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properties (ergodicity, hyperbolicity) and that cause the model losing ability in describing intrinsically multiscale processes.

The Lorenz system [1] has a central role in modern science as it has provided the first example of low-dime

Palmer [17] introduced artificially

$$
\begin{array}{l}
{}_{t} \quad {}^{2} \quad J \quad , \quad {}^{2} \quad g \quad {}_{x} \quad {}^{4} \\
{}_{t} \quad J \quad , \quad \dfrac{T}{H} \quad {}_{x} \quad k \quad {}^{2} \quad \dfrac{}{C_v} \quad {}_{ij} \quad {}_{ij}
\end{array}
\qquad (1a\text{-}1b)
$$

where $g$ is the gravity acceleration, $\vec{v} \quad u, w \quad {}_{z}, {}_{x}$, $T \quad T_0 \quad z \; T/H$, with $H$ uniform depth of the fluid and $T$ imposed temperature difference. The suitable boundary ˜W ™`ïiÀ

See [3] for a detailed derivation in the case $Ec \quad 0$, considering that the sign of the term proportional to $R$ is wrong in both Eqs. 34 and 35. In the case $Ec \quad 0$, the seminal Lorenz system can be derived by severely truncating the system, considering the evolution equation for the real part of $_{1,1}$ and for the imaginary parts of $_{1,1}$ and $_{0,2}$, and performing suitable rescaling (see below). While $_{0,2}$ has no real part because of the boundary conditions [3], neglecting the imaginary (real) part of $_{1,1}$ ($_{1,1}$) amounts to an arbitrary selection of the phase of the waves in the system. An entire hierarchy of generalized Lorenz models, all obeying to this constraint, can be derived with lengthy but straightforward calculations. See, *e.g.*, [4-6] for detailed discussion of these models.

In this work we include the modes $_{1,1}$, $_{2,2}$, $_{1,1}$, $_{2,2}$ in our truncation, and retain both the real and the imaginary parts, whereas the considered horizontally symmetric modes $_{0,2}$ and $_{0,4}$ are, as mentioned above, imaginary. Finally, we assume, in general, a non vanishing value for $Ec$. If we define $_{1,1} \quad X_1 \quad iX_2$ , $_{1,1} \quad Y_1 \quad iY_2$

$$T \quad , \quad S \quad \begin{matrix} R & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 \\ 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & I \end{matrix} \quad \begin{matrix} X \\ Y \\ A \\ B \\ Z \end{matrix} \quad , \tag{5}$$

with $I$ being the 2X2 identity matrix and being the 2X2 rotation m

a

properties of $Z_1$ do not depend on the initial conditions, and agree with those of the $z$ variable of the classical Lorenz system. Moreover, the statistical properties of the quadratic quantities $X^2$ $X_1^2$ $X_2^2$, $Y^2$ $Y_1^2$ $Y_2^2$, and $XY$ $X_1Y_2$ $X_2Y_1$ do not depend on the initial conditions (whereas those of each term in the previous sums do!), and agree (for all values of $r$, , $b$ !) with those of $x^2$ and $y^2$, and $xy$ of the classical Lorenz system, respectively. Analogously, since the do not dep determining a chaotic dynamics for the $A_1, A_2, B_1, B_2, Z_2$ variables and a trivial or periodic behavior for the $X_1, X_2, Y_1, Y_2, Z_1$ variables.

With the "classical" Lorenz parameter values $r \;\; 28$, $\;\; 10$, $b \;\; 8/3$, the variables $X_1, X_2, Y_1, Y_2, Z_1$ obviously have an erratic behavior, whereas the variables $_1, B_2, B_1, B_2, Z_2$ which describe the faster spatially varying wave components, do not feature any time variability when asymptotic dynamics is considered, as they converge to fixed values derivation) of the model discussed above, the fact that we recover the classical results of the 3-component Lorenz system is quite reinsuring. The Kaplan-Yorke dimension [24] of the system is:

$$d_{KY} \quad k \quad \frac{\overset{k}{\phantom{x}}_{j\ 1} \ {}_{j}}{\left|\ {}_{k\ 1}\right|} \quad 4.183 \tag{6}$$

where $k$ is such that the sum of the first $k$ (4, in our case) Lyapunov exponents is positive and the sum of the fi is beyond the scopes of this paper, as we confine ourselves to studying the properties of the system when chaotic motion is realized, thus focusing on the $Ec \quad 0$ limit. We consider $Ec \quad 0,0.02$ .

In physical terms, the coupling allows for a mixing of the phases of the thermal and streamfunction waves: after a $O \quad ^{1}$ time the system “realizes” that the symmetry (4-5) is broken, so that the previously described degeneracies are destroyed and ergodicity is established in the system. Note that, since also in the case of $Ec \quad 0$ the convection does not preferentially act on waves of a specific phase, we have that, pairwise, the statistical properties of and , and of and are identical and do not depend on the initial conditions. The same applies for the and

highlights on these variables, in Fig. 2 we show, from top to bottom, the projection on $A_1$ of three typical trajectories for $Ec \quad 10^{3}$, $Ec \quad 10^{4}$, and $Ec \quad 10^{5}$, respectively. Going from the top via the middle to the bottom panel, the time scale increases by a factor of 10 and 100, respectively. The striking geometric similarity underlines that the time scales of the dominating variability for the slow variables can be estimated as mixing requires $Ec^{1}$ time units. The same considerations apply for the $Y_1$ and $Y_2$ pair of variables.

Further insight can be obtained by looking at the spectral properties of the fast and slow variables; some relevant examples of power spectra for $Ec\ \ 10^{3}$, $Ec\ \ 10^{4}$ and $Ec\ \ 10^{5}$ are depicted in Fig. 4, where we have considered the low frequency range by concentrating on time scales larger than $10/\ _1$. The white noise nature of the fast variable is apparent and so is the overall lack of sensitivity of its spectral properties with respect to $Ec$. Note that any signature of the amplitude modulation, which is responsible for phase mixing observed in Fig. 3, is virtually absent. This shows how crucial physical processes are masked, due to their weakness, when using a specific *metric*

$m,n = 1,1$ and $m,n = 2,2$ , and the imaginary part of the x-symmetric modes $\Theta_{0,2}$ and $\Theta_{0,4}$ . As modes characterized by a faster varying spatial structure and different parity are added to the Lorenz spectral truncation, which describes the dynamics of the real part of $\Psi_{1,1}$ and of the imaginary part of $\Theta_{1,1}$ and $\Theta_{0,2}$ only, the system we consider in this work represents the thermal impact of viscous dissipation, controlled by the Eckert number *Ec*. The presence of a forth parameter - together with the usual Lorenz parameters *r* (relative Rayleigh number), $\sigma$ (Prandl number), and *b* (geometric factor) – marks a crucial difference between this ODEs system and other extensions of the Lorenz system proposed in the literature - see, *e.g.*, [4-6]. Moreover, as can be deduced following the argumentations of Nicolis [18], this system specifically allows for the closed-form computation of the entropy production, as the Eckert number enters into its evaluation. Up to our knowledge, this is the first ODE system obtained as a truncation of the convective equations presented in the scientific literature featuring all these additional properties.

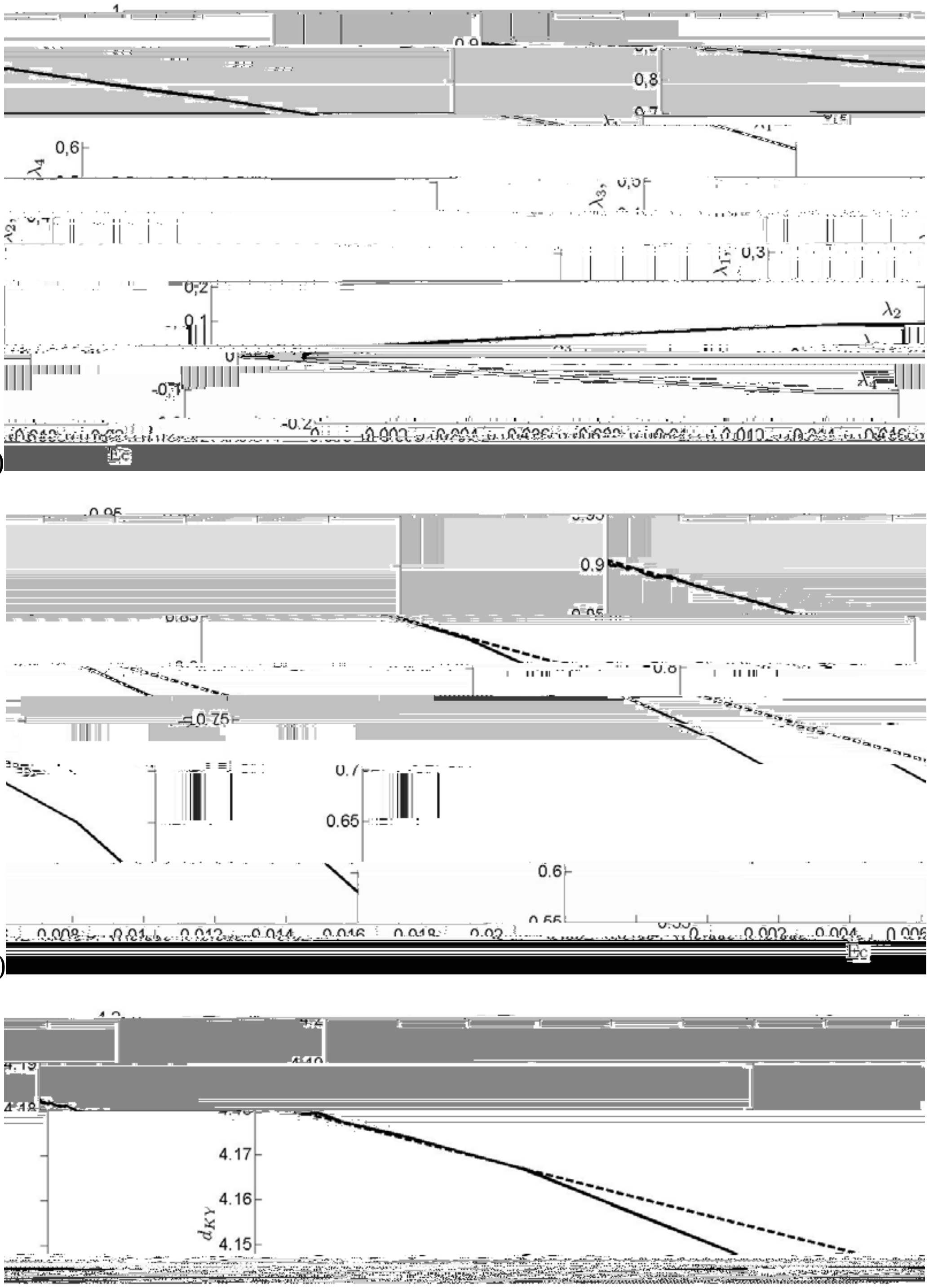
exponent coincides with that of the Lorenz system, and the value of one of the six negative exponents agrees with that of the negative Lyapunov exponent of the Lorenz system. Therefore, the Lorenz system contains already all the interesting unstable dynamics described by this extended ODEs system, and features exactly the same value for the metric entropy. Correspondingly, while the five variables (*fast*) describing the modes $\Psi_{1,1}$, $\Theta_{1,1}$, and $\Theta_{0,2}$ have an erratic behavior, the other five variables (*slow*) converge to fixed values.

2. When $Ec \quad 0$, the symmetry of the system is broken, and coupling occurs between the fast and slow variable over a time scale $O\,Ec^{\;1}$. This is clarified by adopting standard multiscale formalism. If we select, as in the original Lorenz system, $r = 28$, $\sigma = 10$, $b = 8/3$, the system is chaotic for $0 < Ec \quad 0.045$, whereas for higher values of $Ec$ a quasi-periodic regime is realized. In the chaotic regime, the symmetry-break is accompanied by the establishment 12.bg9.o.7976lP ⁄MCID ol2.0dyineShape3nv /.9450 13.0cs: two Lyapui7.76lP ⁄MCII

modulation occurring on time scales of $Ec^{-1}$ which superimposes on the fast dynamics controlled by the $O(1)$ time scale $1/\lambda_1$.

The system introduced in this paper features very rich dynamics and, therefore, may have prototypical value for phenomena generic to complex systems, such as the interaction between slow and fast variables and the presence of long wee2ls40 1.900.81 24 16.98 re8.15547 699.412.0108 0 0

2. extension of the present analysis to higher order truncation ODEs systems, with detailed investigation of the $Ec = 0$ invariance properties, of the impact on these symmetries resulting from setting $Ec > 0$, and the ensuing multiscale analysis. This is especially relevant in the context of the results presented by Franceschini and Tebaldi [26] and Franceschini et al. [27], who emphasized that spectral truncation and modes selections procedures have to be critically addressed.
3. definition, following [7], of the minimal truncated 3D model of convection able to represent the thermal impact of viscous dissipation;



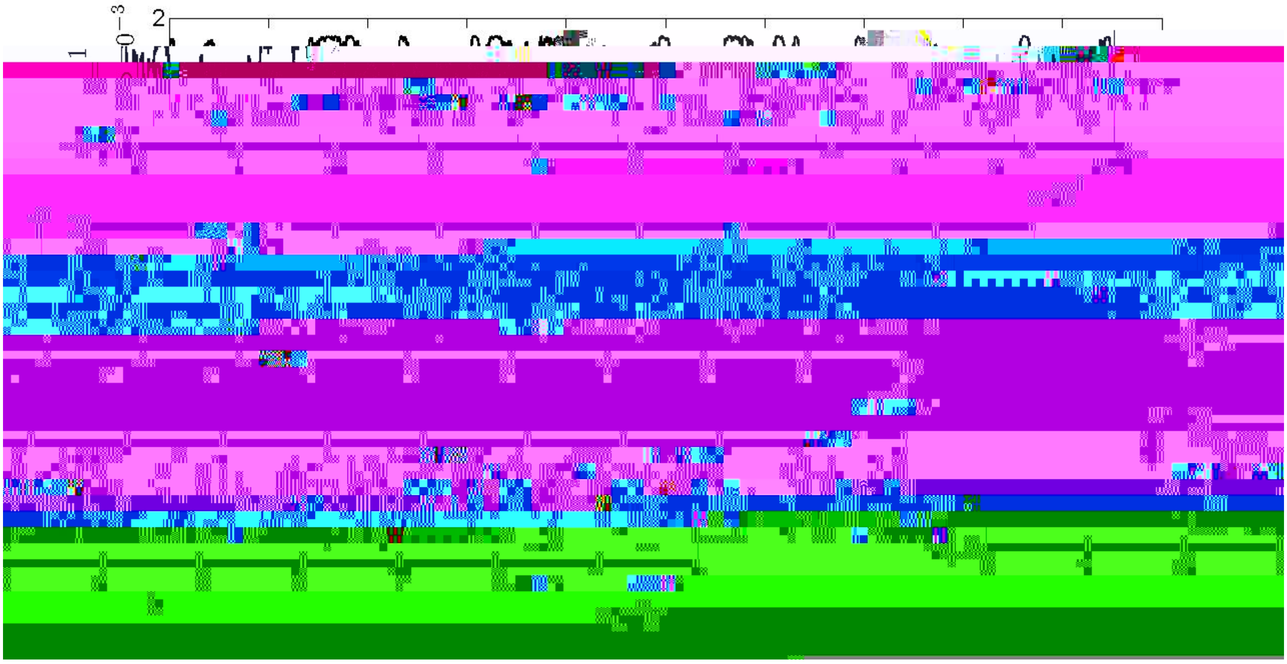
(a)

(b)

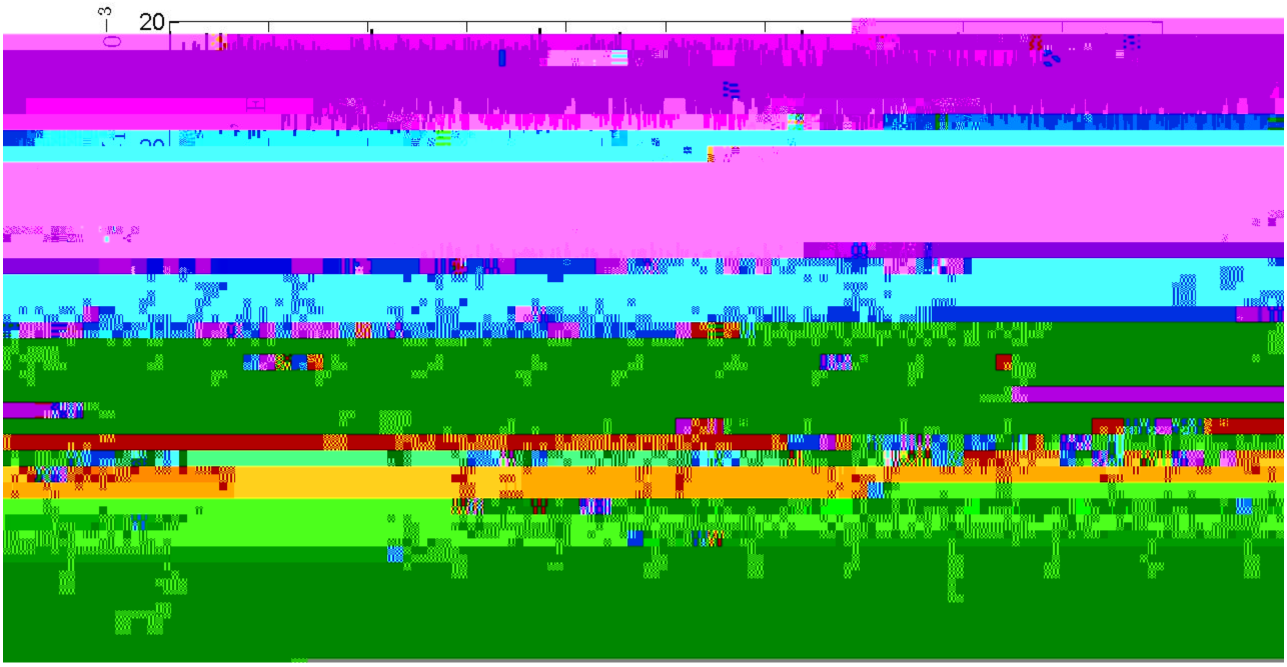
(c)

$h$ $d_{KY}$

$\approx$



*A*



*X*


***A*** ***X***

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