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Department of Mathematics

Finite element modelling of multi-asset barrier options

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Abstract

The main objective of this dissertation is to present a finite element method to compute the price of knock-out barrier options that are depended on the price of two underlying assets. The price evolution of the assets is assumed to follow a geometric Brownian motion and priced by using the Black-Scholes model. The value of the option is formulated within the framework of the Nobel Prize work of Robert C. Merton, Fischer Black and Myron Scholes.

The partial differential equation form of the Black-Scholes model is discretized using a ${P_1}^{NC}$ finite element method and the numerical result is presented using the finite element mesh generator program called Gmsh.

Declaration

I confirm that this is my own work, and the use of all material from other sources has been properly and fully acknowledged.

Acknowledgments

I would like to thank my supervisor Dr. Emmanuel Hanert for his supervision, time and patience during the course of this dissertation. I would also like to thank my colleagues on the MSc course and the other members of the academic staff in the Department of mathematics at the University of Reading.

Finally I would especially like to thank my family and friends for their love, support and encouragement.

Notations

The following notations are used throughout the dissertation:

α - basket constant
c - scaling parameter
i - underlying asset (where i = 1,2 to denote each of the asset)
t - current time
K - strike price
V

Contents

Chapter 1

2. Currencies: For example the value of $100 in pound Stirling.

3. Shares in a company: The value of shares would reflect how well the company is doing. If the company is successful, the value of its share will rise, on the other hand if the company is not doing so well, then the value of the share will drop.

2.3 Derivative

In mathematics, the term *derivative* means an instantaneous change of a quantity with respect to some variable. In finance, *derivative* is an instrument whose value is derived from, or is depended on the value of some underlying asset.

As an example, suppose an investor wishes to buy 100 shares in a company in three months time. The current price (also known as the spot price) is £1 per share. Suppose that the price of the share will increase within the next three month. Clearly, the investor would not wish to buy the share for a higher price then the price it was before. There are three choices in which the investor can keep the price at £1 per share.

1. Buy the share immediately, by paying the spot price.

2. Make an agreement with the company to buy the share at a pre-agreed point in the future for a pre-agreed price. With this agreement, the investor will be obliged to buy the share at that date. This is called a forward contract.

3. Make an agreement with the company to have the right but not the obligation to buy the share at a pre-agreed point in the future. This is called an option.

Choices 2 and 3 are financial derivatives because the price of each contract is depended upon the values of the underlying assets.

2.4 Options

An option is a contract or agreement which would give the holder the **right** but not the **obligation** to buy or sell a specified asset at a fixed price (strike price, denoted by K) up to a fixed period of time (exercise date T). The exercise date is the date at which the option expires.

Since the options gives the buyer a **right** and the seller an **obligation**, the buyer will paid an option premium V, to the seller (writer) for the privilege of purchasing and holding the option. The premium (cost of purchasing an option) of the option is agreed between the buyer and seller of the option. Options have become popular in the financial world, for the following reasons:

1. Options are cost efficient

   For example, an investor may want to purchase 100 shares of a stock with spot price at £150. In total, the investor will have to pay £15,000 for all of these shares. Options provide the opportunity for the investor to purchase the same amount of share exposure but at a much reduced price. The investor can use the options market to choose an option which would mimic and recreate the situation of the stock closely.

   Suppose there is such an opportunity such that the investor can purchase a call option for £50 with a strike price at £50 for each of the 100 shares. The investor would only paid £5000 in total (representing 100 shares). If at the exercise date, the investor would like to exercise their rights to buy all the share at the strike price of £50, then the investor only needed to paid £10,000 (option price + strike price of 100 shares) rather then the £15,000 paid for direct investment.

2. Higher Potential Gain

   The potential gain in using options can be much higher then the potential gain with the usual investment in stocks. This is known as the leverage effect. But a consequence of the leverage effect will be the increase risk of losing all investment. Therefore for options, risks become more important.

3. More flexibility

   Options offer more variety of investment alternatives. Options can be used to recreate many different situations.

4. Opportunity for hedging and speculations

   Hedging is an investment technique that is use with the aim of cancelling or reducing the risk of another investment. Options allow investors to protect their position against price fluctuation and minimised the lost caused by unwanted risks. Speculation involves the trading of any financial assets in an attempt to profit from any price fluctuations. Options are popular with investors because it allowed the opportunity for greater potential gain (but at the risk of magnifying the loss).

5. Systematic method for pricing options

   The price of options can be computed by using the well-known Black-Scholes Model (more details on this model later). Therefore options can be traded with some confidence.

The value S, of a stock is driven by supply and demand. According to Higham (2004) and Hull (2006), the value of an option is influenced by the following five principal factors:

1. The strike price K.
2. The price of the underlying asset S_i in relation to the strike price.
3. The cumulative cost to hold a position in the security. This would include interests and dividends.
4. The time to expiry of the option given by t_e.
5.

The buyer of a call option expects the price of the underlying asset to rise by the exercise date. The seller will received the premium, and will be oblige to sell the asset at the strike price, should the buyer exercise the option to do so. Figure 1 shows the payoff diagram when buying a call option, as viewed by the buyer.

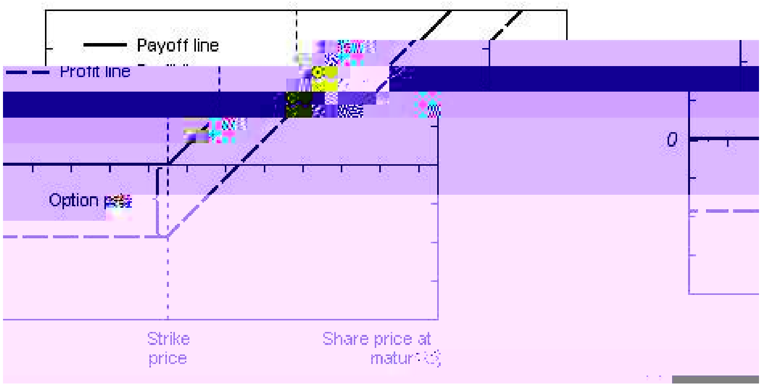

Figure 1 - Payoff Diagram for buying a European call option (source: http://en.wikipedia.org/wiki/Call option)

The buyer of the call option will make the most profit when the value of the underlying asset is increasing and exceed the strike price plus the price paid for the option premium. To illustrate the idea, a simple example is given below:

A simple example of a European call option on a stock

Suppose the price of a stock in a company is currently £40. An investor expects the stock price to rise in the future. The investor buys a call option with the strike price set at £40 with the exercise date 15th November 2007. For this right, the investor will paid the company a premium of £10 for this call option. Now consider the following two scenarios:

1. Stock price rises **above** the strike price (£40)

   Suppose the stock price rises to £60 on the exercise date. The investor will exercise the option to buy the stock for £40. When the stock is purchase, the investor can either keep the stock or sell the stock on for £60. By selling the stock, the investor

The investor can theoretically make unlimited profit. Profit is only made when $S_i > K + V$. This is represented by the profit line in figure 1. The lost to the investor will be limited to the price of the premium initially paid for the call option. In the view of the seller "writer" of the call option, he or she will expect the price of the stock to not rise. Figure 2 shows a graphical interpretation when selling a call option, as viewed by the writer.

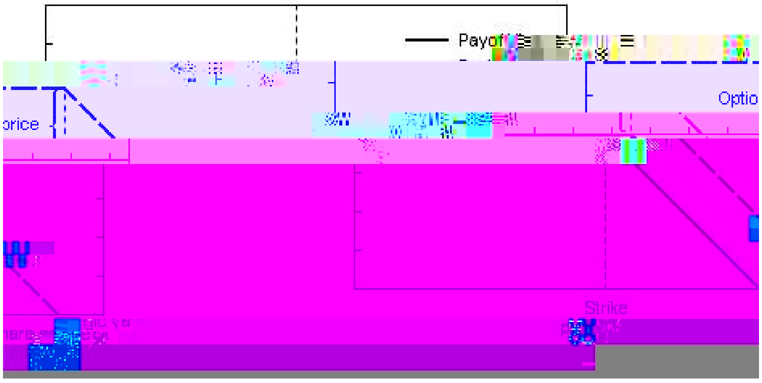

Figure 2 - Payoff Diagram for writing a European call option (source: http://en.wikipedia.org/wiki/Call option)

Now consider the following two scenarios:

1. Stock price rises **above** the strike price

   The writer of the option will make a profit as long as the price of the stock does not exceed the strike price plus the premium received. After that, the writer could theoretically suffer unlimited losses.

2. Stock price stay **below** P717q1strike

2.5.2 Put Option

A put option gives its holder the right (but not the obligation) to **sell** an agreed quantity of a prescribed asset at the strike price at the exercise date. The writer of the option is **obliged** to **purchase** the prescribed asset at the strike price from the holder, should the holder decide to sell. The holder will paid the writer the option premium for the privilege of holding the option.

The buyer of a put option expects the price of the underlying asset to fall by the time of the exercise date. Another reason would be that the buyer wants to protect the price of the asset (generally term a protective put strategy). Figure 3 shows the payoff diagram when buying a put option, as viewed by the buyer.

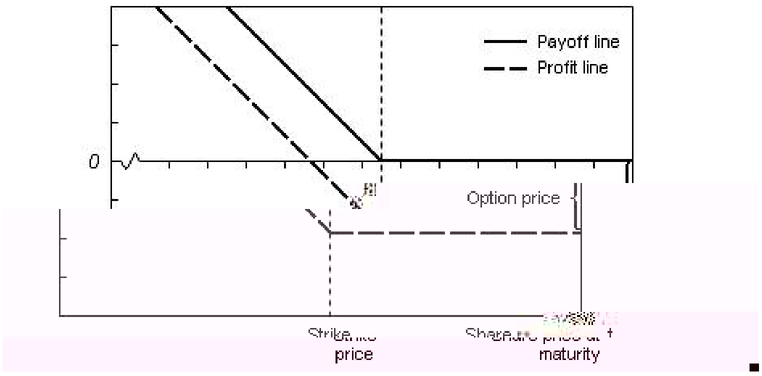

Figure 3 - Payoff Diagram for buying a European put option (source: http://en.wikipedia.org/wiki/Put option)

The buyer of the put option will make the most profit when the value of the underlying asset is decreasing. Therefore a lower stock price means a higher profit. To illustrate the idea, a simple example is given below:

A simple example of a European put option on a stock

Suppose the price of a stock in a company is currently £60. An investor expects the stock price to drop in the future.

The investor buys a put option with the strike price set at £50 with the exercise date 15th November 2007 from a put writer. For this right, the investor will paid the put writer a premium of £10 for this put option. Now consider the following two scenarios:

1. Stock price drops **below** the strike price

   Suppose the stock price drops to £30 on the exercise date. The investor will purchase the stock for £30, and then exercise the put option to sell the stock for

£50 to the put writer. By selling the stock, the investor will have made a profit of £20. The net profit will now be £10, when the cost of the premium of £10 is subtracted.

2. Stock price stay on or **above** the strike price

   Suppose the stock price never drop to £50. The investor will clearly not buy the stock for more then £50 and sell it to the put writer for £50. Therefore the option is not exercised and would expire worthless. In this scenario, the total loss for the holder is limited to the cost of the option premium of £10.

For the put holder, profit is only made when $S_i < K + V$. This is represented by the profit line in figure 3. In view of the put writer, profit is maximised when the price of the underlying asset exceeds the strike price. Figure 4 shows the payo diagram when buying a put option, as viewed by the writer.

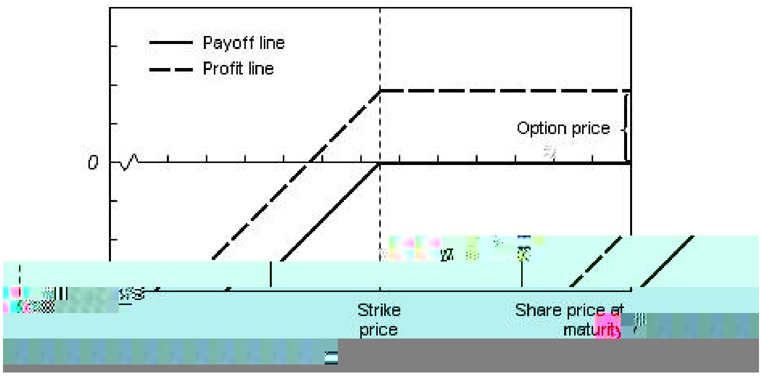

Figure 4 - Payo Diagram for writing a European put option (source: http://en.wikipedia.org/wiki/Put option)

A Summary of a Put Option

Let $P(S_i, T)$ denote the value a standard European put option, with strike price K and exercise date T. Also let S denotes the current value of the underlying asset and t the current time. At the expiry date T, if $K > S_i(T)$ the option holder will buy the asset at

2.6 Exotic Options

Exotic options are alternatives to Vanilla options (see Higham 2004). Exotic options are worthless. After reaching the knockout barrier, any value for S_i will be ignored and the option ceases to exist.

Barriers are usually observed at some discrete barrier observation dates. For example the barrier can be applied for one day every week.

Barrier Shape

According to Pooley et al (2000), for problems with one underlying asset, barriers are typically 'points'. For problems with two underlying assets, the barriers can be any shape

2.7.1 Portfolio

The term portfolio is usually used to describe a collection of investments held by a financial organisation or a private individual. Portfolio may consist of the following combinations:

- assets
- options
- cash invested in a bank

2.7.2 Volatility

Volatility is a measure of the risk and uncertainty of future price movements of an asset. For example, the volatility of a stock price is a measurement of the risk and fluctuation concerning future stock price movements. An asset with a high volatility will be more likely to increase or decrease its value, then an asset with a low volatility. Large volatility arbitrage market is where the market offers no opportunities for arbitrage. Arbitrage will occur in practice, but will not remain in existence for very long as many arbitrageurs will make use of computer systems to take advantage of any arbitrage opportunities that may arise. Due to more buying pressures, this will push up the lower price and due to selling pressure, will lead to a reduction of the higher price. Therefore any price differences will be soon be eliminated.

2.7.4 Correlation

In the financial world, correlation is used to show how two assets moves in relation to

Chapter 3

Black-Scholes Equation

An important model use for pricing European call and put options on stocks is the Black-Scholes Model. In this chapter the background, the derivation and the key ideas of the Black-Scholes model are explained in more details. Many of the descriptions in this chapter are taken from the books by Chriss (1996) and Hull (2006).

3.1 Black-Scholes Model

The Black-Scholes model is a well-known and popular model use to calculate the value of a European option. Ever since its development in 1973 by Fischer Black and Myron Scholes, the model still remains one of the most preferred models and provides the basis of options theory. To compute the value of an option, the model requires the following information for the problem in consideration:

1. The strike price of the option, K
2. The price of the underlying assets, S_i

- There are no transaction costs in trading
- Underlying asset can be traded continuously
- There are no dividends during the life of the derivative
- Arbitrage free market, therefore no arbitrage opportunities
- Short selling of the asset is permitted

The Black Scholes model can be derived from the Black-Scholes partial differential equation (PDE). The PDE provides the framework to compute a fair price for options.

3.2 Deriving the PDE form of the Black-Scholes Model

In order to value an option, a mathematical description of how the underlying asset behaves must be developed. The price of the asset is assumed to follow a stochastic process. This means that the price of the asset will change randomly over time. An example of a stochastic process is the Markov Process. In this process the past history of the asset will be ignored and consider irrelevant. Therefore predictions for the future price will be unaffected by any past price of the asset, as the behaviour of the asset over a short period of time depends only on the current value of the asset.

The asset price is usually assumed to follow a Wiener process, which is a more specific type of Markov process. The Wiener process is a stochastic process where the change in a variable over a short period of time Δt has a normal distribution with zero mean and unit variance. An Itō process is a generalised form of the Wiener process where the random fluctuation is following a normal distribution. For further theory and results regarding Markov, Wiener and Itō processes, we refer to the book by Hall (2006).

The Wiener process is also called Brownian motion. The geometric Brownian model originated in the study of a physical model for the motion of heavy particles suspended in a medium of lighter particles. In Brownian motion, the faster lighter particles will randomly collide with the heavier larger particles, with each collision observed to be random and independent. According to Chriss (1996), for a longer period of time, the particle displacement will be normally distributed, where the mean and standard deviation depends only on the amount of time that has passed. The geometric Brownian motion model can be used to describe the probability distribution of the future value of the stock. In his work, Osborne (1964) showed that the movement of stock prices shared many similar characteristics with the movement of molecules in the Brownian motion model. The derivation of the PDE form of the Black-Scholes Model for one underlying asset is shown in the next section. The same idea is use to derive the PDE form for two underlying assets.

3.3 Deriving for one underlying asset

Using the assumption that the price of the underlying asset follows a geometric Brownian motion, will give the expression

$$\frac{dS}{S} = \mu dt + \sigma dW$$

$$dS = S\mu dt + S\sigma dW \tag{3.1}$$

where S denotes the underlying asset, μ is the drift term (the average rate of increase per unit time of the asset), σ is the volatility of the stock, and dW is a random term with a Wiener process distribution (dW has zero mean and unit variance). The drift term causes the underlying assets to move in a certain direction (see Pooley et al (2000)).

Equation (3.1) also follows the Itō process. This process was name after the discoverer, Kiyoshi Itō. An important result from the Itō process is the Itō's Lemma. This lemma is used to find the differential of a function that follows a stochastic process and plays a very important role in the pricing of derivative. The informal proof of this lemma is shown in Hull (2006). Itō's Lemma is stated as follows:

Suppose a variable x follows the Itō process. Then dx is given by

$$dx = a(x,t)dt + b(x,t)dW \tag{3.2}$$

Now consider a function *G(x,t)*, which is some function that is at least two times differentiable. Then the function *G(x(t),t)* would also follow the Itō's process. Therefore for a function *G(x(t),t)* we have

$$dG(x(t),t) = \left(\frac{\partial G}{\partial t} + a(x,t)\frac{\partial G}{\partial x} + \frac{1}{2}b(x,t)^2\frac{\partial^2 G}{\partial x^2}\right)dt + b(x,t)\frac{\partial G}{\partial x}dW \tag{3.3}$$

The equation given by (3.3) is the specialisation of Itō's Lemma. Now the stock price follows the process given by (3.1). This is similar to equation (3.2), with $a(S,t) = S\mu$ and $b(S,t) = S\sigma$ respectively.

Now let *V(S,t)* denote the value of some particular option with asset of price S and for some time t, where $t \leq T$ (expiration date of the option). Applying the Itō's Lemma to *V(S,t)*, will gave

$$dV(S(t),t) = \left(\frac{\partial V}{\partial t} + a(S,t)\frac{\partial V}{\partial S} + \frac{1}{2}b(S,t)^2\frac{\partial^2 V}{\partial S^2}\right)dt + b(S,t)\frac{\partial V}{\partial S}dW \tag{3.4}$$

Now applying $a(S, t) = S\mu$ and $b(S, t) = S\sigma$ from equation (3.1) to equation (3.4) gives

$$dV(S(t), t) = \left(\frac{\partial V}{\partial t} + (S\mu)\frac{\partial V}{\partial S} + \frac{1}{2}(S\sigma)^2\frac{\partial^2 V}{\partial S^2}\right) dt + (S\sigma)\frac{\partial V}{\partial S} dW \tag{3.5}$$

Now consider a portfolio Π composing of a long option and a short portion of the where r is the risk-free interest rate.

Now by substituting (3.6) and (3.9) into (3.10), we have

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}(S\sigma)^2\frac{\partial^2 V}{\partial S^2}\right) dt = r\left(V - \frac{\partial V}{\partial S}S\right) dt \tag{3.11}$$

Dividing both sides of (3.11) by dt, and rearranging gives

$$\frac{\partial V}{\partial t} + \frac{1}{2}(S\sigma)^2\frac{\partial^2 V}{\partial S^2} + r\frac{\partial V}{\partial S}S - rV = 0 \tag{3.12}$$

The equation given by (3.12) is the Black-Scholes partial differential equation (PDE) for the option price V for one underlying asset. This equation can be used to compute the price of a European option with one underlying asset. For a European option, only the final price of the option at expiration is known. This implies that the Black-Scholes PDE must be solved backward in time to find the initial price of the option. In order to achieve this, it would be necessary to replace the time t by τ using the expression

$$\tau = T - t \tag{3.13}$$

where τ denotes the backward time point of the option.

3.4 Black-Scholes Model PDE for Two Asset Barrier Option

The PDE form of the Black-Scholes PDE for a European option for two underlying assets S_1,S_2 with a knock-out barrier can be expresses as

$$\frac{\partial V}{\partial \tau} - r\sum_{k=1}^{2} S_k\frac{\partial V}{\partial S_k} = \sum_{kl=1}^{2} D_{kl}(t, S_1, S_2)\frac{S_kS_l}{2}\frac{\partial^2 V}{\partial S_k\partial S_l} - \lambda 1_{I\times(\Re^2\setminus\Omega_b)}V - rV \tag{3.14}$$

where V denotes the price of the option, r the risk free interest rate, and λ is some given (large) constant used to set the option price to zero when the barrier is applied.

Equation (3.14) is a two (

$$\begin{pmatrix} \sigma_{11}^2(t, S_1, S_2) & \rho\sigma_{11}(t, S_1, S_2)\sigma_{22}(t, S_1, S_2) \\ \rho\sigma_{11}(t, S_1, S_2)\sigma_{22}(t, S_1, S_2) & \sigma_{22}^2(t, S_1, S_2) \end{pmatrix} \tag{3.15}$$

This matrix depends on the volatilities of the two assets, σ_{11} , σ_{22} and their correlation ρ.

The term $\lambda 1_{I\times(\Re^2\setminus\Omega_b)}V$ in equation (3.14) represents the barrier of the option which is applied at some discrete time intervals. Inside the barrier λ is equal to zero, and outside the barrier the value of λ is equal to 1.

Chapter 4

Discretization and solving the Black-Scholes Equation

In general, the Black-Scholes PDE cannot be solved analytically for exotic options (e.g. Barrier Options). Therefore numerical methods are use to compute the numerical solutions to the PDE equation given by (3.14). This dissertation will use a finite element method to compute the numerical solution to the Black-Scholes PDE (3.14).

4.1 Introduction to Finite Element Method

What is the finite element method (FEM)?

The finite element method is a numerical method that is generally used to numerically solve for the solution of partial differential equations.

Advantages of using FEM for pricing options

When pricing options, the FEM has several advantages over other numerical methods, for example finite difference (FD) methods.

1. Irregular and complex shapes caused by barriers can be more accurately represented by unstructured mesh used by FEM. For structured mesh, it is harder to set the grid points to deal with the complex shapes.

2. FD requires a higher resolution across the domain, and therefore will take longer to compute the numerical solutions. FEM only have high resolution in the domain of interest, such as near the barrier. Away from the barrier, a lower resolution is used.

3. It is harder to incorporate the boundary conditions using FD than by using FEM (see Topper (2000)). Neumann boundary conditions can be naturally incorporated in the FEM formulation.

By integrating by parts both the advection and the diffusion terms, we obtain

$$\sum_{e} \int_{\Omega e}$$

The Galerkin formulation is obtained by replacing V by V^h and $\hat{V}$ by ψ_i in equation (4.6).

$$\sum_e \int_{\Omega e} \frac{\partial V^h}{\partial t}\psi_i d\Omega + \sum_e \int_{\partial\Omega e} \mathbf{a}\cdot\mathbf{n}V^h\psi_i d\Gamma - \sum_e \int_{\Omega e} V^h\nabla\cdot(\mathbf{a}\psi_i)d\Omega$$

$$= \sum_e \int_{\partial\Omega e} D\psi_i\nabla V^h\cdot\mathbf{n}d\Gamma - \sum_e \int_{\Omega e} D\nabla V^h\cdot\nabla\psi_i d\Omega \tag{4.8}$$

where $\sum_e \int_{\partial\Omega e} \mathbf{a}\cdot\mathbf{n}V^h\psi_i d\Gamma$ is the advective flux and $\sum_e \int_{\partial\Omega e} D\nabla\cdot\mathbf{n}V^h\psi_i d\Gamma$ is the diffusive flux. When the P_1^{NC} scheme is used, then the diffusion flux is equal to zero (Hanert et al (2004)). The advective flux is computed in an upwind fashion.

For time integration, a 3rd order Adams-Bashforth scheme is used to solve the Black-Scholes PDE (3.14). The Adams-Bashforth scheme of order 3 can be written as

$$V^{n+1} = V^n + \Delta t\left(\frac{23}{12}F^n - \frac{16}{12}F^{n-1} + \frac{5}{12}F^{n-2}\right) \tag{4.9}$$

where $\frac{\partial V}{\partial t} = F(V,t)$

4.2.2 Barrier Shape

The shape of the barrier is determined by the problem in consideration. For problems with two underlying asset, the barrier can be represented by any shape in the 2D plane according to Pooley et al (2000). The movements of the asset prices would be affected by diffusion. Diffusion itself is caused by the volatilities of assets S_1 and S_2. If $\sigma_{11} = \sigma_{22}$ then the diffusion would have an annular shape. The annular barrier use in this dissertation is given by

$$\Omega = \left\{K_1 < \sqrt{\sigma_1{}^2 + \sigma_2{}^2} < K_2\right\} \tag{4.10}$$

which represents an annular barrier with inner and outer radii equal to the strike price of the assets given by K_1 and K_2 respectively. An annular barrier is used because the volatilities are identical for S_1 and S_2, as seen later in chapter 5. This barrier is shown in the figure below.

The length of the major and the minor axis is determined by major = $c\sigma_{11}$ and minor = $c\sigma_{22}$ where c is some scaling parameter. In this dissertation, the elliptical barrier is horizontal, with the centre located when the price of the assets are both £100. The shape of the barrier is shown in the figure below:

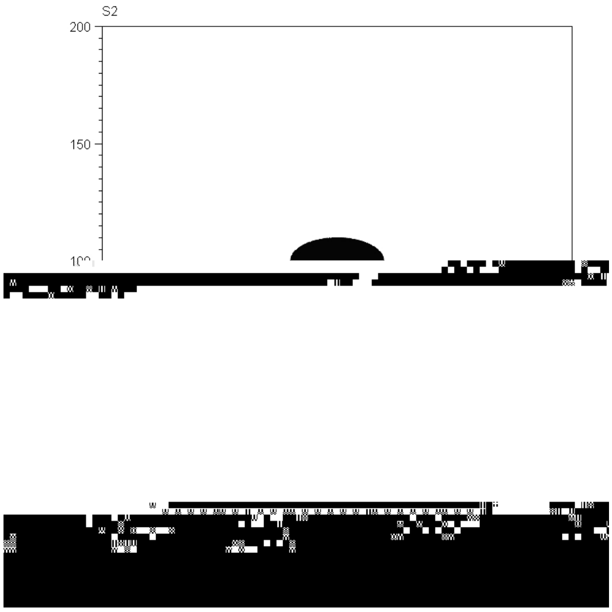

Figure 4.4 - Elliptical barrier shape applied on the basket call option

Inside the barrier (shown in black), the price of the options may have some positive values. But when the asset prices are outside the elliptical barrier (shown in white), then the option would be knock out and immediately ceases to exist. Therefore outside the barrier, the value of the option would be zero.

4.2.3 Mesh

As mentioned earlier, one of the most attractive features of using FEM is its capability to deal with irregular and complex shapes caused by the barrier with high accuracy. This can be done by using a two dimensional unstructured mesh. Options price exhibits a discontinuity near the barrier edge. Therefore a high resolution is required to "capture" this discontinuity. This is done by placing extra nodes closer to the barrier and fewer nodes away from the barrier, where the option value is zero. For an annular barrier, an example of an unstructured mesh that is used to discretized the domain to solve equation (3.14) is shown in the figure 4.5.

In figure 4.6, the unstructured mesh uses 10638 elements, with maximum resolution of £0.279404 and minimum resolution of £11.8018. The mesh has high resolution near the boundary edge to capture the discontinuity of the solutions. As can be clearly seen in the figure above, the barrier option requires fine mesh spacing near and on the barrier to ensure more accurate solutions. Fine mesh spacing is required to capture the discontinuities introduced at each barrier observation dates. Therefore extra nodes are placed closer and inside the barriers to ensure higher resolution inside the domain. Outside the barrier the option price would be zero everywhere, because when the asset price crosses outside the barrier, the option would immediately cease to exist. Therefore fewer nodes would be required outside the barrier.

Chapter 6 will compare the numerical solutions produced using the unstructured mesh shown in figure 4.6 with the numerical solutions produced using the structured mesh shown in figure 4.7 below:

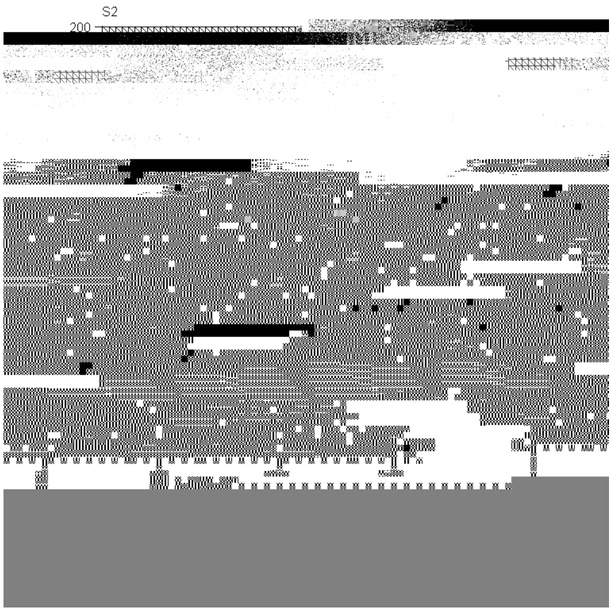

Figure 4.7 - Structured mesh use to discretized the Black-Scholes PDE with elliptical barrier imposed on the option (generated with Gmsh (http://www.geuz.org/gmsh/)

In figure 4.7, the structured mesh uses 10658 elements, with resolution of £2.739. Therefore the mesh has the same resolution all over the domain.

4.3 Computing the solutions of the Black-Scholes PDE equation

A program has been written in C++ which is used to compute the solution for pricing barrier option given by equation (3.14). This program computes the numerical value of the option at the exercise date of the option (when $t = T$) and then solves backward the Black-Scholes PDE (3.14) to compute the price of the option at the initial time (when $t = 0$). The program Gmsh is used to display the graphical output of the solutions for each type of options. The numerical solutions of pricing options are shown in chapter 5 for an annular barrier and in chapter 6 for an elliptical barrier.

Chapter 5

Numerical Solutions of Pricing Options

5.1 Neumann Boundary Condition

In the first section the options are computed using the homogeneous Neumann boundary condition given by $\underline{\nabla} V \cdot \underline{n} = 0$. All figures shown in this section is produced using the program Gmsh (source: http://www.geuz.org/gmsh/). The numerical solutions for each type of option are shown below.

5.1.1 Max Put Option

The figure at time $t = 0$ in figure 5.1 shows the numerical solution for the max put option at the start of the lifetime of the option. This is when the barrier is first applied to the max put option. As expected for a put option, the option is not exercised when the asset price for both of the asset S_1 and S_2 is more then the strike price of K = £25). This means that the option expires worthless and have the value of zero in this region. This can be clearly seen in the larger blue space. It can also be observe that when the value for both of the assets crosses the lower barrier level of £20, then the option immediately expires worthless. This is because the knock-out barrier causes the option to immediately expire worthless as soon as the value of the underlying asset crosses the barrier. The effect of the barrier can be clearly seen in the lower left corner of the figure. It can also be observed that the cost of the max put option is highest when the price of both the two underlying asset is between £17.50 and £20. This occurs very close to the lower limit of the barrier. The peak value of the option at this time is £7.21. This occurs when the values of both assets are between £15 and £17.50.

The figure at time $t = \frac{T}{3}$ in figure 5.1 shows the max put option at a third (at time $t = \frac{T}{3}$) of its lifetime. It can be seen that the peak value for the price for the option has slightly increased from £7.21 at the start of the option to the value of £7.87, at t

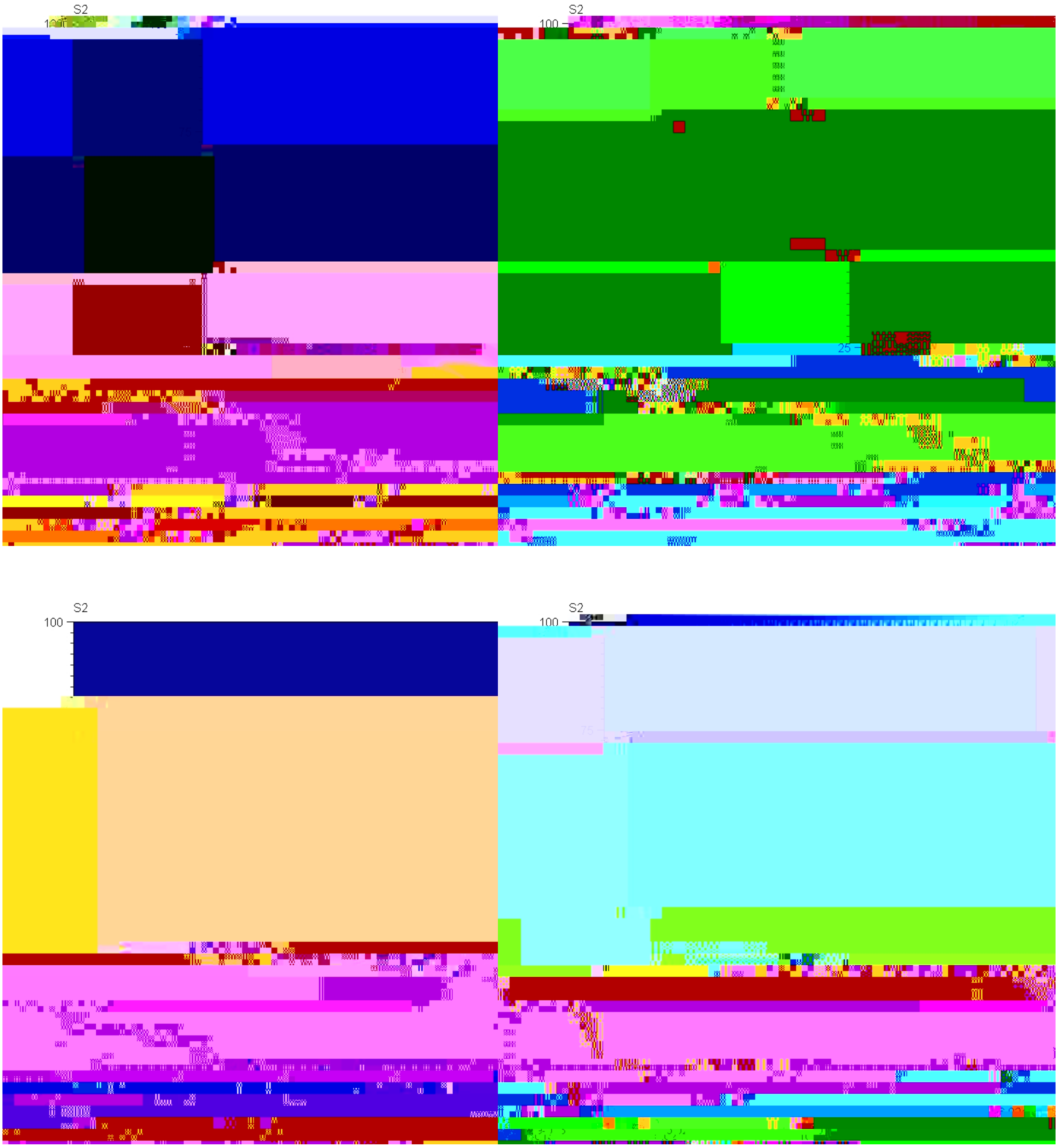

Figure 5.1: Max Put Option

are as close as possible to the lowest limit of the barrier.

5.1.2 Min Put Option

The figure at time $t = 0$ in figure 5.2 shows the numerical solution for the min put option at the start of the lifetime of the option. This is when the barrier is first applied to the min put option. The min put option is greatly affected by the barrier because when the values for the assets are below the lower limit of the barrier (£20) or above the upper limit of the barrier (£35) the option is knock out by the barrier and immediately ceases to exist. It can also be observed that the cost of the min put option is highest when the price of one of the asset is between £25 and £27.50 and the other asset is between £1 and £5. This is concentrated near the lower barrier limit as expected for a put option. The peak value of the option at this time is £23.70.

The figure at time $t = \frac{T}{3}$ in figure 5.2 shows the min put option at a third (at time $t = \frac{T}{3}$) of its lifetime. It can be seen that the peak value for the price of the option has slightly increased from £23.70 at the start of the option to the value of £24.30, at $t = \frac{T}{3}$. The location for higher values of the option remains located where the price of one of the asset is between £25 and £27.50 and the other asset has the value between £1 and £2.50.

The figure at time $t = \frac{2T}{3}$ in figure 5.2 shows the min put option at a two-third (at time $t = \frac{2T}{3}$) of its lifetime. As comparison, it can be seen that the peak value for the price for the option price has increased from £24.30. at the third of the lifetime of the option to the value of £24.60, at two-third of the lifetime of the option. This is a slight increase in the price of the option. The highest values for the min call option occur when one of the assets has values between is between £22.50 and £30 and the other asset has the value between £1 and £2.50.

The figure at time $t = T$ in figure 5.2 shows the numerical solution for the min put option on the exercise date of the option. The option price has increased from £24.60, at two-third of the lifetime of the option to the peak option price of £24.70. This is the highest price for the option in its whole duration of its lifetime. This value is located when the value of one asset is between £20 and £35, with the value of the other asset between £1 and £5. When S_1 , $S_2 >$ £25 the price for the min put option is zero because both of the assets prices $S_1, S_2 > K$, therefore nothing will be gained from exercising the option. Because the option is not exercised, the option expires worthless.

5.1.3 Basket Put Option

The figure at time $t = 0$ in figure 5.3 shows the numerical solution for the basket put option at the start of the lifetime of the option. This is when the barrier is first applied to the basket put option. The basket put option is greatly affected by the barrier because the option is exercised from £0 to £45. The location for the highest values of the basket put option occurs when the price of one of the asset is close to £25 with the other asset

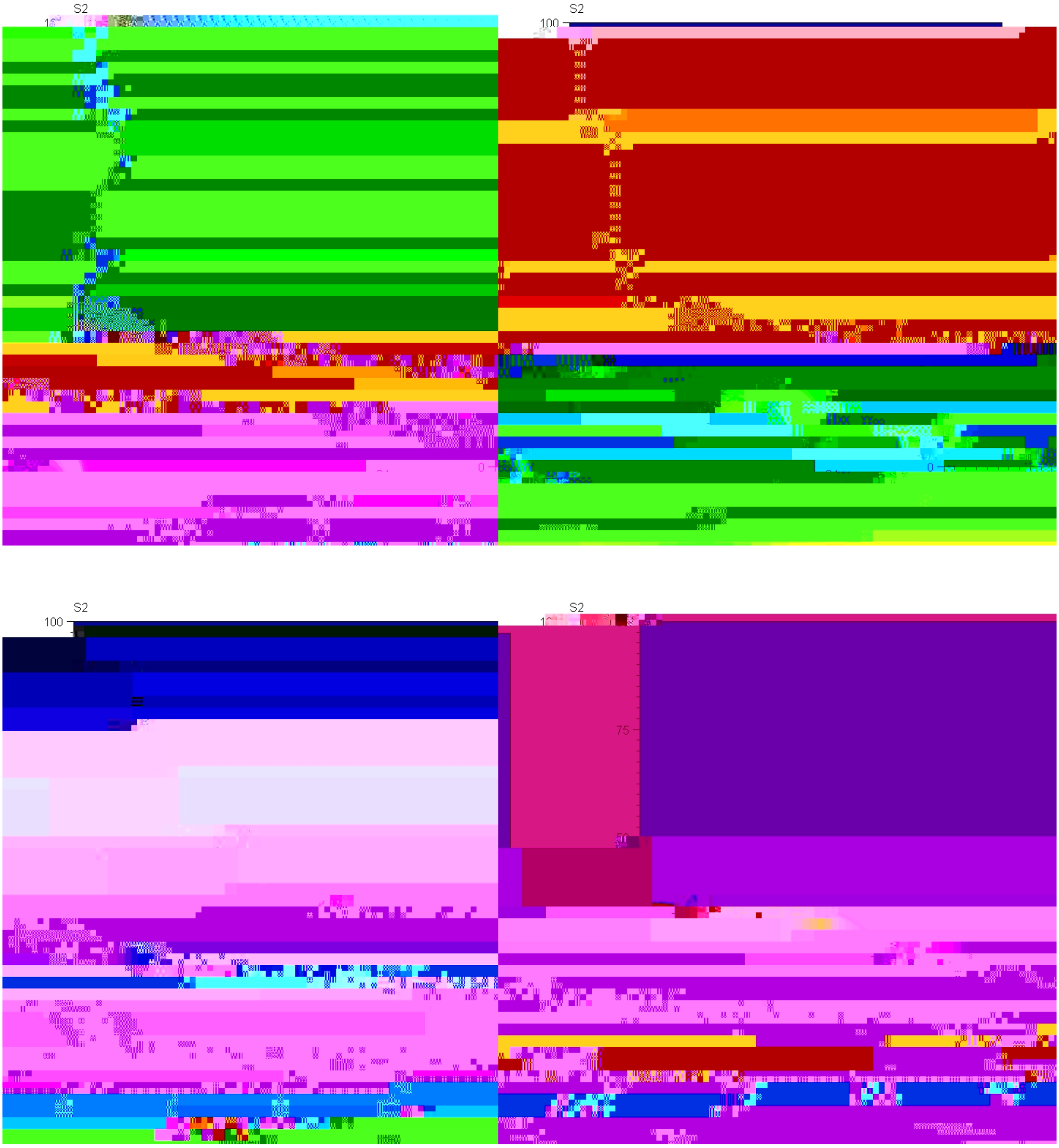

Figure 5.2: Min Put Option

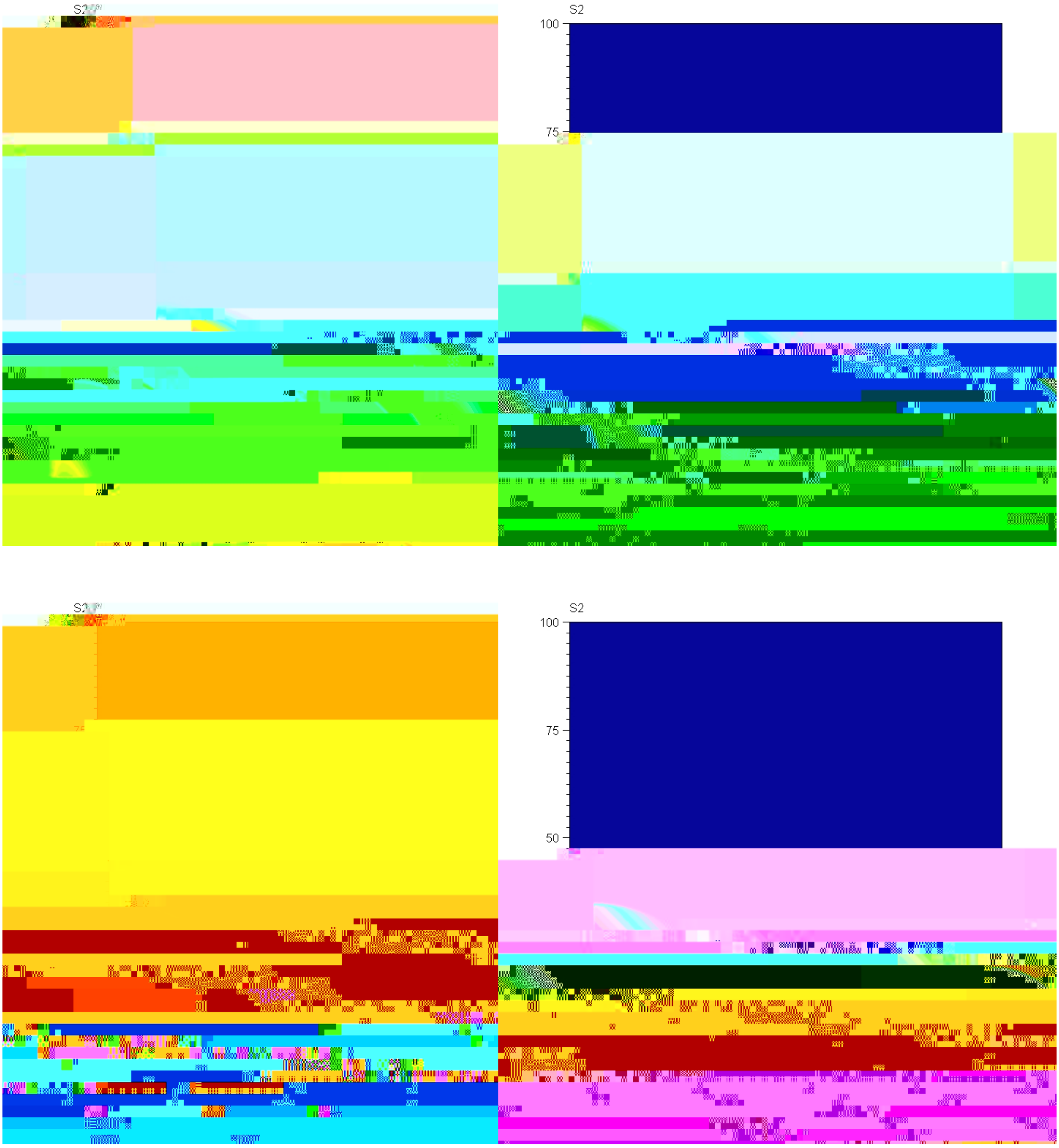

Figure 5.3: Basket Put Option

close to zero. This is concentrated closer to the lower limit of the barrier. The peak value of the option at this time is £11.30.

The figure at time $t = \frac{T}{3}$ in figure 5.3 shows the basket put option at a third (at time $t = \frac{T}{3}$) of its lifetime. Comparing the results, it can be seen that the peak value for the price of the option has slightly increased from £11.30 at the start of the option to the value of £11.90, at $t = \frac{T}{3}$. The location for higher values of the option remains located when the price of one of the asset is close to £25 with the other asset close to zero.

The figure at time $t = ^{2T}$

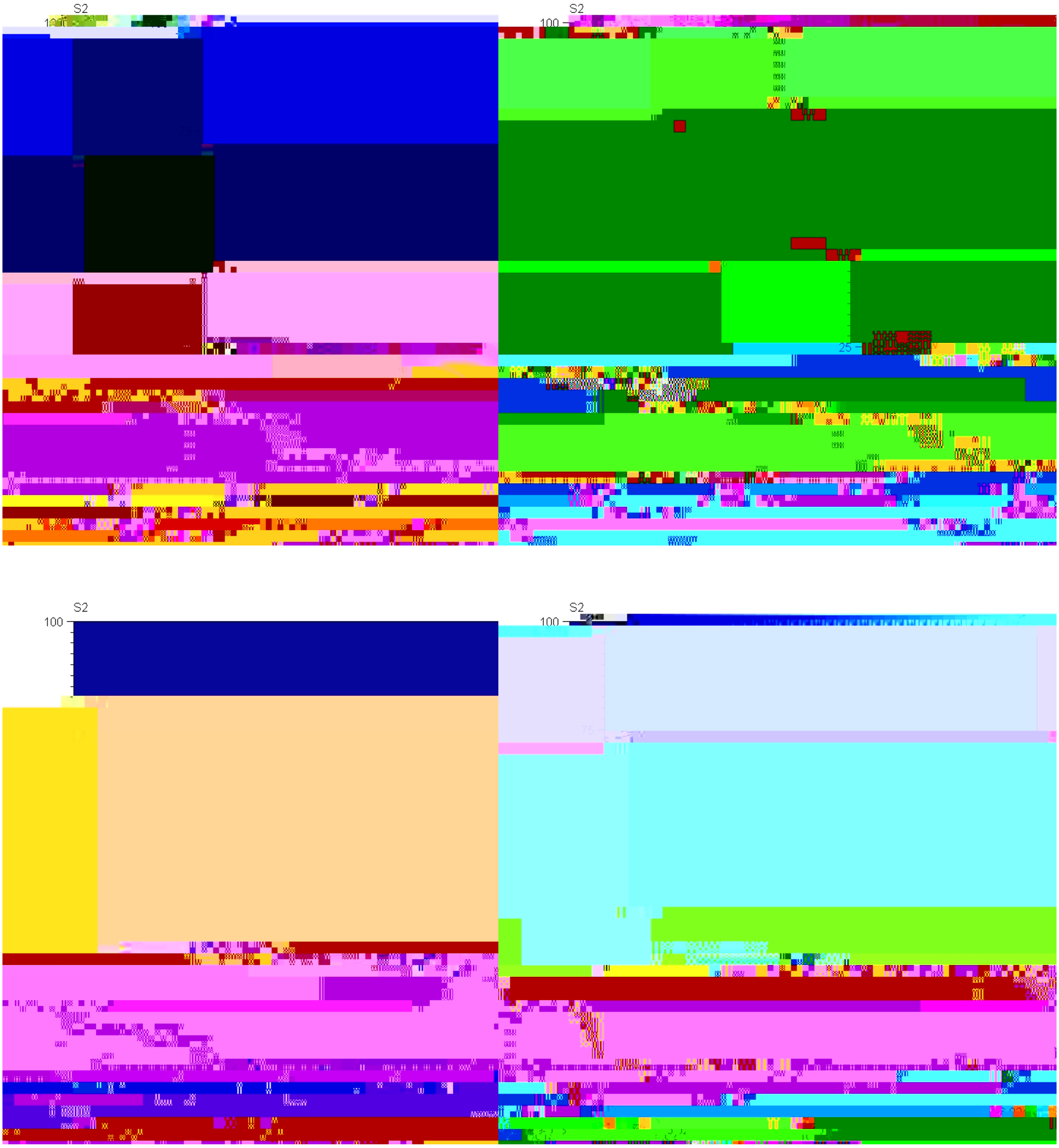

Figure 5.4: Max Put Option

boundaries is found to have no impact on the numerical values for the option during the option lifetime. As a result, the peak value for the option price is the same as the peak value produced using Neumann boundary conditions (see figure 5.1). Therefore for the max put option, there is no effect when imposing Dirichlet or Neumann boundary conditions

5.2.2 Min Put Option

The numerical results computed using Dirichlet boundary conditions shown in figure 5.5 has many similarities with the numerical results computed using Neumann boundary conditions shown in figure 5.2. For example, the highest values for the option price are located near both the S_1 and S_2 axis.

The option price on the boundary is £25 at time $t = T$. This value is imposed on the boundary for all time. But imposing Dirichlet boundary conditions on the min put option is not realistic because it implies that the option peak price remains at £25 for all time. As a result the peak option is higher then the peak option obtain using Neumann boundary conditions (see figure 5.2).

5.2.3 Basket Put Option

The numerical results computed using Dirichlet boundary conditions shown in figure 5.6 has many similarities with the numerical results computed using Neumann boundary conditions shown in figure 5.3. For example, the peak value for each figure is the same. The option price on the boundary at time $t = T$ is imposed on the boundary for all time. The effect of imposing Dirichlet boundary conditions has no impact on the numerical solutions of the basket put option. The peak option value for $t = 0$, $t = \frac{T}{3}$, $t = \frac{2T}{3}$ and $t = T$ are £11.30, £11.90, £12.60 and £14.20 respectively.

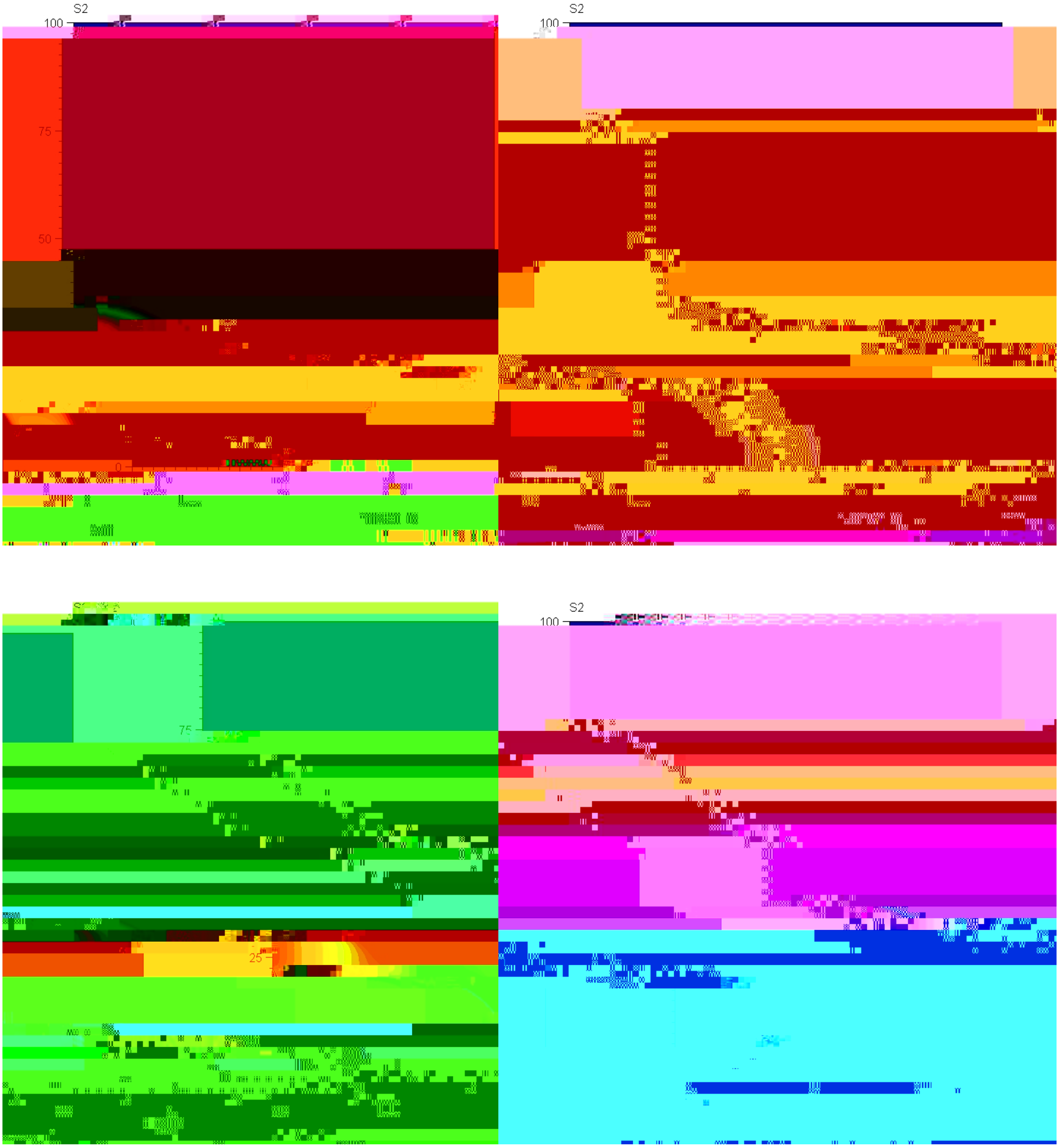

Figure 5.5: Min Put Option

Figure 5.6: Basket Put Option

5.3 Summary

It is clearly shown that the peak option price is lowest at the start of every option. This is because there would be a higher risk that the price of the assets will be more likely to change by the time of the exercise date. Due to this increased risk, an investor would have little information on the future unknown path that each asset would take until the exercise date. Therefore with little information, a buyer would be less likely to buy the option. Therefore the writer of the option will charge a lower option price to try to entice an investor to buy the option. As the time to expiration is reduced, the option price gets higher. Therefore at the end of the option, the price of the option would be at its highest value in its lifetime. The price of the asset is not likely to change much just before the exercise date, therefore an investor will be confident in buying the option at this time. As a consequence, the price of the option will be at its highest in its lifetime. This is one of the reasons for the use of barriers. Barriers help to reduce the cost of purchasing the option, especially at the start of the option. The annular barrier is not suited for computing call options. Since call options is only exercised $S > K$, then this occur outside the upper limit of the barrier (£35). Therefore for call options, the barrier would knock out the options. As a consequence, call options will have the values of zero everywhere. This is the reason for the omission of computing call options in this chapter.

The annular barrier is suited for computing put options. For put options, the options is exercised only when $S < K$. Therefore all the put options are affected by the barrier. As seen earlier, when the values for the underlying assets is below the lower barrier limit (£20), or higher then the upper barrier limit (

Chapter 6

Investigation into the effects of barriers in pricing options

K

In chapter 5, the numerical solutions for max, min and basket types of put options are produced using a annular barrier with inner and outer radii equal to K_1

- barrier applied daily
- basket constant: $\alpha = 0.5$

6.2 Numerical Results

All figures shown in this section is produced using the program Gmsh
(source: http://www.geuz.org/gmsh/). The numerical solutions for the basket call are shown in figure 6.1.

6.2.1 Numerical Results using unstructured mesh

An unstructured mesh (see figure 4.6) is used to compute the numerical solution for the basket call option in this section.

The figure at time $t = 0$ in figure 6.1 shows the numerical solution for the basket call option at the start of the lifetime of the option. The basket constant α is taken to be 0.5. This is when the barrier is first applied to the basket call option. The peak value of the option is £0.14 which is located where the price of assets S

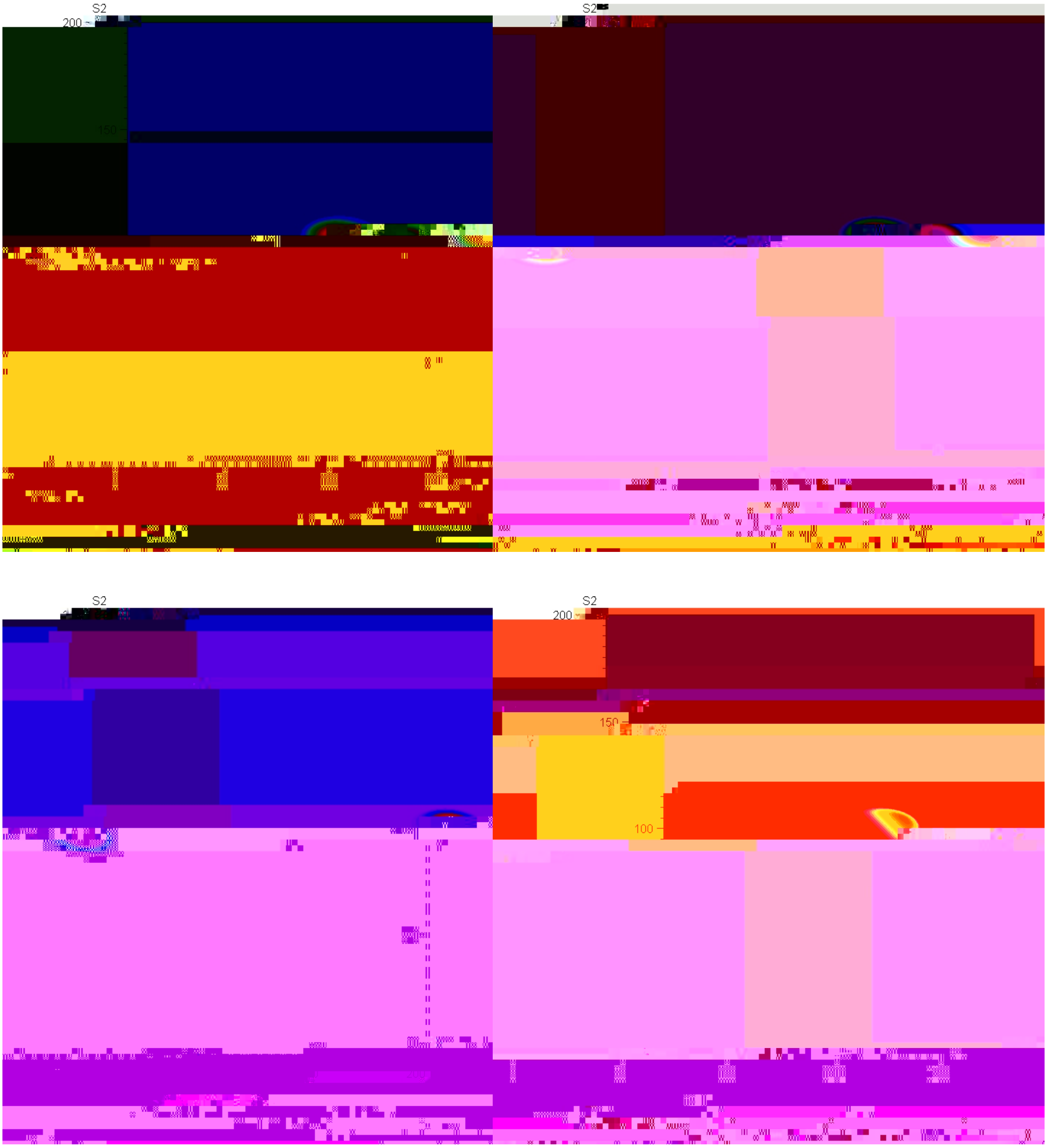

Figure 6.1 - Basket Call Option using unstructured mesh

Also the figure shows that when both of the assets have values less than £100, the option price has the value of zero. A reason for this price would be that when the barrier is lifted, the basket call option is not exercised in this region, therefore the option expires worthless.

Summary

The maximum option price is located at the centre of the elliptical barrier during most of the option lifetime because at the centre, the price of the assets S_1 and S_2 are both furthest away from the edge of the barrier, therefore it is less likely to be knock out by the barrier. The peak option price of the basket call option is lowest at the start of its lifetime. As before, this is because there would be a higher risk that the price of the assets will be more likely to change by the time of the exercise date. As a consequence of this risk, the writer of the option will charge a lower option price to try to entice an investor to buy the option. Otherwise the buyer would be less likely to buy the option. As the time to expiration is decreasing, the option price gets higher, since the risk of the assets changing its values is decreasing. Therefore at the end of the option, the price of the option would be at its highest value in its lifetime. The price of the asset is not likely

When comparing the result produced in figure 6.2, with the results produced in figure 6.1, the two results shares similar characteristics.

- Both results shows the option would be knocked out if the values of the underlying asset crosses the barrier
- The peak option values for both results are both concentrated near the centre of the barrier
- There is a similar pattern in the distribution for the price for the options. Approaching the centre of the barrier from the barrier edge would lead to an increase in the option price.

Chapter 7

Conclusion

Barriers introduced discontinuities in the solution at each discrete barrier observation dates. FEM allows the use of an unstructured mesh to accurately compute the solutions by adding extra nodes with smaller spacing near the barrier limits, to capture the discontinuities. Adding extra nodes will improve the accuracy of the solutions in the regions of interest.

This dissertation looked at the results computed by imposing two different types of barrier shapes on the options. It is found that put options are more suited to the annular barrier imposed on the options. The put options are computed using both Neumann and Dirichlet boundary conditions. Imposing Dirichlet boundary condition on the boundary of the S_1 and S_2 axis can affect the peak option price of the Min Put option during its lifetime. But there are minimal impact on the numerical values produced by the Max Put and the Basket Put options.

Results for the basket call option computed by Pooley et al (2000), was successfully reproduced using the ${P_1}^{NC}$ finite element method and applying an elliptical barrier. When the barrier shape is reduced, more area of the option would have a higher chance of breaching the barrier. Therefore it can be expected that there would be a decrease in the option prices, due to this higher risk. Also if the barrier is rotated, there could also be a higher chance of breaching the barrier. This can lead to a reduction in the prices of the options. Conversely, the option prices would increases if there is a lower risk of breaching the barrier.

Also this dissertation looked at the accuracy of the option price by comparing the numerical solutions produced using structured and unstructured meshes. It is found that using a structured meshes will gave higher values for the option price, when compared with the option price computed on an unstructured mesh.

For further research, I could look the effects of changing the size and rotation of the barrier in more detail.

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