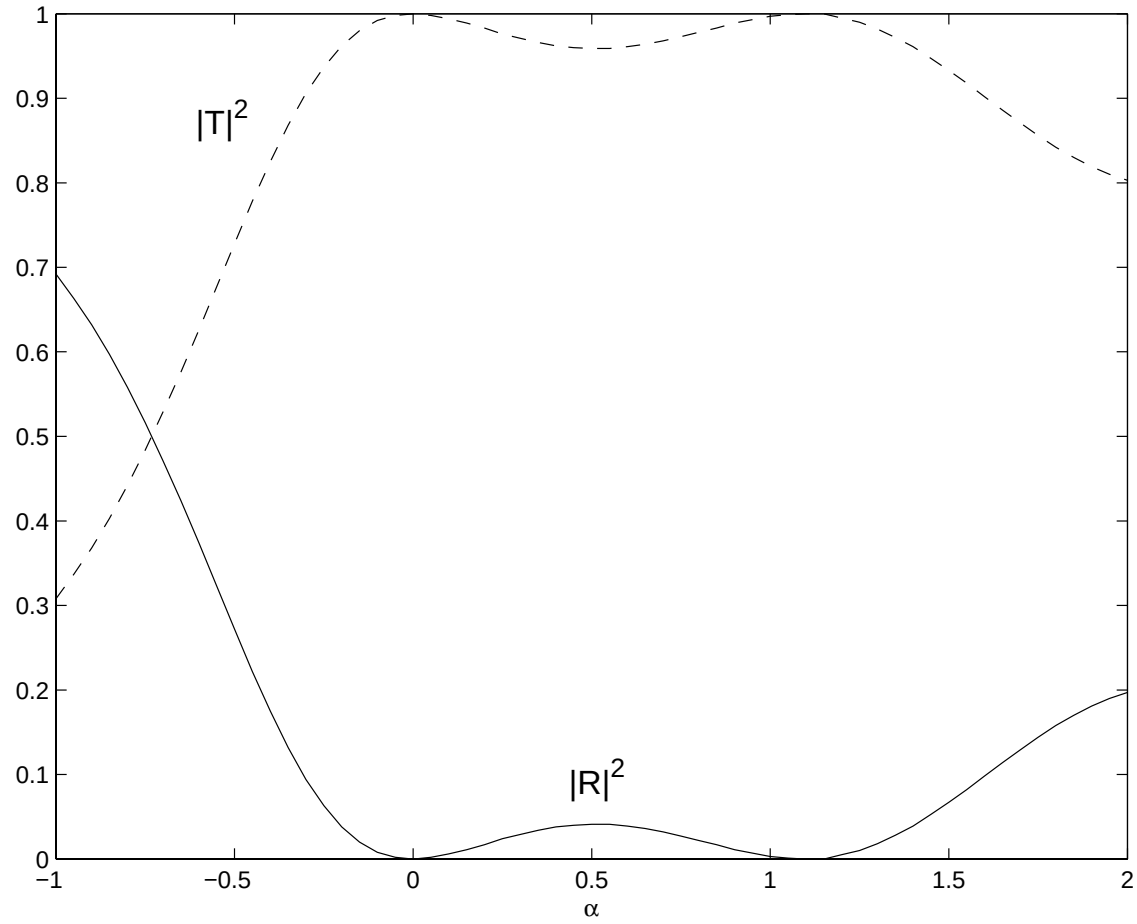


Figure 7.5: Energy proportion approximations against α

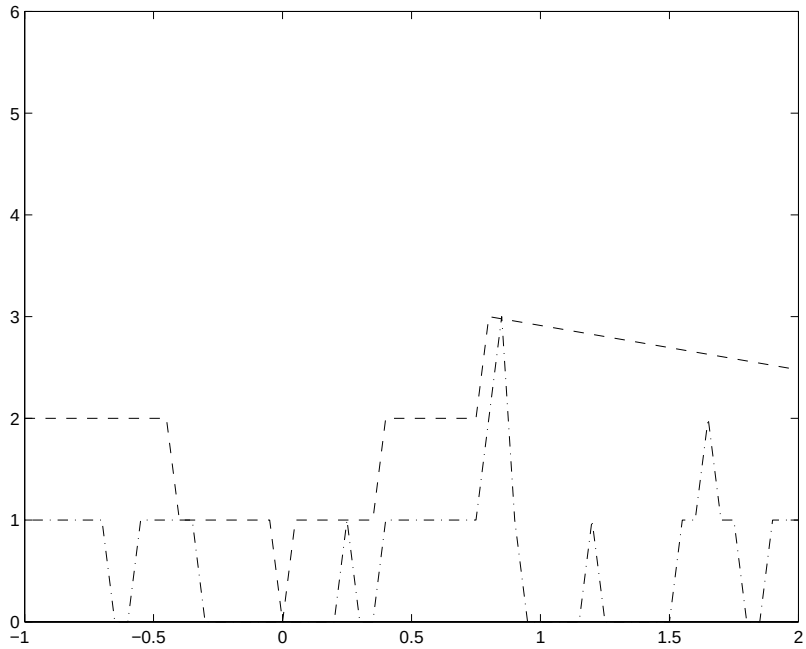
Note that, figure 7.5 shows that check ii, the conservation of energy property

$$|R|^2 + |T|^2 = 1,$$

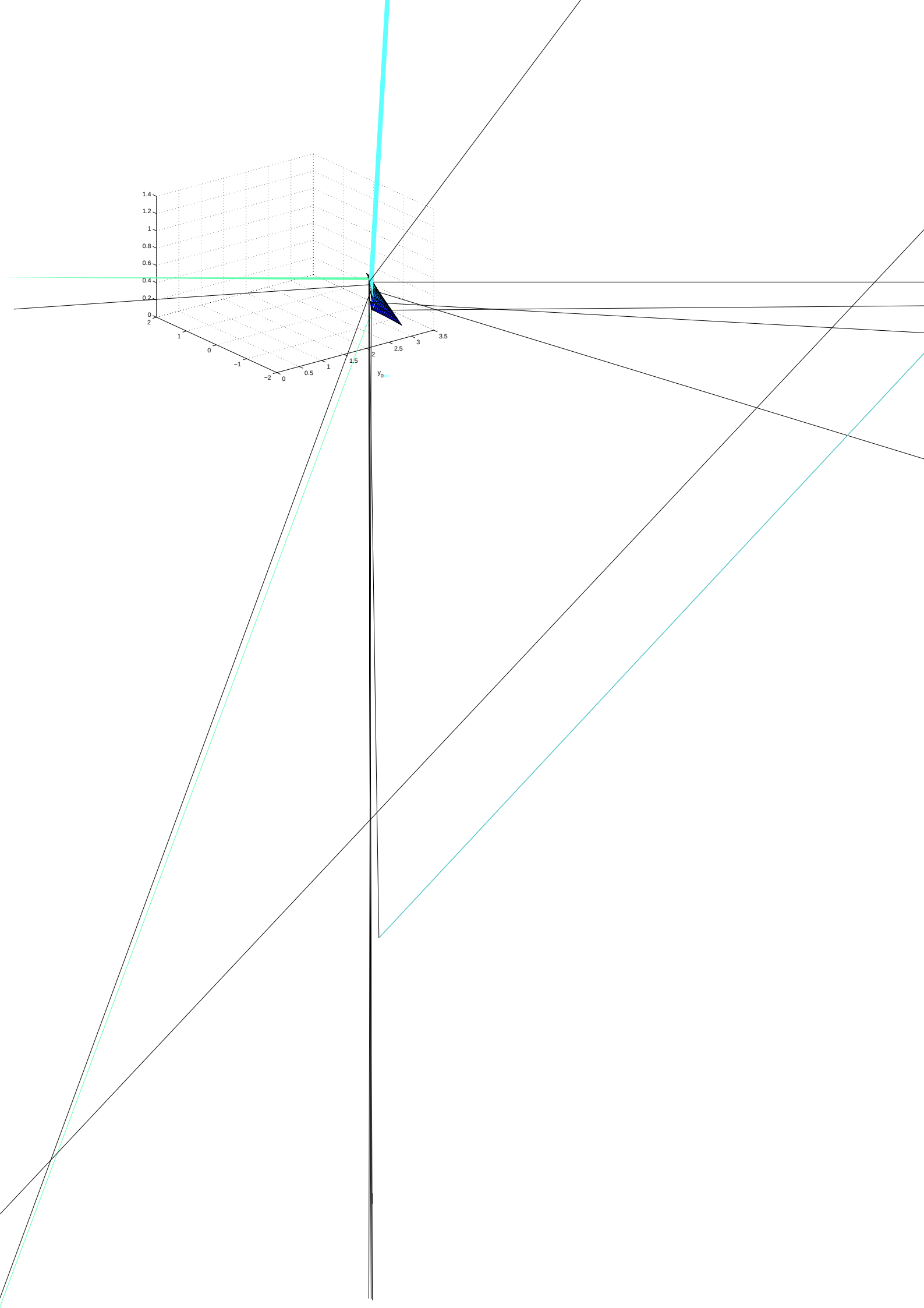
is satisfied. Checks i and v were also satisfied by al:ˆ:Φr6y?9z7ˆˆ6al ˜˜˜D˜):Φr6ximations?9z7, ?9zV

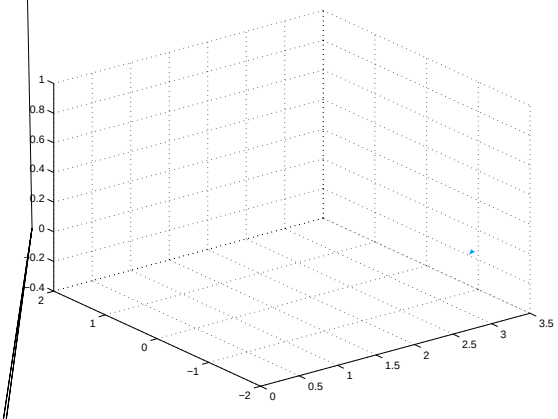
The overall tendency seems to be that, the greater the size of the obstacle, the greater the amount of reflected energy.

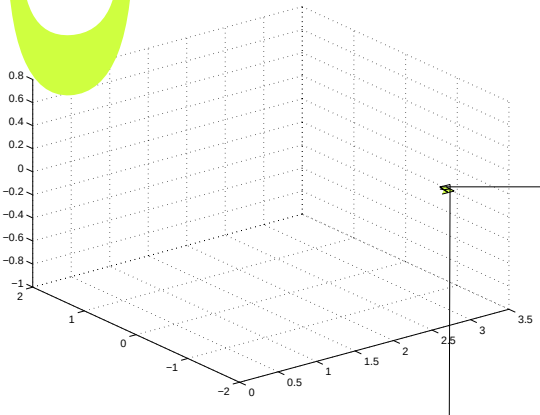
In figures 7.6-7.7, we return to investigating the behaviour of the re-iterated Galerkin method. Figure 7.6 plots the size of trial space one used against the number of re-iterations until the norm of the residual error in problem 1 reaches order 10^{-4}. Figure 7.7 is the corresponding graph for problem 2. Both figures 7.6 and 7.7, also plot the number of re-iterations needed for $|R|^2$ and $|T|^2$ to converge to 3 decimal places.

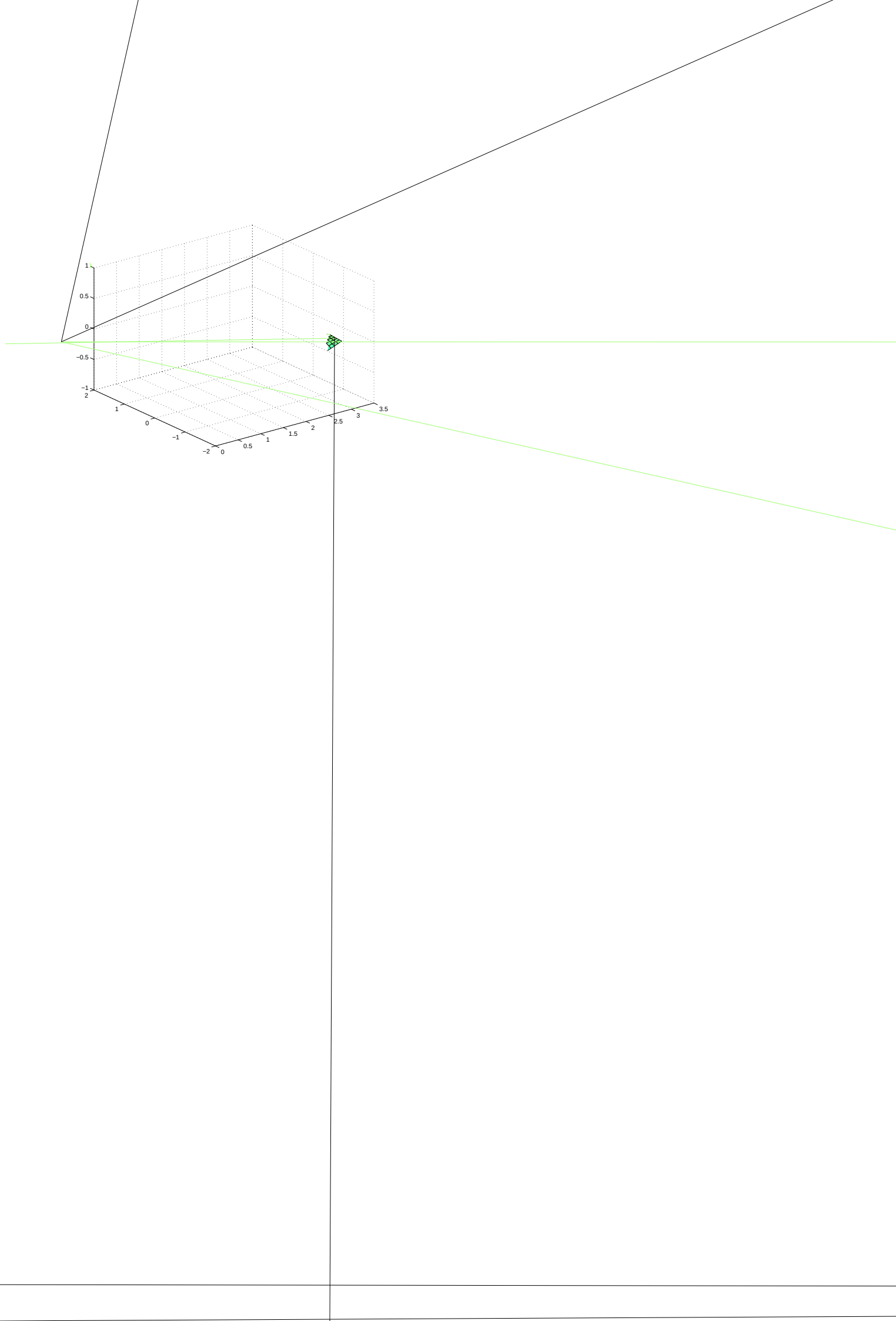

6
5
4
3
2
1
0
−1
−0.5
0
0.5
1
1.5
2

It is of interest to see the approximations to ϕ


1.4
1.2
1
0.8
0.6
0.4
0.2
0
2
1
0
−1
−2
0
0.5
1
1.5
2
2.5
3
3.5
$y_{0,x0}$


1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
2
1
0
−1
−2
0
0.5
1
1.5
2
2.5
3
3.5


0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
2
1
0
−1
−2
0
0.5
1
1.5
2
2.5
3
3.5


1
0.5
0
−0.5
−1
2
1
0
−1
−2
0
0.5
1
1.5
2
2.5
3
3.5

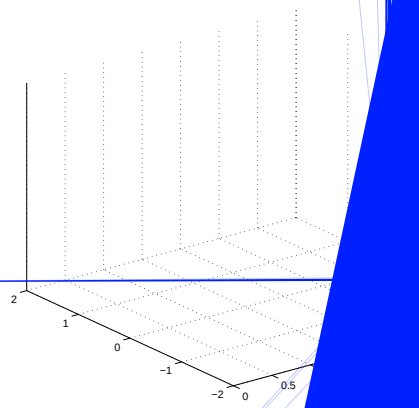
In order to test whether the re-iterated Galerkin method, and the forms it, are robust, they are applied to a variation on the problem that has estigated in this chapter.

7.7 P r bolic x dependence

By making the subtle change

$$k\ x, y) = k_0 + \frac{1}{\pi}\left(\ \frac{\pi}{m})^2 - x^2\right) \sin^2\ y) \tag{7.20}$$

we introduce a new property to the problem, in that the wa ndence, in the obstacle, is inversely parabolic, rather than inversely trigonor e does little to the overall size and shape of k. In particular, all symmetrie evious problem remain, and all preparatory investigation into trial spaces ar h, it could quite reasonably be presumed that the re-iterated Galerkin met n a very similar fashion on this problem as it did on the previous.



As always, we begin by satisfying ourselves that the re-iterated Galerkin method is required, by applying Sloan iteration to the problem. Table 7.21 con7.)21HHv

Table 7.22: e-iterated Galerkin

$\alpha = 2$; $\mathcal{S}_N = \mathcal{S}_3$; refinement: 40×40
Trial space 1 : 7.9); Trial space 2: 7.10)

e-iterate	$\|\hat{r}_{1,n}\|$	$\|\hat{r}_{2,n}\|$	$\frac{\|\hat{r}_{1,n}\|}{\|\hat{r}_{1,n-1}\|}$	$\frac{\|\hat{r}_{2,n}\|}{\|\hat{r}_{2,n-1}\|}$	$\|R\|^2$	$\|T\|^2$

Figure 7.64 plots the approximations to $|R|^2$ from both the problem in which the x dependence of $k\ x, y)$ was trigonometric, against the current problem, in which the dependence is parabolic. This is done

C a ter 8

Conclusions

The body of our work has been in familiarising ourselves with the ways a particular aspect

erator equation in the Hilbert space of Lebesgue integrable functions. Most of the issues raised were of a computational nature, with attention centring on how an efficient program could be constructed to implement the re-iterated Galerkin method for the acoustics problem. In §1 we introduced the imaginary dimension to the problem, and this meant that when writing a program the integral equation had to be split into two, in order that the program worked only with real numbers. The integral operator also required special attention due to the singular infinite series that appears in the kernel. e also touched on the errors that are introduced by writing a program to perform computations, and the effects on the accuracy of approximations made using the re-iterated Galerkin method. It was seen that the re-iterated Galerkin method is well suited to the effects of these errors, as it is sensitive to the error in the current approximation, and hence does not accumulate errors. Given more time and effort, this could have been proved in a rigorous manner.

The program was implemented using trial spaces motivated by a discussion of the symmetries of the two integral equations. This discussion also helped us refine the setting of the integral equations. To counter the lac

there exists an unique solution. This is if the chosen parameters coincide with an eigenvalue of the Galerkin approximation. More specifically, for a chosen trial space, and set of parameters, it may not be possible to invert the matrix required to produce the Galerkin approximation. The eigenvalues of this matrix are approximations of the eigenvalues of the exact equation. By changing the trial space these approximations will move, and the problem disappear, unless the problem was an accurate indication of an eigenvalue of the exact equation. The subject of eigenvalues is an ongoing area of research.

In §7, frequent reference was made to the possibility of an investigation of the optimality of trial spaces. The choice of trial space is fundamental to the success or failure of an application of the re-iterated Galerkin method. For the problems with which we dealt, it was possible to achieve a fast rate of convergence from a fairly small trial spaces, chosen in a logical rather than knowingly optimal fashion. However, by simple changes in certain parameters, the spectral radius of the operator $\tilde{\mathcal{G}}M$ could be made large enough that a more prudent choice of trial spaces would be required to maintain the low computational cost that the re-iterated Galerkin method emphasises. The difference in the effect of the re-iterated Galerkin method on problems that involve operators, with different size spectral radii, was well documented in the results of §7. An extension in this area could involve comparing the truncated double Fourier series used here, against other trial spaces, for instance, Legendre polynomials, or it could be in deciding on a more optimal way of extending the truncated Fourier series, than the 'triangular' approach we have adopted. The orthogonality relations of the trigonometric functions used in the kernel as well as the trial functions, would undoubtedly have an influence over this optimality

A endix A

The MatLab code used to apply the re-iterated Galerkin method.

Functions are represented by finite dimensional vectors that hold point values taken at the nodes of a regular cartesian mesh. This structure was outlined in §6. The functions that appear are contained in appendices C-F.

```
clear;

global X2d
global rect
global alpha
global m
global k_0
global beta_0
global X_vals
global f_5
global basis1
global basis2
global mx_term
```

```
STORE(1,4:8) = 0;
```

comme t – Define the limits of the rectangular region D'...

```
A = -pi/m;
B = -A;
C = 0;
D = pi;
```

comme t – `xp`=X_p and `yp`=Y_p.

```
xp = 40;
yp = 40;
```

comme t – `X_vals` contains the values x . Likewise `Y_vals` contains the values y .

```
X_vals = zeros(xp,1);
for loop=1:xp
   X_vals(loop)=((2*loop -1)*(B-A)/(2*xp)) + A;
end

Y_vals = zeros(yp,1);
for loop=1:yp
   Y_vals(loop)=((2*loop -1)*(D-C)/(2*yp)) + C;
end
```

comme t – `X2d` contains the points $\boldsymbol{x}$.

```
X2d = zeros(xp*yp,2);
for loop=1:yp
   X2d(xp*(loop-1) +1:xp*loop,1)=X_vals(:);
   X2d(xp*(loop-1) +1:xp*loop,2)=Y_vals(loop);
end
```

comme t – `rect`=Q.

```
rect = ((B-A)/xp)*((D-C)/yp);

free_term(:,1) = cos(beta_0*X2d(:,1)).*sin(X2d(:,2));
free_term(:,2) = sin(beta_0*X2d(:,1)).*sin(X2d(:,2));
```

comme t – `mesh` is the number of nodes in the rectangular mesh.

```
mesh = length(X2d(:,1));
```

comme t – `basis1` contains vectors defining the trial functions for problem 1.

```
basis1 = zeros(mesh,1);
```

```
basis1(:,1) = sin(X2d(:,2));
basis1(:,2) = cos(m*X2d(:,1)).*sin(X2d(:,2));
basis1(:,3) = sin(3*X2d(:,2));
```

comme t – `basis2` contains vectors defining the trial functions for problem 2.

```
basis2 = zeros(mesh,1);
basis2(:,1) = sin(m*X2d(:,1)).*sin(X2d(:,2));
basis2(:,2) = sin(2*m*X2d(:,1)).*sin(X2d(:,2));
basis2(:,3) = sin(m*X2d(:,1)).*sin(3*X2d(:,2));
```

comme t – Calculate the dimensions of the two trial spaces ...

```
dim1 = length(basis1(1,:));
dim2 = length(basis2(1,:));
```

comme t – f_5 $= A_{X,Y}$.

```
f_5 = log_5(X2d(:,1));
```

comme t – The `G_basis` matrices contain vectors defining the trial functions having been operated on by $\tilde{\mathcal{G}}M$.

```
for loop=1:dim1
   G_basis1(:,loop) = tilde_G_modif(basis1(:,loop));
end

for loop=1:dim1
   A_basis1(:,loop) = basis1(:,loop) - G_basis1(:,loop);
end
```

comme t – `matrix1` and `matrix2` are the matrices used to calculate the Galerkin approximations.

```
for loop = 1:dim1
for loop2 = 1:dim1
   matrix1(loop,loop2) = A_basis1(:,loop2)'*basis1(:,loop);
end
end

matrix1 = rect*matrix1;

for loop=1:dim2
   G_basis2(:,loop) = tilde_G_modif(basis2(:,loop));
end

for loop=1:dim2
```

A

```
clear A_sigma_1; clear A_sigma_2;
```

comme t – This follows the theory of §6.2.

```
energy_mat = eye(4);
energy_mat(1:2,3:4) = (1/(beta_0*pi))*L;
energy_mat(3:4,1:2) = -(1/(beta_0*pi))*L;
energy_vec(1:2,1) = L(1:2,1);
energy_vec(3:4,1) = L(1:2,2);
energy_c = energy_mat*energy_vec;

R = (1/(beta_0*pi))*(i*(energy_c(1)-energy_c(4)) - (energy_c(2) + energy_c(3)));
T = 1 + (1/(beta_0*pi))*(i*(energy_c(1) + energy_c(4)) - (energy_c(3)-energy_c(2)));
```

comme t – Check ii...

```
energy_cons = abs(R)^2 + abs(T)^2;

energy_R = abs(R)^2
energy_T = abs(T)^2

STORE(4,6) = R;
STORE(4, ) = T;
STORE(4,8) = energy_R;
STORE(4,9) = energy_T;
STORE(4,4:5) = 0;
clear energy_mat; clear energy_vec;
```

comme t – Checks i & v.

```
check_1 = inner_product(sigma1 , Mult(free_term(:,2)));
check_2 = inner_product(sigma2 , Mult(free_term(:,1)));
check_error = l2_norm(check_1 - check_2);

re_iterate_res_error1 = iterate_error1;
re_iterate_res_error2 = iterate_error2;
clear iterate_error1; clear iterate_error2;
```

comme t – **Re-iteration**.
comme t – Can do any number of re-iterations, here 1000 is the chosen number.
comme t – The code within the loop mimics the above code.

```
 for loop=1:1000
```

comme t – The `previous` are used in the spectral radius approximations.

```
   previous1 = re_iterate_res_error1;
```

```
    inner_product(A_sigma_1 , Mult(sigma2));
   L(2,1) = L(1,2);

   energy_mat = eye(4);
   energy_mat(1:2,3:4) = (1/(beta_0*pi))*L;
   energy_mat(3:4,1:2) = -(1/(beta_0*pi))*L;
   energy_vec(1:2,1) = L(1:2,1);
   energy_vec(3:4,1) = L(1:2,2);

   clear energy_c;
   energy_c = energy_mat / energy_vec;

   R = (1/(beta_0*pi))*(i*(energy_c(1)-energy_c(4)) - (energy_c(2) + energy_c(3)));
   T = 1 + (1/(beta_0*pi))*(i*(energy_c(1) + energy_c(4)) ...
    - (energy_c(3)-energy_c(2)));

   energy_cons = abs(R)^2 + abs(T)^2;
   energy_R = abs(R)^2
   energy_T = abs(T)^2

   STORE(4+loop,6) = R;
   STORE(4+loop, ) = T;
   STORE(4+loop,8) = energy_R;
   STORE(4+loop,9) = energy_T;

   clear energy_mat; clear energy_vec;

   check_1 = inner_product(sigma1 , Mult(free_term(:,2)));
   check_2 = inner_product(sigma2 , Mult(free_term(:,1)));
   check_error = l2_norm(check_1 - check_2);
end
```

A endix B

The

```
yp = 40;

X_vals = zeros(xp,1);
for loop=1:xp
   X_vals(loop)=((2*loop -1)*(B-A)/(2*xp)) + A;
end

Y_vals = zeros(yp,1);
for loop=1:yp
   Y_vals(loop)=((2*loop -1)*(D-C)/(2*yp)) + C;
end

X2d = zeros(xp*yp,2);
for loop=1:yp
   X2d(xp*(loop-1) +1:xp*loop,1)=X_vals(:);
   X2d(xp*(loop-1) +1:xp*loop,2)=Y_vals(loop);
end

rect = CGvoYZiB*GF;(0u*YTah ;eti f
```

```
end
end

matrix = rect*matrix;

F = basis_ip(free_term);

inv_mat = inv(matrix);

c = inv_mat*F;
p = basis*c;

residual = free_term - p + G_basis*c;
Galerkin_res_error = l2_norm(residual)

STORE(3,2) = Galerkin_res_error;

iterate_error = Galerkin_res_error;
clear Galerkin_res_error;
```

comme t – **Repeated loan iteration**

```
for loop=1:1000
   clear previous;
   previous = iterate_error;

   clear iterate_error;
   p = residual + p;

   clear residual;
   residual = free_term - p + tilde_G_modif(p);

   iterate_error = l2_norm(residual)

   spectral_rad_est = iterate_error / previous

   done = loop

   STORE(3+loop,1) = loop;
   STORE(3+loop,2) = iterate_error;
   STORE(3+loop,3) = spectral_rad_est;
end
```

A endix C

The multiplication operator M. The unwanted function is merely deleted.

```
function var = Mult(f)

global X2d
global alpha
global m
global k_0
```

comme t – The trigonometric x dependence...

```
var = 2*k_0*alpha*(cos(m*X2d(:,1)/2).^2).*((sin(X2d(:,2))).^2 ).*f +...
    (alpha^2)*(cos(m*X2d(:,1)/2).^4).*((sin(X2d(:,2))).^4).*f;
```

comme t – The parabolic x dependence...

```
l = pi/m;
parab = (l^2)-(X2d(:,1).^2);
parab = parab/pi;

var = 2*k_0*alpha*parab.*((sin(X2d(:,2))).^2).*f +...
    (alpha^2)*(parab.^2).*((sin(X2d(:,2))).^4).*f;
```

A endix D

The approximation of the operator $\tilde{\mathcal{G}}M$.

The modifications that led to the structure of $\tilde{\mathcal{G}}M$ used is outlined in §6.3. Integrals are approximated using the rectangular midpoint rule.

.1

comme t – The function that approximates $\tilde{\mathcal{G}}M$.
comme t – It merely combines other functions.

```
function var = tilde_G_modif(f)

global f_5

var = -sine_kernel(f) + ...
    modified_series(f)+...
    (log_2(f)/4)-...
    (log_3(f)/4)-...
    (log_4(f)/4)-...
    (Mult(f).*f_5)/4;

var = var/pi;
```

.2

comme t – The part of the kernel defined as $\pi M\ a_0$).

```
function var = sine_kernel(f)

global X2d
global rect
global beta_0

g = Mult(f);
dim = length(f);
```

```
var = zeros(dim,1);

for loop=1:dim
   var = var + sin(beta_0*abs(X2d(loop,1)-X2d(:,1)))*...
    sin(X2d(loop,2))*g(loop);
end

var = var.*sin(X2d(:,2));
var = var*rect;
var = var/beta_0;
```

.3

comme t – The approximate series $M\mathcal{S}_N)\ x, y \mid x_0, y_0)$.

```
 function var = modified_series(f)

 global X2d
 global rect
 global k_0
 global mx_term

 g = Mult(f);
 dim = length(f);
 var = zeros(dim,1);
```

comme t – The first term is done separately...

```
for loop = 1:dim
   var = var - exp(-abs(X2d(loop,1)-X2d(:,1))).*sin(X2d(loop,2))*g(loop);
end

var = rect*sin(X2d(:,2)).*var;

for loop2=2:mx_term
   int_term =zeros(dim,1);
   gamma_n = sqrt((loop2land2)-(k_0land2));
for loop = 1:dim
   int_term = int_term + g(loop)*sin(loop2*X2d(loop,2))*sin(loop2*X2d(:,2)).*...
    (loop2*exp(-gamma_n*abs(X2d(loop,1)-X2d(:,1)))-...
    gamma_n*exp(-loop2*abs(X2d(loop,1)-X2d(:,1))));
end
   int_term = int_term*rect;
   int_term = int_term/(gamma_n*loop2);
   var = var + int_term;
end
```

.4

comme t – The part of the kernel defined as $M\mathcal{L}_1)\ x, y \mid x_0, y_0)$.

```
function var = log_2(f);

global X2d
global rect

dim = length(f);
g = Mult(f);
var = zeros(dim,1);

 for loop=1:dim
   var = var + g(loop)*log( 1 - 2*cos(X2d(loop,2) + X2d(:,2)).*...
    exp(-abs(X2d(loop,1)-X2d(:,1))) +...
    exp(-2*abs(X2d(loop,1)-X2d(:,1))));
end

var = var*rect;
```

.5

comme t – The part of the kernel defined as

```
    +((X2d(loop,2)-Xv(:,2)).^2))));
end

var = var*rect;
```

.6

comme t – The part of the kernel defined as $\mathcal{L}_3\ x, y \mid x_0, y_0)\big(\ M\psi)\ x, y) -\ \ M\psi)\ x_0, y_0)\big)$.

```
function var = log_4(f)

global X2d
global rect

dim = length(f);
g = Mult(f);
var = zeros(dim,1);

for loop=1:dim
```

comme t – The vector `Xv` is used to artificially implement the limit 6.8).

```
   Xv=X2d;
   Xv(loop,1)= X2d(loop,1) - 1;

   var = var + ...
    (g(loop)-g(:)).*...
   log(((X2d(loop,1)-Xv(:,1)).^2)+((X2d(loop,2)...
    -Xv(:,2)).^2));
end

var = var*rect;
```

.7

comme t – The part of the kernel approximated by a modified rectangular mid-point rule.
comme t – In §6.3 this function is denoted as $\mathscr{L}$.
comme t – The value of this function depends only on the refinement.

```
function var = log_5(f)

global X2d
global rect
global X_vals

dim = length(f);
```

A endix E

Functions that output a vector of inner products of a function with either trial space 1 or trial space 2.

They are used to produce the 'right hand side' vectors that appear in the Galerkin approximations.

E.1

comme t – Used in problem 1.anmnmnmnm:c6anfUflk1iU w d?;ranmnmnmnz.

```
end

var = var*rect;
```

A endix F

Functions that approximate a L_2 D')-inner product and L_2 D')-norm. As always, approximations are made using the rectangular midpoint rule.

F.1

```
function var = inner_product(f,g)

global rect
```

comme t – ritten knowing that all functions to be encountered are real-valued.

```
var = f'*g;
var = var*rect;
```

F.2

comme t – The norm function is a trivial extension of the inner product function.

```
function var = l2_norm(f)

var = sqrt(inner_product(f,f));
```

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