THE UNIVERSITY OF READING

DEPARTMENTS OF MATHEMATICS AND METEOROLOGY

4D-Var for high resolution,

Abstract

[illegible]

Declaration

con y o n o nd e e o e o o e o ce
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Acknowledgements

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Contents

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Mode o p nd po e pec t n u^r x_j^r

n πx_j^r n πx_j^r n π x_j^r n π x_j^r

Acronyms

N — N e c e e p ed c on

LAM — L ed e ode

LBC — L e o nd y cond on

d en on on d on

d en on on d on n e

n — n e e n e

c e e o e n o

o e n o

Data Assimilation Notation

$\mathbf{x}$ — e ec o

$\mathbf{x}^a$ — n y ec o

$\mathbf{x}^t$ — e e ec o

$\mathbf{x}^b$ — c o nd e p o e e

$\mathbf{y}$ — o e on ec o

$\mathbf{h}$ — o e on ope o

$\mathbf{H}$ — ne on o o e on ope o $\mathbf{h}$

Model Notation

u	e pe e e
$\mathbf{x^P}$	en p coo d n e
$\mathbf{x^L}$	LAM p coo d n e
t	e
c	Ad ec on e oc y
σ	on con n
$\mathbf{x^P}$	en d p c n
$\mathbf{t^P}$	en e ep
$\mathbf{x^L}$	LAM d p c n
$\mathbf{t^L}$	LAM e ep
N	N e o p en dpo n
M	N e o LAM dpo n
$\mathcal{T}$	N e o p en ode e ep n on ndo
S	N e o LAM e ep n on ndo
h	o o LAM o p en d p ce
τ	o o LAM o p en e ep
D	N e o p en ode d p ce co e ed y LAM
B	en dpo n co e pond n o LAM o nd y
B	en dpo n co e pond n o econd LAM o nd y
b	d o e zone

Chapter 1

Introduction

The ultimate problem in meteorology [18, p. 6].

o d e en een ed o de c e p o e o o ec n e
C p Acco d n o B e ne o p od ce e e o ec

1.1 Motivation

[illegible]

[illegible]

[illegible]

en pe ec o e on nd no c o nd e LAM d on c nno c p e e c y d e o e o n od ced o LBC

o n od ced o LBC e d e ence n e o on e een nd LAM c e LAM d on o e e o o en e

n LAM d on e en c e on e n do n o LAM e ed on o o en e o

Chapter 2

Data Assimilation

2.1 Types of data assimilation

2.1.2 Variational schemes

n on on o e nd n y $\mathbf{x}^a$ c n e co

nc on e e p o e co nc on n ed d ec y n n

e e c e o nd n y

A c e 7 nd n y y n n n co nc on

$$\mathbf{J} \; \mathbf{x} \quad \mathbf{x}$$

d e ence e een co nc on nd co nc on o

e dependence n o e on e A c e n o e

n on ndo nd o e on e o ed o e d ed n e

ode e on e e o ed o d n e nd n e p ed c ed y ode

e co p ed o e on e o e on e d e y o

nd n n y $\mathbf{x}^a$ c e n ode o d o $\mathbf{x}^a$ e e e

ec o y o o e on nd c o nd

2.1.3 The Kalman Filter

n con o on ec n e ed y n e

d en on e en e d oo o o e on o e ed o e

e ndo o e e ep o e n o e on e ep

y e ep o e op n y e c o e on e en p e o

n o on

c n e e ed n o e o ec deno ed $\mathbf{f}$ nd n y deno ed $\mathbf{a}$

o ec e co p e o e o ec

$$\mathbf{x}_k^f \quad \mathbf{M}_{k-1}\mathbf{x}_{k-1}^a,$$

nd e o co nce o ec

$$\mathbf{P}_k^f \quad \mathbf{M}_{k-1}\mathbf{P}_{k-1}^a\mathbf{M}_{k-1}^T,$$

e $\mathbf{M}_k$ ne ode $\mathbf{t_k}$ nd $\mathbf{P}_k$ e o co nce

$\mathbf{t_k}$ c e nn n o on ndo $\mathbf{P_0}$ $\mathbf{B}$ e $\mathbf{B}$ e o

co nce ed n o p c y e e ne ec ed ode e o

e o e o co nce o ec e c n e nc ded n o on

n y e co p e o n n co p on

$$\mathbf{K}_k \quad \mathbf{P}_k^f\mathbf{H}_k^T \quad \mathbf{H}_k\mathbf{P}_k^f\mathbf{H}_k^T \quad \mathbf{R}_k \quad ,$$

e n y

$$\mathbf{x}_k^a \quad \mathbf{x}_k^f \quad \mathbf{K}_k \quad \mathbf{y}_k - \mathbf{H}_k\mathbf{x}_k^f \quad ,$$

nd e o co nce o n y

$$\mathbf{P}_k^a \quad \mathbf{I} - \mathbf{K}_k\mathbf{H}_k \quad \mathbf{P}_k^f,$$

e $\mathbf{x}_k^a$ n y e $\mathbf{t_k}$ oc ed e o co nce $\mathbf{P}_k^a$

n e e en e o op y o e o e on e en y p o de e ne n ed e e o e nd e o co nce

2.1.4 Why 4D-Var is the method of choice

One d n e o on o y c n nd e nd ec o e on n o e n n need o e e ope on y c n o n o e on o de e n non ne nne nd p o ec n o on o de p ce o o e on p ce nd ce e non ne o e on ope o

o d n e o e on c n e d ed n e e p ce y n de dyn c e n no on y e o e on e n ed pp op e e o e e o e on e oc on c n e ed n one on pe od 7 c n e c dyn c y con en n o on o e e o o e on 7 n n o o e on o on c e con en dyn c e on c n e pen ed e po ene no e o o e on e ed o e d n e e e e e o n e

o op y o e o e on e en y n e e en e p o de e ne n ed e e o e nd e o co nce e e e e een

Bo nd ene o n e o ed e o co nce 7 o e e
o d n e o d nce c o nd e o co nce n
e o co nce o ec ep e o co nce o ec p o de o dependen
c o nd e o co nce end o e ndo n e
n con n co nce done n

on ed e ode c en y n MA p n eo o o c

A ency e o on Me O ce ode c en y n

o o n o nd e e e dy p n p ed y

Me eo nce o ed e o on ed e 7 d

on e con n e on n e n e on c

d n e o t_T o $\mathbf{t}$ e d o n e on

$$\boldsymbol{\lambda} \quad {}_{+1}$$

$$\boldsymbol{\lambda}_k \quad \mathbf{M}_k^{\mathbf{T}} \boldsymbol{\lambda}_{k+1} \quad \mathbf{H}_k^{\mathbf{T}} \mathbf{R}_k^{-} \quad \mathbf{y}_k - \mathbf{h}_{\mathbf{k}} \ \mathbf{x}_k \ , \quad \mathbf{k} \quad \mathcal{T}, \ldots, \ .$$

d en o $\mathbf{J}_0$ n e n en y

$$\mathbf{J}_0 \ \mathbf{x}_0 \quad -\boldsymbol{\lambda}_0.$$

c n no e ed n n e e e on e e d en o $\mathbf{J}$ o e c c ed $\mathbf{M}$ no n o d ne ode nd $\mathbf{M}^{\mathbf{T}}$ d o n ode

on o nd n L n e p e e e o nd n ne e n n ec on

c c o on pp o c o o n n n nde nd n con c on o con ned n on e p n doe no need o e ne ed nd c n py e en

$$
\begin{aligned}
J(\mathbf{x}_0) &= \frac{1}{2}(\mathbf{x}_0 - \mathbf{x}^b)^T \mathbf{B}^{-1}(\mathbf{x}_0 - \mathbf{x}^b) \\
&\quad + \frac{1}{2}\sum_{k} (\mathbf{y}_k - \mathbf{H}_k\mathbf{x}_k)^T \mathbf{R}_k^{-1}(\mathbf{y}_k - \mathbf{H}_k\mathbf{x}_k), \\
&\overset{\text{def}}{=} J_b + J_o,
\end{aligned}
$$

nd con n

$$\mathbf{x}_k = \mathbf{M}_{k-1}\mathbf{x}_{k-1}.$$

e J_b de ned o e c o nd e nd J_o de ned o e o e on e

2.3.1 Minimisation of the cost function J

o n e co nc on e e e n on o o e c ce nc de Ne on o con e d en e d e ed de c p on o e e d c n e o nd n o ec on o e pe nd e e o e no de c ed e o e e e ype o d en de cen o e e e o J nd d en ∇J o e c c ed e c e on nd e e o e need o con de e c n e e ed

e on o co nc on e e y o d y py c c n e on J_b c n e c c ed d ec y o c c e J_o e py e o n en ne ode $\mathbf{M}_k$ o e o e o o on o d n e e c n n c c e nd o e

$$\mathbf{d}_k = \mathbf{R}_k^{-1}(\mathbf{y}_k - \mathbf{H}_k\mathbf{x}_k),$$

e $\mathbf{d}_k$ e no n no ed dep e e dep e c n n e ed o c c e J_o e e o con on

$$J_o(\mathbf{x}) = \frac{1}{2}\sum_{k} J_{ok}(\mathbf{x}),$$

e

$$J_{ok}(\mathbf{x}) = (\mathbf{y}_k - \mathbf{H}_k\mathbf{x}_k)^T \mathbf{d}_k.$$

e on o den c n o e done n o p den o $\mathbf{J_b}$ c n
p y e o nd y c c n $\nabla\mathbf{J_b}$ d ec y

$$\nabla\mathbf{J_b} \quad \mathbf{B}^{-1} \; \mathbf{x}_0 - \mathbf{x}_b \, .$$

o c c e $\nabla\mathbf{J_0}$ d ec y o d e co p on y n e e o d e e $\mathbf{N}$
o d ode n d o n o $\nabla\mathbf{J_0}$ o e c c ed one d o n
ode n e need o c o e

$$\nabla\mathbf{J_0} \; \mathbf{x} \qquad - \sum_{k} \nabla\mathbf{J_{0k}} \; \mathbf{x} \, ,$$

$$- \sum_{k} \mathbf{M}_1{}^{\mathbf{T}} \ldots \mathbf{M}_k{}^{\mathbf{T}} \mathbf{H}_k{}^{\mathbf{T}} \mathbf{d}_k ,$$

$$- \; \mathbf{H}_0^T \mathbf{d}_0 \quad \mathbf{M}_1^T \; \mathbf{H}_1^T \mathbf{d}_1 \quad \mathbf{M}_2^T \; \mathbf{H}_2^T \mathbf{d}_2$$

$$\mathbf{M}_3^T \; \mathbf{H}_3^T \mathbf{d}_3 \qquad \mathbf{M}^T \mathbf{H}^T \mathbf{d} \qquad .$$

on c n no e c c ed o o e y n n d o n
e $\boldsymbol{\lambda}$ o ze o n e $\boldsymbol{\lambda}$ $+1$ n e on e n ep
c d o e ep e c ep dd n o c n e $\mathbf{H}_k^T \mathbf{d}_k$ o $\boldsymbol{\lambda}_\mathbf{k}$
e o e pp y n d o n ode o e $\boldsymbol{\lambda}_{\mathbf{k}-}$ ence e on eco e

$$\nabla\mathbf{J_0} \; \mathbf{x} \qquad -\boldsymbol{\lambda} \, .$$

d en o $\mathbf{J_0}$ een o nd e n ne e e e on
e o nd n ec on e on de ed y e d o
L n e p e

e e d o n ode de ned n po e o n en ne ode e $\mathbf{M}^{\mathbf{T}}$
Ope on y e e d o n no con c ed e p c y o n po e o
n en ne ode d o n ode nd e ed no con de ed
n ne ec on

2.3.2 The adjoint model

Ope on y d c e e ode ne ed o e n en ne ode nd n
d c e e d o n e on e con c ed o e d c e e ne ed e on
d o n de ed d ec y o n en ne ode code n n o c
d o n e d

o e e e c n o e o o d c e e n en ne e on o nd n po e e o ze o y e o e ed n

pe ec y e e e o py n po e $\mathbf{M}$ o ene e d o n o done n o c e C p e

Once d o n con c ed need o e e ed e e o d e en pec c need o e e ed nd e e done n ep e d

The Adjoint test

n d o n e on e een de ed d ec y o de code d o n code need o e e ed o e y p od c n co ec d o n o do e e de n on

$$\langle \mathbf{A}\mathbf{b}, \mathbf{c}\rangle \quad \left\langle \mathbf{b}, \mathbf{A}^{T}\mathbf{c}\right\rangle,$$

e $\mathbf{A}$ ne ope o $\mathbf{A}^{T}$ d o n $\mathbf{b}$ nd $\mathbf{c}$ e ec o nd c e $\langle \ldots, \ldots \rangle$ deno e n nne p od c

e e o o de $\mathbf{M}$ nd d o n $\mathbf{M}^{T}$ e c n e e on e y pp y n de o n cond on $\delta\mathbf{x}_0$ o p od ce n e n pp y n d o n o n e

d o n co ec e d e

$$\langle \mathbf{M}\,\delta\mathbf{x}_0, \mathbf{M}\,\delta\mathbf{x}_0\rangle \; - \; \left\langle \delta\mathbf{x}_0, \mathbf{M}^{T}\mathbf{M}\,\delta\mathbf{x}_0\right\rangle \quad .$$

The Gradient test

e e y d o n p od ce co ec d en o co nc on o do e e d en e d e dy e d o e d o n de o e p e

A y o e p n on o co nc on $\mathbf{J}$ e

$$\mathbf{J}\;\mathbf{x} \quad \alpha\overline{\delta\mathbf{x}} \quad \mathbf{J}\;\mathbf{x} \quad \alpha\overline{\delta\mathbf{x}}^{T}\nabla\mathbf{J}\;\mathbf{x} \quad O\,\alpha\;,$$

e α c nd $\overline{\delta\mathbf{x}}$ ec o o n en o c n e e n ed o e nc on o α

$$\varphi\ \alpha \quad \frac{J\ x \quad \alpha\overline{\delta x} \ - J\ x}{\alpha\overline{\delta x}^{T} J\ x} \qquad O\ \alpha\ . \qquad 7

n ec on 7 e de on ed o n e co nc on e pec o x
o e e i e od e e p o e o ep e en n nd n e n B
o N p y n e e B yp c y o O o
o e co e p o e co nc on n e d po ed n e o d e en con o
e y e n o con o e n o 7

Chapter 3

The discrete Fourier transform

n c p e e n od ce d c e e o e n o nd po e pec c n e n e e y e e d o nde nd n n y ene ed y d on e de on e e e o po e pec nd y e e o p ope e o nd y en e o e oo o nde nd n y

3.1 Definition

ne co ne nd n y e ponen e p ope y y e o on o e e e o d c e e e y p ced po n 7 p

Le a

if s Nm, m Z

otherwise,

where ξ

o nc on f_j e f_j $f\ \xi_j$ de ned o e

$$\mathrm{DFT}\ f_j \quad \hat{f}_k \quad \sum_{j}^{N-1} f_j e^{-i\ \pi jk/N}, \qquad k \quad , \ , \ldots, N - \quad ,$$

e k en e nd N

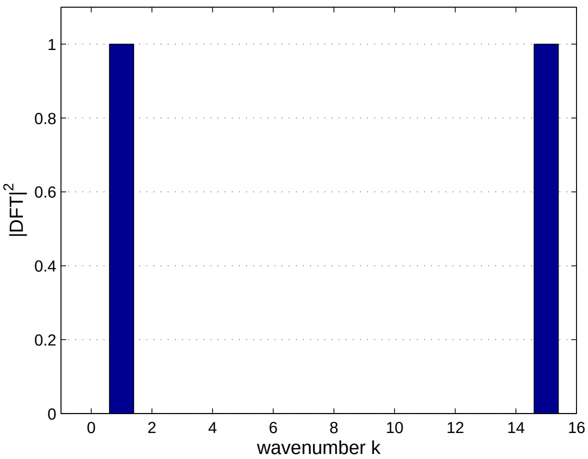


Figure 3.1: The power spectrum of sin(ξ_j), scaled by the factor $2/N$.

c c n e e n ed o e

$$\hat{f}_k = \frac{1}{2i} \sum_{j}^{N-1} e^{-i\xi_j (k-\kappa)} - \frac{1}{2i} \sum_{j}^{N-1} e^{-i\xi_j (k+\kappa)} .$$

n o de o e e e on e e c co p e e ponen

e p e e po e pec o n ξ_j on do n x ,
e ξ_j πx_j j , , $N-$ x_j j/N nd N

By e n κ n e on e ee ee po e pec o n ξ_j p o ed n e c p de k nd k $N-$ nd ze o e e y e e e By pp y n c n c o $/N$ o e on e o o n n p de o one

o p e e p e e c n e n o ee e ne o po e pec po e pec p o de e d o o e e e e o coe c en n n e y

e e ny no n p ope e o nd de c p on o e c n e o nd n ec on nd p nd 7 p on o e e p o e e o e p ope e nd e e o e e e ee n po e pec e de on e c n e ed oo o nde nd n y ene ed y d on

3.3 Properties of the DFT

Linearity

ne nc on p c o nc on

$$y_j \quad af_j \quad bg_j,$$

e a nd b e c o y_j

$$\hat{y}_k \quad a\hat{f}_k \quad b\hat{g}_k.$$

Sy etry of the Co ple Conjugate

o e e ence f_j j , , $N-$ n ene e e ence o N co p e n e n p c $\hat{f}_k$ nd $\hat{f}_{N-k}$ e e ed y

$$\hat{f}_{N-k} \quad \hat{f}_k^{*}$$

o k , ks

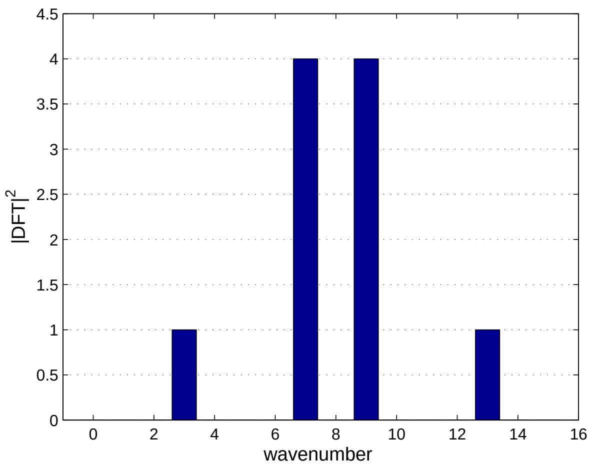
Phase

n ene o nc on $\mathbf{f_j}$ c n e en

$$\mathbf{f_k} \quad \mathbf{f_k^R} \quad \mathbf{i f_k^I},$$

e $\mathbf{f_k^R}$ e p nd $\mathbf{f_k^I}$ n y p

o e e o n e en nc on



4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
|DFT|2
0
2
4
6
8
10
12
14
16
wavenumber

po e pec

p ope y o po e pec o o e o den y c n c o n n n y p od ced y d on

3.4.3 Phase

A ed n ec on odd nc on n o o n y p on y nd e en nc on n o o e p on y nc on c n e odd o e en o c n e de p o co n on o o odd nd e en p By con de n e nd n y p o ep e y e c n nde nd e e p e o nc on o e p e $\mathbf{y_j}$ n $\kappa\boldsymbol{\xi_j}$ n odd nc on nd e o e p e y n y coe cen de on ed n c c on done n ec on o e e e no con de ne e p e e c n ee no e p ope y o

o e c n e ed o nde nd p e e e nc on $\mathbf{y_j}$ n $\boldsymbol{\xi_j}$ $\boldsymbol{\varphi}$ e $\boldsymbol{\varphi}$

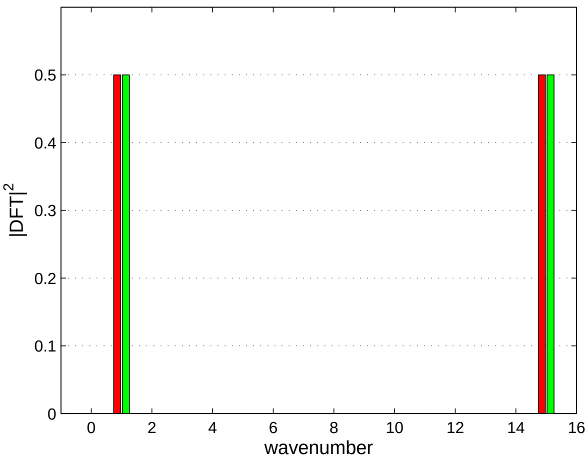


Figure 3.3: The power spectrum of the real (red) and imaginary (green) parts of the DFT of sin($\xi_j + \pi/4$), scaled by the factor $2/N$.

3.5 Summary

e e n po e o e n o o n o on o o on y p y pe o n nd con de n e e o po e pec n e on o no n p ope e o . By con de n ne nd co ne e o nc on p ce e e n en e nd p e n p de nd p e o co ponen o nc on c n e o nd o nd po e pec e e o de on ed e d o c n e done e e d e een de on ed e on c ne e e p e c n e y e pp ed o o e co p c ed nc on

Chapter 4

Limited Area Models

e e ny d e en ed e ode LAM c en y e n n ope on y
e cen e o nd o d e e n y n od c n

n ed ode M Me O ce n on ed e do n e o y 7

e e e c nd o ec n ode ency co o on 7 C en o p ne nc de NCA N on Oce n c nd A o p c Ad n on NOAA N on Cen e o n on en

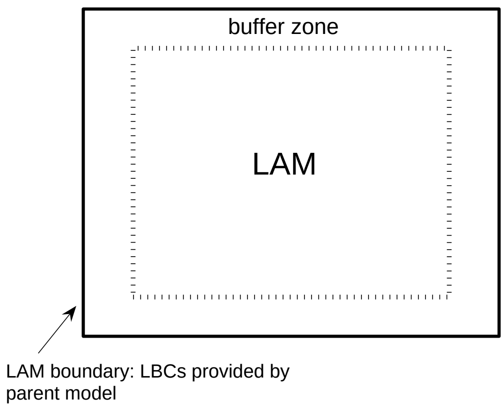


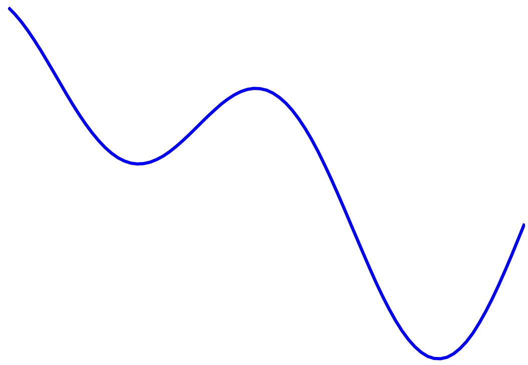
Figure 4.1: Schematic of the LAM domain and bu er zone.

LAM n e o on nd A MON e e o on o A d n zon e o on o o nd A OM n d e M M MA c en y e o on

Ano ed p ope y o ode e o nd y cond on LBC e p o ded o e co e e o on p en ode n o de o e o on on n e o o LAM do n o e p e c ed y p en ode o nd e e zone p e en ed o nd e o LAM

A ode e e on c o e zone c e o e on e α_i nd d o e **b** e e een ode o e p e n y e **b** nd α p e ne n e po on 7 e n LAM **b** de n e e ence y e nd de e o α α_i – co $\pi i/b$ A e e on zone LAM o p e en n e en on zone o o n pe od c y on LAM do n e d c ed n ec on 7

A LAM e LBC o p en ode o e e e ency c y e p o ded d e e een ode n y e LBC e pec ed y p en ode e e y co e d e ep 7 n con n M M LBC e on y en n e nd end o on ndo nd e n e po ed e een e o e n e po on co d de de cc cy o ode o ce on o nd e o



LBC pp ed y p en de c n e o o ce o e o n LAM d e o e e e on

zon e c nd e po e o on o p en de ene y co e n o LAM nd LBC e o e n e po ed o LAM d e e y e ep en p en de n p o d n LBC pe ec e e n e po on e o n od ced n n e po ed o LAM e o on

e y o e d e ence n de e p nd p e on e en e o on e n o de c e d e ence n de o p nd e d e ence o nd e y c e p o e nd d en c c n n ence LAM n e o

p en de y p y e on o o e e on y e n p c e e occ n n on p oc on o de e op n oo e y o oo e o e p e nd e o e n ed o LAM o nd y n e ce

e p o e nd o e d c ed y A e e p o e oc ed LAM e y c n o e e ce ed n d on d c ed n ne ec on

One y o ne e e p o e o p y oc e o nd e c en y o e o e o o o c n e e c e o c nno e n ed n d o o nd y n ndo o o ec o e o on o ec n e e p y p c c do n c n on y e ed ze nd e e n c en co p e e o ce o o o o nd e o e c en y y

A d c ed n ec on one d p e n ed ope on y e o e zone e zone end o o on ne o nd e nd e o e o d ny p p c co d e ec e de e o e ndependen y y o n ny d c epenc e e een de o nd e

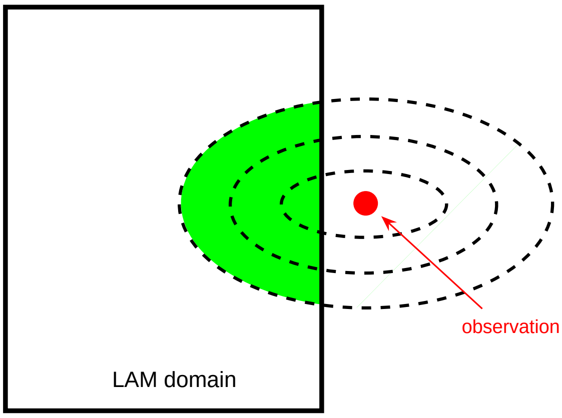


Figure 4.3: Diagram illustrating the observational information lost from observations outside the LAM domain. The red dot is an observation and the dotted lines indicate the spreading of this observational information by the matrix B. The green area is the part of the LAM domain that would be influenced by the observation if it were visable to the LAM.

o e c n c e e ncon enc e e een e e ep e en ed n LAM nd co e ode p o d n LBC o e p e po on o e on y e d e en n o do n

A d c ed n ec on con o e n o ed o e o e zon co e on n e e n o on on c e con o e n o e n e e de ned n pe

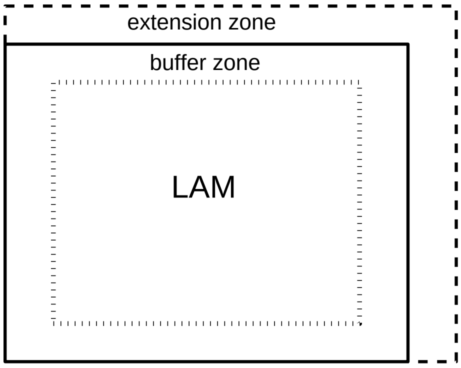


Figure 4.4: The extension zone of the HIRLAM model, as shown in [35]

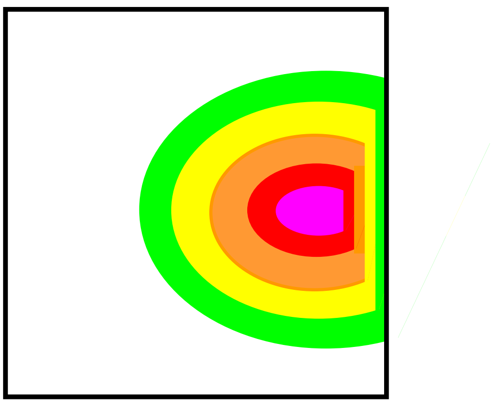


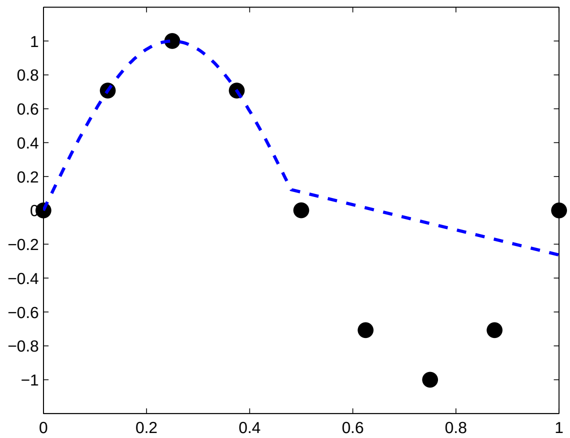
Figure 4.5: Diagram illustrating a feature being 'cut-off' by the LAM boundary. The colours indicate the size of the increment, white being zero increment and pink being the largest increment.

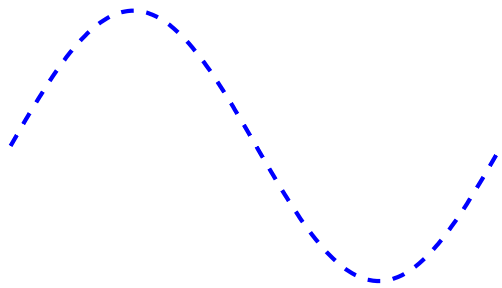
[illegible]

[illegible]

e een o od y LAM c o nd y dd n on n y nc e en o co e n y e ed o c e e e n pec ed en c e N n y n LAM n on y done o c e e n N e p e en ed n c n p o e nc e en n e on o o nd y nc e en o nd y e ze o o p no en e e n c o e d no o e p o e n LAM on n o e e en y n ode p o d n LBC e e p od ced n co e ode n y e e o LAM n y Ope on y y no e c e o e e e po e o e co e ode o ec o p o de e c e no een e ed

An e n e e d p opo ed y e p o p o e con ency o LAM n y LBC p o ded y p en ode y nc d n n e e n co nc on e e e e d nce e een e o nd y e o n y nd LBC o o ode n o e o on eo e y eco end on n nd 7 LB


1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.2
0.4
0.6
0.8
1



Chapter 5

A 4D-Var algorithm for a LAM domain

n C p e e n od ce ene ed e ode LAM do n nd d c e

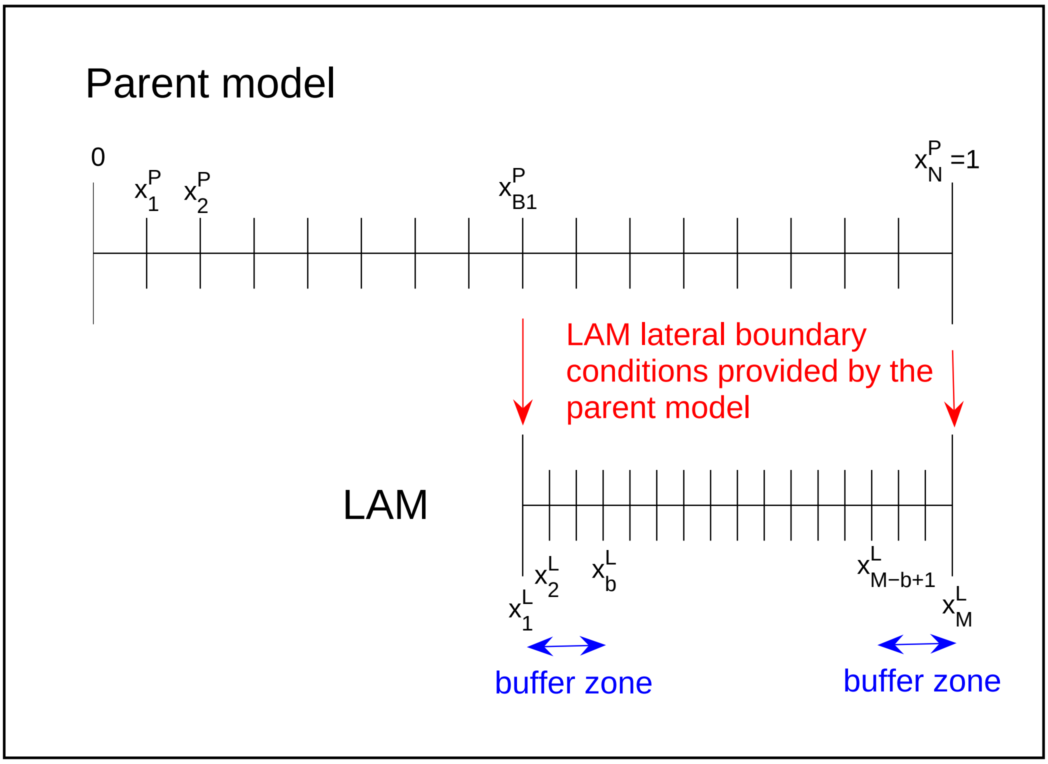


Figure 5.1: Diagram of model domains.

ndo

e o nd y cond on LBC o LAM e p o ded y p en ode n o de o e o on on n e o o LAM do n o e p e c ed o nd e e e zone p e en ed o nd e o LAM e zone co e **b** LAM dpo n nd e e on c e ed e e e e on ec e ed ope on y 7 d c ed n ec on e e ne n e po on nc on

$$\alpha_i \quad - \quad i/b \, ,$$

e **b**

5.2 Modifications to the 4D-Var algorithm for the LAM domain

A d c ed n ec on o ene e n n y n e n e co nc on 7

$$J(\mathbf{x}) = \frac{1}{2}(\mathbf{x}-\mathbf{x}^b)^T \mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}^b) + \frac{1}{2}\sum_{k}^{T} (\mathbf{y}_k - \mathbf{H}_k(\mathbf{x}_k))^T \mathbf{R}_k^{-1}(\mathbf{y}_k - \mathbf{H}_k(\mathbf{x}_k)),$$

ec o ode e on

$$\mathbf{x}_{k+1} = \mathbf{M}_k \mathbf{x}_k.$$

o e e LAM e e on pe o ed n e zone c e e o p en ode o no po e o e ode e on nc d n od c on o nd e n o e c n e e e LAM ode e on

$$\mathbf{x}_{k+1} = \tilde{\mathbf{M}}_k \mathbf{x}_k + \mathbf{P}\,\mathbf{x}^p_{k+1},$$

e $\mathbf{x}_k$ no e ec o on LAM do n e e k $\tilde{\mathbf{M}}_k$ od ed e e k e cco n o c n c o n e zone d e o e e on $\mathbf{P}$ o c n c o o n ence o o e n e zone d e o e e on nd $\mathbf{x}^p_k$ ec o o e o p en ode e e k

o p c y e e $\tilde{\mathbf{M}}_k$ e e e nd e c n e o e ep ce $\tilde{\mathbf{M}}_k$ n e on con n $\tilde{\mathbf{M}}$ A n $\tilde{\mathbf{M}}$ ep e en ee po n d c e on c e o o

$$\mathbf{x}_{i,k+1} = \eta\,\mathbf{x}_{i-1,k} + \gamma\,\mathbf{x}_{i,k} + \mu\,\mathbf{x}_{i+1,k},$$

e e

$$\widehat{\mathbf{M}} = \begin{pmatrix}
\mathbf{0} & \mathbf{0} & & & & & \cdots & & & & & \mathbf{0} \\
\psi_1\eta & \psi_1\gamma & \psi_1\mu & \mathbf{0} & & & & \cdots & & & & \mathbf{0} \\
\mathbf{0} & \psi_2\eta & \psi_2\gamma & \psi_2\mu & \mathbf{0} & & & \cdots & & & & \mathbf{0} \\
 & & & \ddots & & & & & & & & \\
\mathbf{0} & & \mathbf{0} & \psi_{b-1}\eta & \psi_{b-1}\gamma & \psi_{b-1}\mu & \mathbf{0} & & \cdots & & & \mathbf{0} \\
\mathbf{0} & & & \mathbf{0} & \eta & \gamma & \mu & \mathbf{0} & \cdots & & & \mathbf{0} \\
 & & & & & & \ddots & & & & & \\
\mathbf{0} & & & & & \mathbf{0} & \eta & \gamma & \mu & \mathbf{0} & & \mathbf{0} \\
\mathbf{0} & & & & \cdots & \mathbf{0} & \psi_{b-1}\eta & \psi_{b-1}\gamma & \psi_{b-1}\mu & \mathbf{0} & & \mathbf{0} \\
\mathbf{0} & & & & \cdots & & \mathbf{0} & \psi_{b-2}\eta & \psi_{b-2}\gamma & \psi_{b-2}\mu & \mathbf{0} & \mathbf{0} \\
 & & & & & & & & & \ddots & & \\
\mathbf{0} & & & & \cdots & & & & \mathbf{0} & \psi_1\eta & \psi_1\gamma & \psi_1\mu \\
\mathbf{0} & & & & \cdots & & & & & & & \mathbf{0}
\end{pmatrix}, \tag{5.2}$$

nd

$$\mathbf{P} = \begin{pmatrix}
\phi_0 & \mathbf{0} & & & & & \cdots & & & & & \mathbf{0} \\
\mathbf{0} & \phi_1 & \mathbf{0} & & & & \cdots & & & & & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \phi_2 & \mathbf{0} & & & \cdots & & & & & \mathbf{0} \\
\vdots & & & & & & & & & & & \\
\mathbf{0} & & & \mathbf{0} & \phi_{b-1} & \mathbf{0} & & \cdots & & & & \mathbf{0} \\
\mathbf{0} & & & & \mathbf{0} & \mathbf{0} & \mathbf{0} & & & & & \mathbf{0} \\
 & & & & & \ddots & & & & & & \\
\mathbf{0} & & & & \mathbf{0} & \mathbf{0} & \mathbf{0} & & & & & \mathbf{0} \\
\mathbf{0} & & & & & \cdots & \mathbf{0} & \phi_{b-1} & \mathbf{0} & & & \mathbf{0} \\
\mathbf{0} & & & & & \cdots & & \mathbf{0} & \phi_{b-2} & \mathbf{0} & & \mathbf{0} \\
\vdots & & & & & & & & & & & \\
\mathbf{0} & & & & & \cdots & & & & \mathbf{0} & \phi_1 & \mathbf{0} \\
\mathbf{0} & & & & & \cdots & & & & & \mathbf{0} & \phi_0
\end{pmatrix}, \tag{5.3}$$

e $\boldsymbol{\varphi}$ φ_i^{b-} ec o o e e on coe cien nd $\boldsymbol{\psi}_i$ $-\boldsymbol{\varphi}_i$ e c p e e e o p ep

ce c n e n o o ode e on c n e no e ec on $\mathbf{J}_b$ e on e c e e ec on $\mathbf{J}_o$ e on e o e e no con de J_0 o e ode e on e e

$$\mathbf{x}_1 \quad \mathbf{M}\,\mathbf{x}_0 \quad \mathbf{P}\,\mathbf{x}_1^p,$$

$$\mathbf{x}_2 \quad \mathbf{M}\,\mathbf{x}_1 \quad \mathbf{P}\,\mathbf{x}_2^p \quad \mathbf{M} \quad \mathbf{M}\,\mathbf{x}_0 \quad \mathbf{P}\,\mathbf{x}_1^p \quad \mathbf{P}\,\mathbf{x}_2^p,$$

$$\mathbf{x}_3 \quad \mathbf{M}\,\mathbf{x}_2 \quad \mathbf{P}\,\mathbf{x}_3^p \quad \mathbf{M} \quad \mathbf{M} \quad \mathbf{M}\,\mathbf{x}_0 \quad \mathbf{P}\,\mathbf{x}_1^p \quad \mathbf{P}\,\mathbf{x}_2^p \quad \mathbf{P}\,\mathbf{x}_3^p,$$

$$\vdots$$

$$\mathbf{x} \quad \mathbf{M}\,\mathbf{x}_{\;1} \quad \mathbf{P}\,\mathbf{x}^p \quad \mathbf{M} \quad \mathbf{M} \quad \mathbf{M}\,\mathbf{x}_0 \quad \mathbf{P}\,\mathbf{x}_1^p \quad \mathbf{P}\,\mathbf{x}_2^p \quad \mathbf{P}\,\mathbf{x}^p.$$

n e n e n o J_0 o e on 7 e

$$\mathbf{J_0} \quad \frac{}{\mathcal{T}} \sum_{\mathbf{k}}^{\mathbf{T}} \mathbf{y}_k - \mathbf{H}_k \; \mathbf{x}_k \;{}^{\mathbf{T}} \mathbf{R}_k^{-1} \; \mathbf{y}_k - \mathbf{H}_k \; \mathbf{x}_k \;,$$

$$- \; \mathbf{y}_0 - \mathbf{H}_0 \; \mathbf{x}_0 \;{}^{\mathbf{T}} \mathbf{R}_0^{-1} \; \mathbf{y}_0 - \mathbf{H}_0 \; \mathbf{x}_0$$

$$- \; \mathbf{y}_1 - \mathbf{H}_1 \; \mathbf{x}_1 \;{}^{\mathbf{T}} \mathbf{R}_1^{-1} \; \mathbf{y}_1 - \mathbf{H}_1 \; \mathbf{x}_1$$

$$- \; \mathbf{y}_2 - \mathbf{H}_2 \; \mathbf{x}_2 \;{}^{\mathbf{T}} \mathbf{R}_2^{-1} \; \mathbf{y}_2 - \mathbf{H}_2 \; \mathbf{x}_2$$

$$- \; \mathbf{y} - \mathbf{H} \quad \mathbf{x} \;{}^{\mathbf{T}} \mathbf{R}^{-1} \; \mathbf{y} - \mathbf{H} \quad \mathbf{x} \;,$$

$$- \; \mathbf{y}_0 - \mathbf{H}_0 \; \mathbf{x}_0 \;{}^{\mathbf{T}} \mathbf{R}_0^{-1} \; \mathbf{y}_0 - \mathbf{H}_0 \; \mathbf{x}_0$$

$$- \; \mathbf{y}_1 - \mathbf{H}_1 \; \mathbf{M}\,\mathbf{x}_0 \; \mathbf{P}\,\mathbf{x}_1^p \;{}^{\mathbf{T}} \mathbf{R}_1^{-1} \; \mathbf{y}_1 - \mathbf{H}_1 \; \mathbf{M}\,\mathbf{x}_0 \; \mathbf{P}\,\mathbf{x}_1^p$$

$$- \; \mathbf{y}_2 - \mathbf{H}_2 \; \mathbf{M} \; \mathbf{M}\,\mathbf{x}_0 \; \mathbf{P}\,\mathbf{x}_1^p \; \mathbf{P}\,\mathbf{x}_2^p \;{}^{\mathbf{T}} \mathbf{R}_2^{-1}$$

$$\mathbf{y}_2 - \mathbf{H}_2 \; \mathbf{M} \; \mathbf{M}\,\mathbf{x}_0 \; \mathbf{P}\,\mathbf{x}_1^p \; \mathbf{P}\,\mathbf{x}_2^p \;,$$

$$- \; \mathbf{y} - \mathbf{H} \quad \mathbf{M} \cdots \mathbf{M}\,\mathbf{x}_0 \; \mathbf{P}\,\mathbf{x}_1^p \; \mathbf{P}\,\mathbf{x}_2^p \cdots \mathbf{P}\,\mathbf{x}^P \;{}^{\mathbf{T}} \mathbf{R}^{-1}$$

$$\mathbf{y} - \mathbf{H}$$

e $\delta\mathbf{y}_k$ $\mathbf{y}_k - \mathbf{H}_k\mathbf{x}_k^b$ e o no e ode con n

$$\delta\mathbf{x}_{k+1} \quad \mathbf{M}_k\delta\mathbf{x}_k. \qquad 7$$

o e on e y o e e d en n e o nc e en d en o $\mathbf{J_b}$ e pec o $\delta\mathbf{x}$ eco e

$$\nabla\mathbf{J_b} \quad \mathbf{B}^{-1}\delta\mathbf{x}_0,$$

nd d en o $\mathbf{J_o}$ e pec o $\delta\mathbf{x}$

$$-\nabla\mathbf{J_o} \quad \mathbf{H}_0^T\mathbf{d}_0 \quad \mathbf{M}^T \mathbf{H}_1^T\mathbf{d}_1 \quad \mathbf{M}_2^T \mathbf{H}_2^T\mathbf{d}_2 \qquad \mathbf{M}^T\mathbf{H}^T\mathbf{d} \qquad ,$$

no

$$\mathbf{d}_k \quad \mathbf{R}_k^{-1} \; \delta\mathbf{y}_k - \mathbf{H}_k\delta\mathbf{x}_k \; .$$

5.4 A gridpoint and a spectral scheme for the LAM domain

5.4.1 The gridpoint scheme

discrete intervals with grid points numbered j , , , ,N and f f$_N$, the Fourier sine transform is defined to be

$$\text{sine transform } f_j \quad \hat{f}_\kappa \quad \sum_{j}^{N} f_j \; \text{n}\; \pi j\kappa / \; N \;,$$

where κ is the wavenumber [68].

ne n o nd p ope e e d c ed n ec on

pec e on o o e e e ed o o pec c e pec e on de e oped ec e pec c e ed ope on y on LAM y Me O ce d c ed n ec on

Bo dpo n nd pec c e e ed n By n o dpo n nd pec ed d on c e e c n co p e o p o o e d n en den c d nd n e e e ec o ne n o

o c n e d on c e o pec one e need o pe o con o e n o d c ed n ec on con o e n o ed e ne n o nd n o nc e en o p c o pec p ce e de ne con o e n o

$$z \quad Wx,$$

nd n n e e n o

$$x \quad Uz,$$

e W $R^{N\times N}$ ne n o nd U $R^{N\times N}$ n e e ne n o

n no on ne n o W c n e en

$$W \quad W_{j\kappa} \;, \quad j \quad ,\ldots, N$$

No e e o ne con o e e con de e ec on co nc on e o e on no n e o $\mathbf{z}$ n e d o $\mathbf{x}$

$$\mathbf{J}\ \mathbf{z}_0 \quad \mathbf{U}\boldsymbol{\delta}\mathbf{z}_0\ {}^{\mathbf{T}}\mathbf{B}^{\ 1}\ \mathbf{U}\boldsymbol{\delta}\mathbf{z}_0 \quad {}_{\mathbf{k}}\ \boldsymbol{\delta}\mathbf{y}_k - \mathbf{H}_k\mathbf{U}\boldsymbol{\delta}\mathbf{z}_k\ {}^{\mathbf{T}}\mathbf{R}_k^{\ 1}\ \boldsymbol{\delta}\mathbf{y}_k - \mathbf{H}_k\mathbf{U}\boldsymbol{\delta}\mathbf{z}_k\ .$$

c n e en

$$\mathbf{J}\ \mathbf{z}_0 \quad \boldsymbol{\delta}\mathbf{z}_0{}^{\mathbf{T}}\ {}^{1}\boldsymbol{\delta}\mathbf{z}_0 \quad {}_{\mathbf{k}}\ \boldsymbol{\delta}\mathbf{y}_k - \mathbf{H}_k\mathbf{U}\boldsymbol{\delta}\mathbf{z}_k\ {}^{\mathbf{T}}\mathbf{R}_k^{\ 1}\ \boldsymbol{\delta}\mathbf{y}_k - \mathbf{H}_k\mathbf{U}\boldsymbol{\delta}\mathbf{z}_k\ ,$$

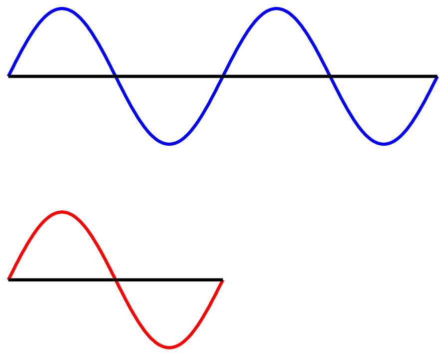
e

$$^{1} \quad \mathbf{U}^T\mathbf{B}^{\ 1}\mathbf{U}.$$

e no con de e ec con o e n o on d en o co nc on d en o $\mathbf{J}_{b}$

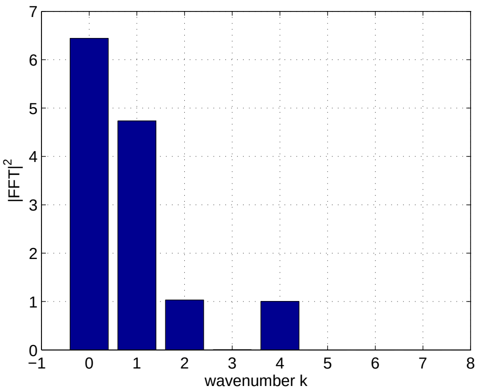
e $\boldsymbol{\theta}$ R $\boldsymbol{\vartheta}$ R $\mathbf{x}_j$ j $\mathbf{x}$ j , , , $\mathbf{N}$ − $\mathbf{x}$ $\mathbf{L}/\mathbf{N}$ d p c n

nd $\mathbf{L}$ en o do 1 n e $\mathbf{x}$, $\mathbf{L}$ $\mathbf{L}$ d

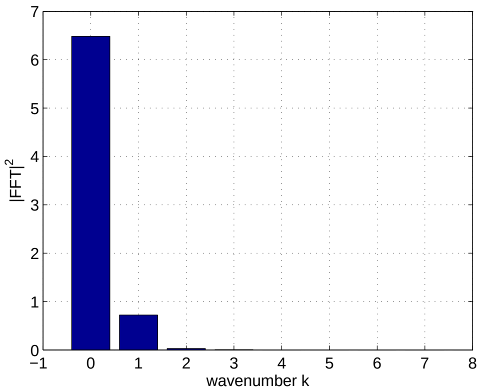


ed n e o c e o LAM do n c o
p en e **m** A c n e een ne e on p en do n
en e o e en n p en do n e e on
LAM do n en e on y one e en n LAM do n

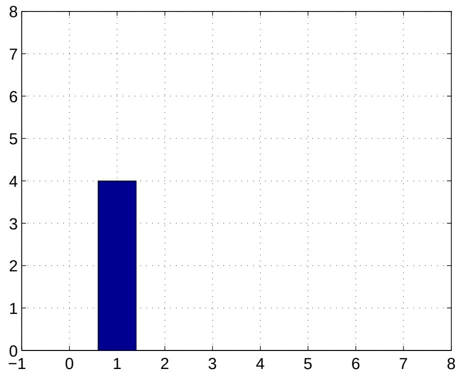
No e nde nd e ence e een en e on do n **L**
nd e on do n **L <** o e e e c n con de en **L <**
no n e e e pec p de o ppe n po e pec



(a)



(a)



n $\pi j/N$ e e con de de n on o n o e de

nce f_j ne co n on o ne e o f_j c n e en

$$f_k \quad a_k \quad b_k \quad c_k \quad d_k,$$

e

$$a_k \quad \frac{}{i} \sum_{j}^{N-} e^{-i\pi j\, k-L\, /N} - e^{-i\pi j\, k+L\, /N},$$

$$b_k \quad \frac{}{i} \sum_{j}^{N-} e^{-i\pi j\, k- \ L\, /N} - e^{-i\pi j\, k+ \ L\, /N},$$

$$c_k \quad \frac{}{i} \sum_{j}^{N-} e^{-i\pi j\, k- \ L\, /N} - e^{-i\pi j\, k+ \ L\, /N},$$

$$d_k \quad \frac{}{i} \sum_{j}^{N-} e^{-i\pi j\, k- \ L\, /N} - e^{-i\pi j\, k+ \ L\, /N},$$

e con de k By n k nd L / n o e on e e b c d o Le o e e a no n o o o on y e on o e e d e en e en n n e on o nd pope e o e e cop c a c n e en

$$a \quad \frac{}{i} \sum_{j}^{N-} e$$

o c e e o on $\mathbf{N}$ e

$$\mathbf{a} \quad \frac{}{n\ \pi/} \quad co\ \pi/ \quad - co \quad \pi/$$

c o d e pe on po e pec $\mathbf{k}$ o p de co e pond n o e e ee $\mathbf{k}$ n e nd e

Ne e con de $\mathbf{k}$ By n $\mathbf{k}$ nd $\mathbf{L}$ $/$ n o e on e n e $\mathbf{c}$ $\mathbf{d}$ o Le nd o e $\mathbf{b}$ $-\mathbf{iN}$ A n e e $\mathbf{a}$ c nno e p ed n o on y e on nd e

$$\mathbf{a} \quad \bar{\mathbf{i}} \sum_{\mathbf{j}}^{\mathbf{N}-} \left(\mathbf{e}^{-\mathbf{i}\pi \mathbf{j}/\mathbf{N}} - \mathbf{e}^{-\mathbf{i}3\pi \mathbf{j}/\mathbf{N}} \right).$$

o $\mathbf{k}$ e e

$$\mathbf{f} \quad \mathbf{a} \quad \mathbf{b}.$$

A n o c e e o on $\mathbf{N}$ e

$$\mathbf{a} \quad \bar{\mathbf{i}} \sum_{\mathbf{j}} \left(\mathbf{e}^{-\mathbf{i}\pi \mathbf{j}/} - \mathbf{e}^{-\mathbf{i}3\pi \mathbf{j}/} \right)$$

c o d e n p de n po e pec $\mathbf{k}$ co e pond n o e e ee $\mathbf{k}$ n e A o

$$\mathbf{a} \quad \mathbf{b} \quad \bar{\mathbf{i}} \sum_{\mathbf{j}} \left(\mathbf{e}^{-\mathbf{i}\pi \mathbf{j}/} - \mathbf{e}^{-\mathbf{i}3\pi \mathbf{j}/} \right) - \mathbf{i}$$

c o d e n p de n po e pec $\mathbf{k}$ co e pond n o e e ee $\mathbf{k}$ n e

e con de $\mathbf{k}$ n e y e e $\mathbf{b}$ $\mathbf{d}$ $\mathbf{c}$ $-\mathbf{iN}/$ nd

$$\mathbf{a} \quad \bar{\mathbf{i}} \sum_{\mathbf{j}}^{\mathbf{N}-} \left(\mathbf{e}^{-\mathbf{i}3\pi \mathbf{j}/\mathbf{N}} - \mathbf{e}^{-\mathbf{i}\pi \mathbf{j}/\mathbf{N}} \right).$$

o $\mathbf{k}$ e e

$$\mathbf{f} \quad \mathbf{a} \quad \mathbf{c}.$$

o c e e o on $\mathbf{N}$ e

$$\mathbf{a} \quad \bar{\mathbf{i}} \sum_{\mathbf{j}} \left(\mathbf{e}^{-\mathbf{i}3\pi \mathbf{j}/} - \mathbf{e}^{-\mathbf{i}\pi \mathbf{j}/} \right),$$

N | | p de n po e pec

doe y N nc e e o e y o e o on on co d e n c n nc e n N o o nc e e p de k y O e o on nc e e p de k pp o c

o n e e o y o on e n o on y en o k e no con de de n on o o ene LAM do n o en L

6.3.2 A general case

con n o o e e e pp o on o nc on ny no n p ope e nc d n con e ence e o pec c c e o nc on d c e e o e e e n e po on con e ence p ope e e y o e o con n o c e O p c n e e e c d c e e c e e e y p o c e o con n o c e 7 p 7 on

Theorem 4.1 For any $f \in W_p^r(0, 2\pi)$ with $r > 1/2$, there exists a positive constant C, independent of N, such that

$$\| f - P_N f \|_{L^2(0,2\pi)} \le C N^{-r} \| f^{(r)} \|_{L^2(0,2\pi)} .$$

nhu

where $W_p^q(0, 2\pi)$ (

2 0 ξ J 536710.9091 Tf3.514984 719.6R43 1r(68699 10.9091 T 4.22969 0 Td [( )0.352268]TJ /R157 7.970= Tf 4.22969

y e ze o p de **k** o n e e e no pe o ne n o
on nc on f_j o e on

6.4 The sine transform

o ec on o pe od c nc on **f** f_j de ned on en do n d ded
n o e y p ced d c e e n e d p o n n e ed **j** , , , N nd
f f_N o e ne n o de ned o e

ne n o f_j f_k $\frac{N}{N}$ f_j n $\pi jk/N$,

e **k** en e

o e e o **N** ne n o c n e c c ed n e c en code
e o o e e d e o ze o ec o e n con de ed e
e c n p y pe o ne n o p c on de c ed n
ec on

o n de nd e e ne n o e y con de n c c on o
de n on

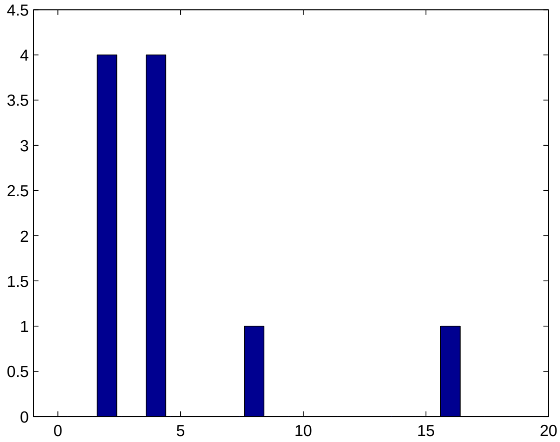
6.4.1 Calculating the sine transform from its definition

e e n y con de n e ec o ne n o on ene ne e e
d d o n ec on e e ene ne e en e **θ** nd
p de **α**

$$\phi_j \quad \alpha \quad n \quad j$$

n e den y

n


4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
5
10
15
20

Con de n **k**

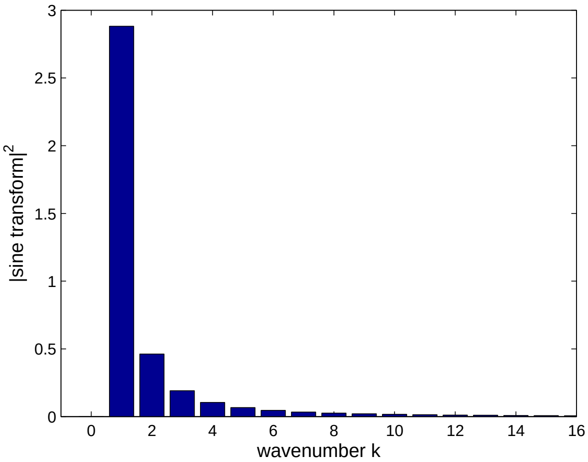
e o do n e ec on en e c ed y ne n o n e en e con ned y f_j co e pond o e en e n po e pec n L / e o e e n e d con de L /

6.4.4 The sine transform when $L = 1/4$

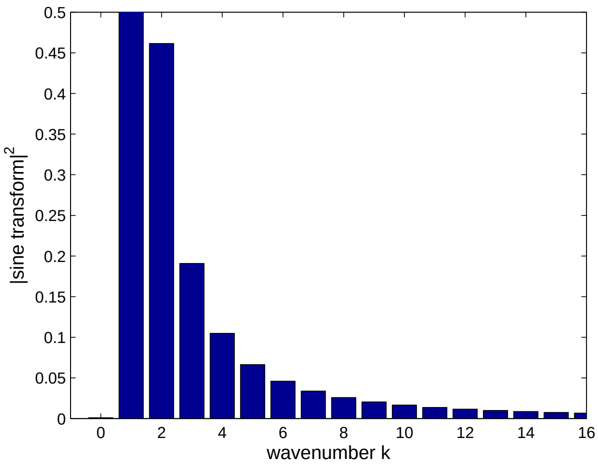
Be o e e pe o ne n o e need o no e e pec o ec on e no en e n f_j on do n L / e κ / , / , , nd o ec on e no ne n o p od ce pe n po e pec k κ e e o e e pec pe n po e pec k / , , , o e e p de c nno e ned o en e e no e n e n $\pi j L/N$ e n f_j oo on o e ep e en ed nd e e e o e nc e o e e e ed y ne n o

po e pec o f_k o e do n L / N n n e A c n e een e con de e p de k o e e e o e p de n e o d e pec k nce n $\pi j L/N$ e n f_j one e pec c n e pec e e con de ne n o o e ep e y po e pec n n e A c n e een o y o p de k c c de o p de o en e c c de e ec d e o con e ence p ope e o ne n o d c ed o n ec on

o nde nd e e e e con de n de n on o ne n o o e on n e n



(a)



(b)

Figure 6.6: (a) The power spectrum of the sine transform of 2 sin$(2\pi x_j)$ over the domain L = 1/4, scaled by a factor of 2/N. (b) Close up of 0 $-$ 0.5 amplitude section.

k nd en c c d n do n o e pec n co p on en n e

ne n o n de k o c y e o ze o nd o y o

on e n o on n e d en o en e k

Chapter 7

A D linear advection-diffusion model

7.1 The advection-diffusion equation

$$u_t \quad cu_x \quad \sigma u_{xx}, \qquad t \qquad , \qquad (7$$

Con de n nc on e o 7 7 e c n ee e on 7 o econd o de cc e n p ce o de cc e n e pp o on o od ed e on

$$\frac{\partial u}{\partial t} \quad c\frac{\partial u}{\partial x} \quad \sigma \quad \frac{Pe}{} \quad \frac{\partial \ u}{\partial x}, \qquad 7$$

e **Pe** **c** **x/σ** d ec e n e p Co p n e pp

o

$$\frac{t \sigma \quad c \quad x}{x} \quad , \qquad (7 \quad)$$

c en cond on o y

o no n e e e σ n e on (7) n e e

$$\nu \quad ,$$

nd d y cond on o d ec on e on p e n e d e c n e on (7) n e e

$$\mu \quad -,$$

nd d y cond on o e on p (7

No e no cc cy nd y cond on o c e e c n pp y d c e e e on o p en nd LAM do n

7.3 The model design

e no e d c e e e on (7 o pp o e e on (7 on p en nd LAM do n On p en do n e e u^P x^P t^P e pe e p ce nd e coo d n e e pec e y nd on LAM e e u^L x^L t^L

7.3.1 The parent model

p en do n x^P , d ded p o con n N p dpo n nd e e $\mathcal{T}$ p en e ep On p en d e on (7 eco e

$$u^P_{j,n+} \quad \nu^P \quad \mu^P \; u^P_{j-\;,n} \quad - \nu^P - \mu^P \; u^P_{j,n} \quad \mu^P u^P_{j+\;,n}, \qquad (7$$

e

$$\nu^P \quad \frac{c \quad t^P}{x^P} \quad \text{nd} \; \mu^P \quad \frac{\sigma \quad t^P}{x^P},$$

o nd y cond on

$$u^P_{,n} \quad u^P_{N,n}. \qquad (7$$

e e $u^P_{j,n}$ u^P x^P_j, t^P_n x^P_j j x^P nd t^P_n n t^P d p c n x^P $/N$ nd e ep t^P $. /\mathcal{T}$

7.3.3 Convergence of the model

No e e ode e need o c ec con e e nd con e n o e p e e e n cond on

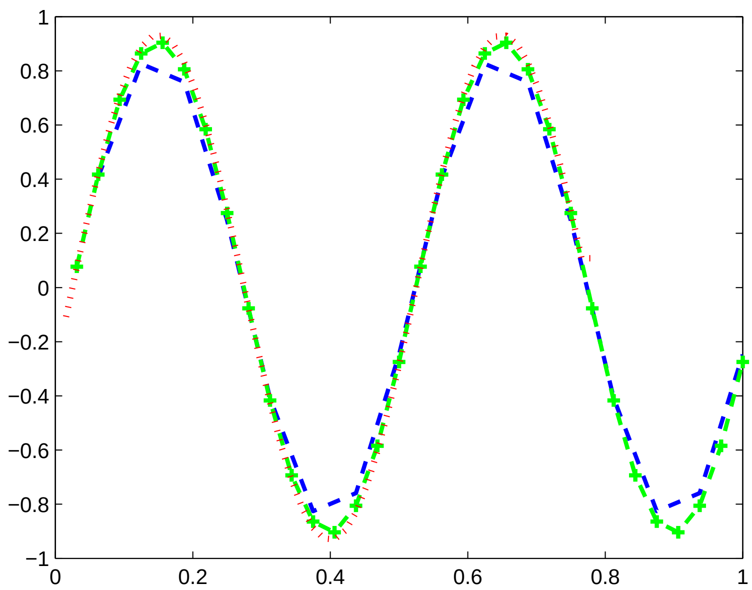
$$u\ x, \quad n \quad \pi x\ , \tag{7}$$

en n n o e on 7 e

$$u\ x, \quad n \quad \pi x \quad \sum_{n\ -\infty}^{\infty} a_n e^{i\ \pi n x}. \tag{7}$$

n o e o on y e on p e

$$a_n \quad \begin{cases} -i/ & \text{en } n \\ i/ & \text{en } n \\ & \text{o e} \end{cases} \tag{7}$$


1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.2
0.4
0.6
0.8
1

Chapter 8

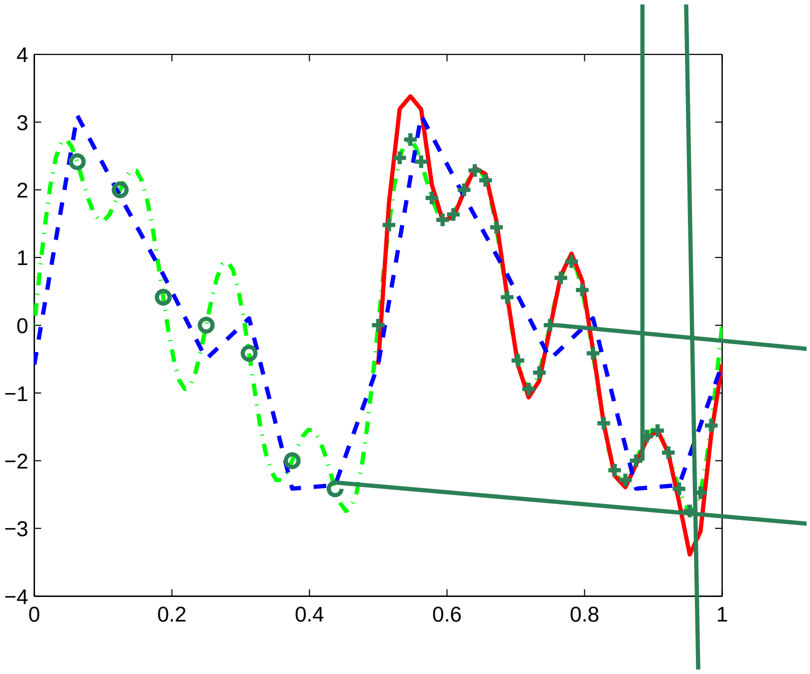
The representation of different scales in a 4D-Var analysis on a LAM domain

do n e e en oo o e e o ed y p en ode e o d e pec e o e e ed o e cc e y y LAM d on

n cond on o p en c e

$$u^P\; x_i^P, \qquad n\; \pi x_i^P\; .$$

e nd o n y e n p y c p ce n e t A c n e een e o on o LAM co p ed p en o o p c p e o on e e n c nno e c p ed



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0
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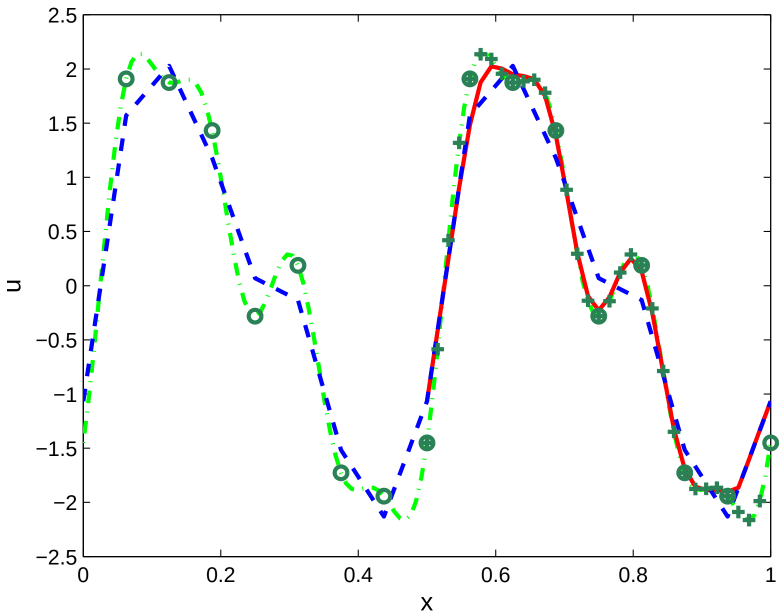
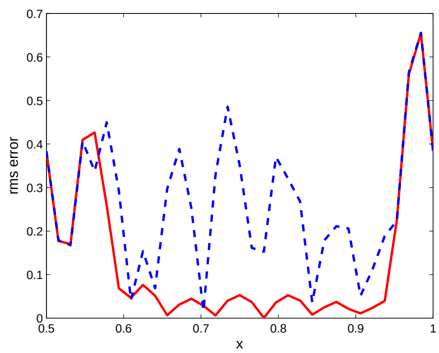
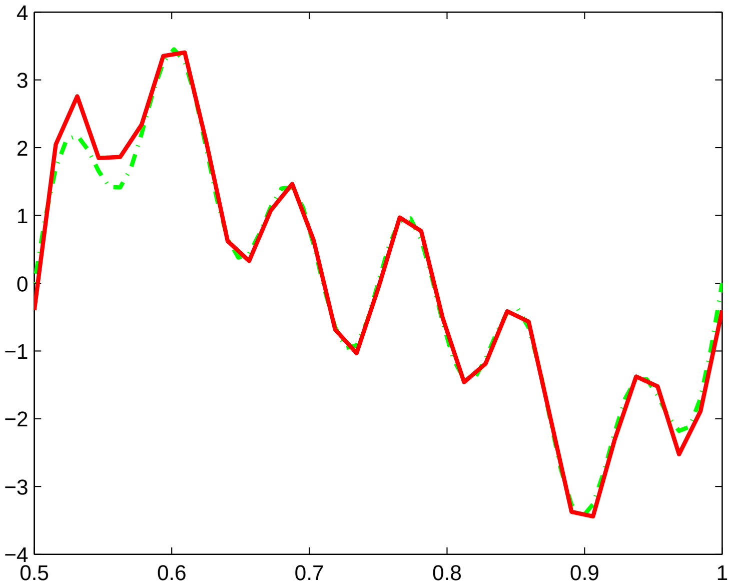


(a) Model outputs at the middle of the assimilation window.



e LAM con de ed o e cc e ec e c p e **k** e
ed y p en o **k** e LAM on y n y e e n
p en nd o **k** e c y n y o e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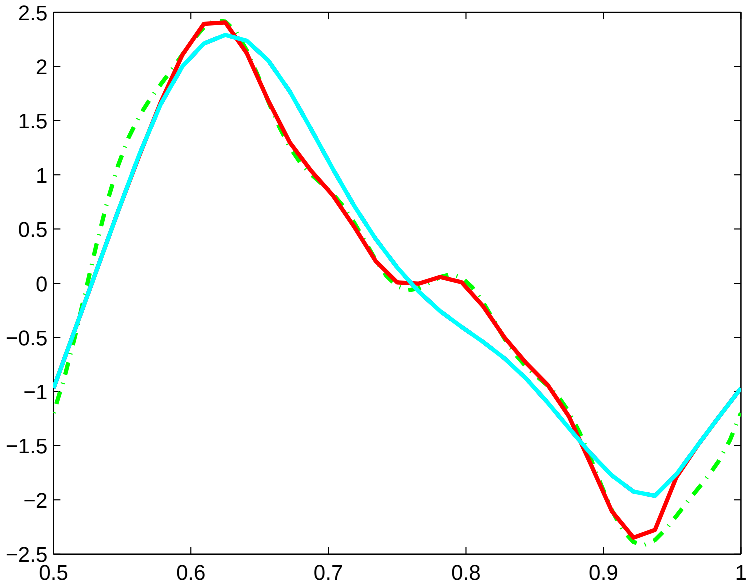

2.5
2
1.5
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0.5
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−0.5
−1
−1.5
−2
−2.5
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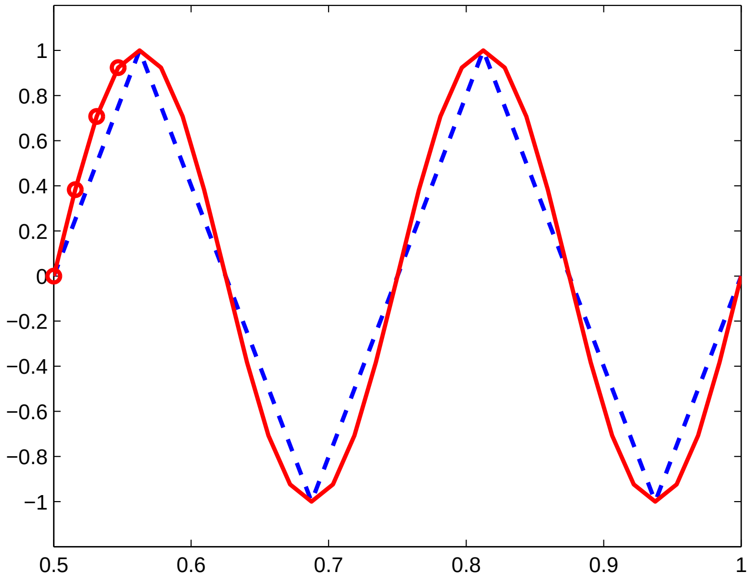
0.8
0.9
1

o ed een p en n y d e LAM n y ene ed n o e on ed nd LAM ode n d on e n o de o con de ene o e o on o e on LAM n y ene ed o e on een y p en e o e o on o e on o p o ed p n

Con de n **t** e e n y no n LAM ode n d on c p en n y e c y n p y c p ce e o d e pec d e o p en n y p o d n n cond on o LAM o e e e con de po e pec o e LAM do n n n e e ee po e pec o e o o p doe no c n pec p ce d e ence c ed y e o on o o p A y ppe e n p y c p ce LAM ode n o p en n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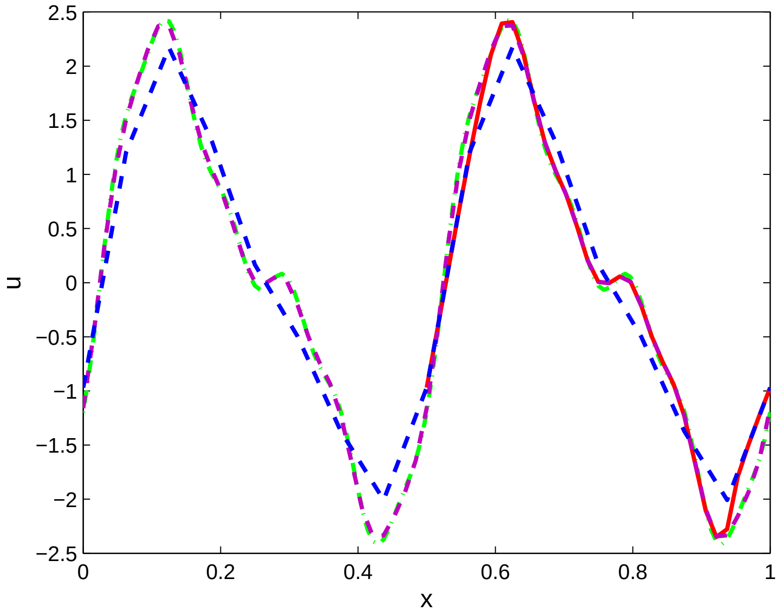
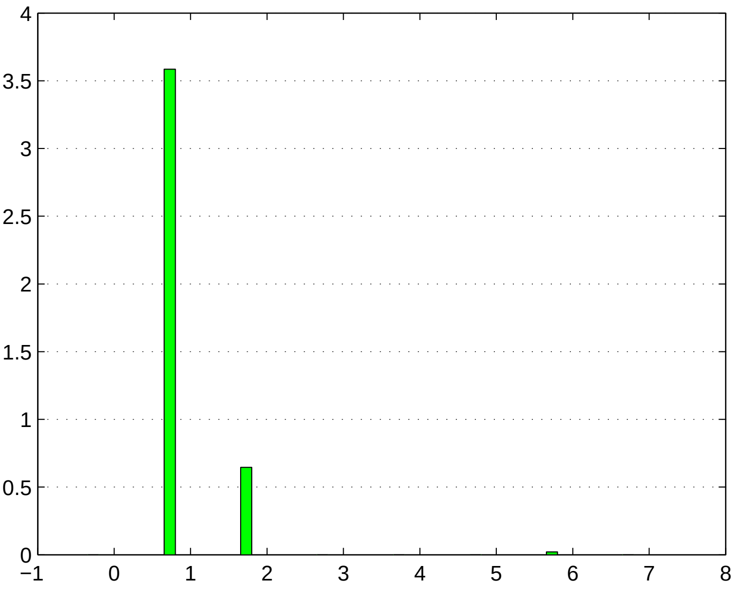
0.8
0.9
1

e o no ce o e LAM n y ene ed o e o on o e on ed o c p e e o on de nd e n e o p en n y d e o n o on con en o o e o on o e on On p en d e o on de n o e on ed o o en e n c e o n π x_j^r e n o on ed o en e **k** n e e e o e o on o e on on LAM d e no e o on o e o e c e e e e o on n o on c e e ed

e no con de dd e o on ndo e ee n e LAM ode n d on no on e c p en n y d e o e n n d e en e o on o e e ode n d on doe c LAM on n o e o on o e on c e y

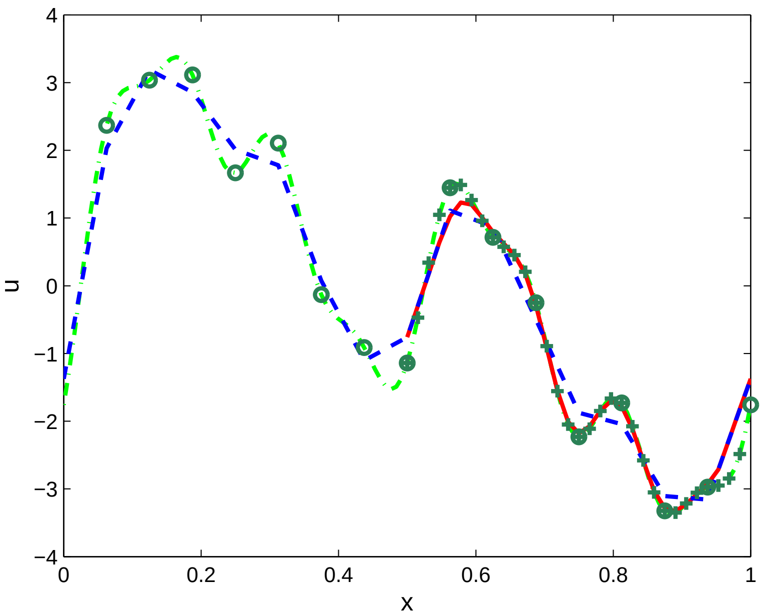


(a) Model outputs at the middle of the assimilation window.

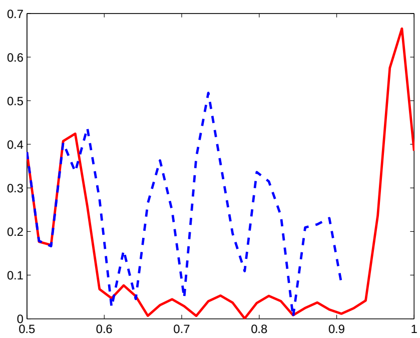


en e d e o e LBC

e e o n od ced ec e o e LBC e n po n po n o no e ec e e ed o e e on e e no con de o e e on e e e ed on



(a) Model outputs at the middle of the assimilation window.



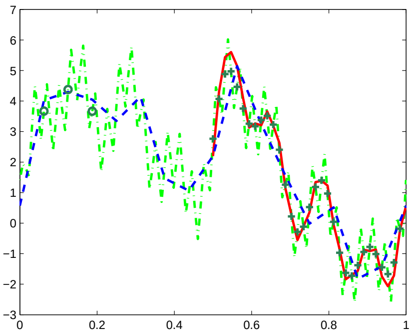
po e pec o e n y e co p on n n de
e een nd n y e po n c p de oc on

n e c e ee LAM c p n ed n o on **k**
e e p en n y doe poo e o o e e p en n y o e
cc e **k** nd n cc c e **k** e o e n de n
o n y e LAM o e e e e p en nde e e

e no con de dd e o on ndo e ee o e
LAM n y n ppe o o e c o e y n p en
con ed y e o n n e A e o e e e o n
LAM e c o e o o nd e c ed y e o n od ced o LBC
Loo n po e pec n n e c e ee dd e o
on ndo LAM c o e c o e y en e

e nd o n y e n p c p ce n e
t LAM oo o e cc e n p en ppe o e pp n
pe o o e c o e y o e e e e con ned n
c nno e e o ed e en y LAM nd e e o on o c on n
e d c o d e cc cy o n y e y p y p o n e
e e o e con de e o o o n y e p o ed n e A c n
e een LAM n c n y o e cc e co p e o o ed dpo n
nd e o e e e o n LAM e c o e o o nd e

cc cy o n y e c n o e nde ood y con de n po e pec
po e pec o nd o n y e n n e c A e


7
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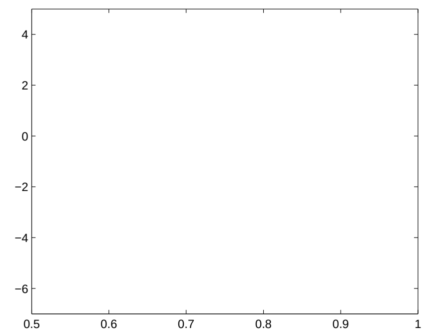
4
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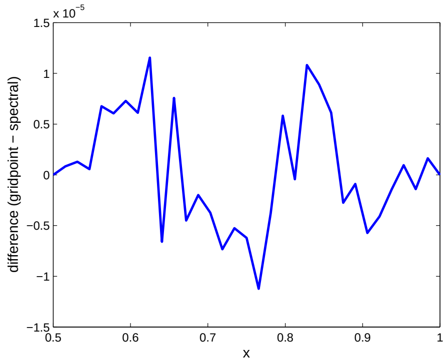
c n e een o ee o O e o e n o po n n e
on y pec LAM n y p o ed

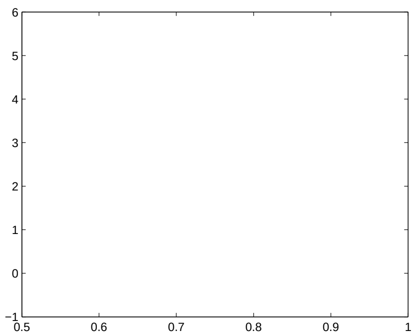
n e nd d p o ed on o LAM nd p en
n y e n e nd dd e o on ndo e pec e y
n o e ee LAM o e c o e y e o p c o
e o on de ed y p en con ed y po e pec
p o ed n e c nd e

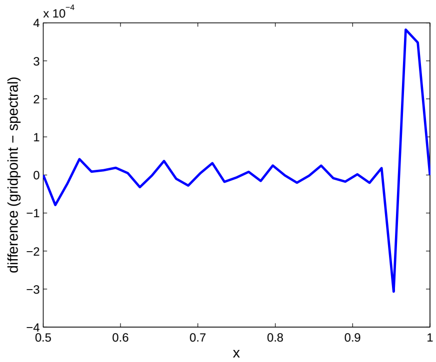
o no n e occ ed n ec on nd LAM
o e cc e n en e p c o en e ed y p en o e
en e n de c y o e cc e n p en n LAM o
e p e k p en o e cc e n LAM o e

po e pec n n e c nd e o o o e ne



(a) Difference between the spectral and grid-point analysis.





(a) Di erence between the spectral and grid-

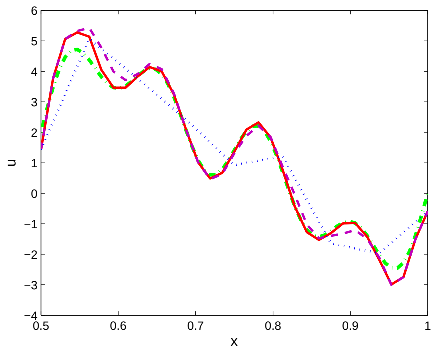
e odeop t eee pen ny

o e copen e n e on o pdc ne A d n d y

dd e o on ndo e den A o e LAM n y

o e cc e cc o en e LAM do n

e e $\mathbf{x}_0^{bP}$ e p en e o on c o nd $_1$e $\mathbf{t}$ nd $\mathbf{x}_0^{b}$ e LAM e o on c o nd $_1$e $\mathbf{t}$



(a) Model outputs at t = 0.

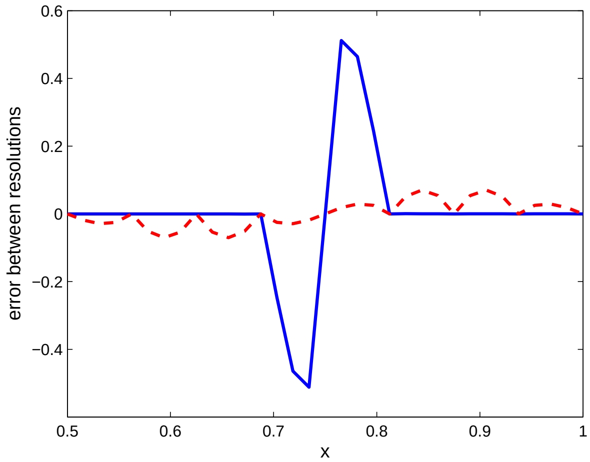


Figure 8.19: Error between model outputs with di erent resoltions for sine and tanh waves.

nd dpo n o o n nd ne e A c n e een ne e d e ence o do n e n e ee e dpo n o e e en n e doe d e d e y c e n c n n e o ne e e n cc cy c e no c e d e ence een n y

o e e pe e ee po n o ene e c o nd e o de n e e o on o on o e c ed o o e e de e n n e y e o on e n o e o on de o p 7 7 o 7 d

n y o e e e o ed o do LAM d on e e
e o on o e on

e o no e e e e en pe ec o e on nd no c o nd e n
co nc on LAM n y c nno co ec y econ c d e o
n ence o c o nd ec o y LBC nd e zone e e o n
e e o c ed y LBC ec o en e o LAM pec

e o ode en e d e e o on LAM n y o
e o n od ced y y n o co pen e o d e ence n d on e co p ed
o e e o occ on en e en e e
ec ed o e e de ee d e o e c e d n y o e p d y

n o e on con n no on o e on e n do no
ode e on e e ed on o o en e o y o
n on o on e e con ned n pec o ode n p c
n n n c n o n o on e n o on ed on o
en e k n con n n o e ne n o k y
7 e 7 7 n 7 e o ze o nd r e d 7 7 n eey o e y o o e o 7 7 7 7 7

n 7 7 e 7 7 7
e o

n e o on o on o e c ed o p y n e po n co e de n n od ce e o n o LAM n y o e e no ed LAM e n cyc ed nd c o nd co n o p e o LAM o ec on y o nd y co nd on e n p o ded y p en de n c o nd e dy e o on nd e no on e n e

o C p e c o nd e o co nce een e o ze o e no o on o con de pec c e non ze o c o nd e o co nce e n e e e c n con o d on ec d e en en e y n p n c o nd e o co nce n pec p ce

Chapter 9

Manipulating scales in a 4D-Var analysis using the background error covariance matrix Σ

No e nde nd e e no on o d e en c e p o ec ed n con o e $\delta\mathbf{z}$ e $\delta\mathbf{z}$ e ed o $\delta\mathbf{x}$ y e on $\mathbf{B}$

$$\delta\mathbf{z} \quad \mathbf{W}\delta\mathbf{x},$$

e $\mathbf{W}$ ne n o co nc on no en y e on

$$J \; \mathbf{z}_0 \quad \delta\mathbf{z}_0{}^{\mathbf{T}} \; {}^{1}\delta\mathbf{z}_0 \quad \boxed{\mathbf{T}} \; {}_{\mathbf{k}} \; \delta\mathbf{y}_k - \mathbf{H}\,\mathbf{U}\delta\mathbf{z}_k{}^{\mathbf{T}}\mathbf{R}_k^{1} \; \delta\mathbf{y}_k - \mathbf{H}\,\mathbf{U}\delta\mathbf{z}_k \; ,$$

e $\mathbf{U}$ n e e ne n o nd ^{1} en y e on

$$^{1} \quad \mathbf{U}^{T}\mathbf{B}^{1}\mathbf{U}.$$

o e on e ee o pec c e e c n con de c o nd e o co nce n p y c p ce $\mathbf{B}$ o n pec p ce

c o nd e o co nce con o c o e on n o

9.1 How the matrix B corresponds to choices of the matrix Σ

e e o o de ne c o nd e o co nce n pec p ce o o pec y nce on d e en en e n e d o p y c oc on po n o con de c ce o e e o co e pond n **B**

e con de γ d γ_i nd μ d μ_i o y e dpo n e

γ_i . i , . i , . . . , . i , . . . , . i 7, . . . , i , . . . ,

nd

μ_i i , . i , . . . , . i , . . . , . i 7, . . . , . i , . . . , .

n γ e e p ced e y nce on e c e nd e nce on c e co e pond o e o d e pec o do n o ode con n n e c e o c c o nd e o n o e on o n ence e c e A c n e een n e co e pond n $\mathbf{B}_\gamma$ d p y n o c n c e e n co e on e n e e en e o d pe ood n nd n n e on

n μ e e p ced e nce on e c e nd nce on c e e n co e pond n $\mathbf{B}_\mu$ e e n o e d on **B** co e on on e c e c n e een n e

c n e een y con de n n e o o **B** no n c e nc on nd e no on p e d y **B**

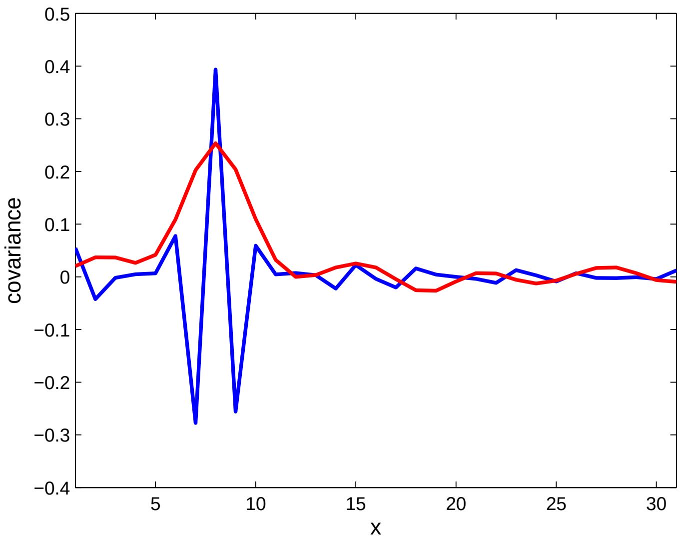


Figure 9.2: Plot of row 8 of the matrices $\mathbf{B}_\gamma$, in blue, and $\mathbf{B}_\mu$, in red.

c e nc on o $\mathbf{B}_\gamma$ nd $\mathbf{B}_\mu$ e p o ed n e A e ee e ne co e pond n o $\mathbf{B}_\gamma$ o c e p d y c e n nc e en on c e n co p on ed ne co e pond n o $\mathbf{B}_\mu$ oo de p e d c e n nc e en on e c e e e p o n e y e e c n e c c e e ec ed n LAM n y y o c ce o e o e o e n d e en c ce o n ode n e [1] e n n ec on

9.2 Initial tests with a non-zero background error covariance matrix Σ

e e n y n e n y o c ce o e c n con o c en e e ec ed y d on

e n pec c e [1] no c o nd e n co nc on [1] nce on c e [1] d α_i nd [1] d β_i e

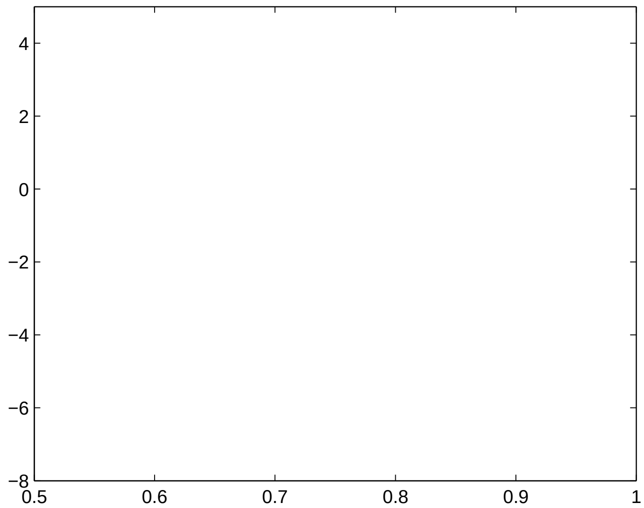
$$\alpha_i \quad i < M/ \quad i \quad M/ ,$$

no c o nd e o e c e nd nce on c e nd

$$\beta_i \quad \begin{cases} & i < M/ \\ & i \quad M/ , \end{cases}$$

no c o nd e o c e nd nce on e c e **M** n e o LAM dpo n

n o e pe en on y n o c n e [1] e e y n e e ep e e e ence ec o y **t** ne co n on o ne e coe c en one nd en e **κ** , , , ,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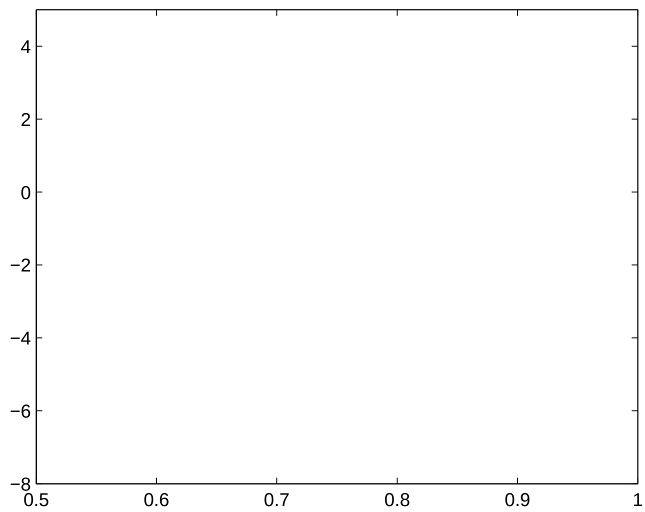
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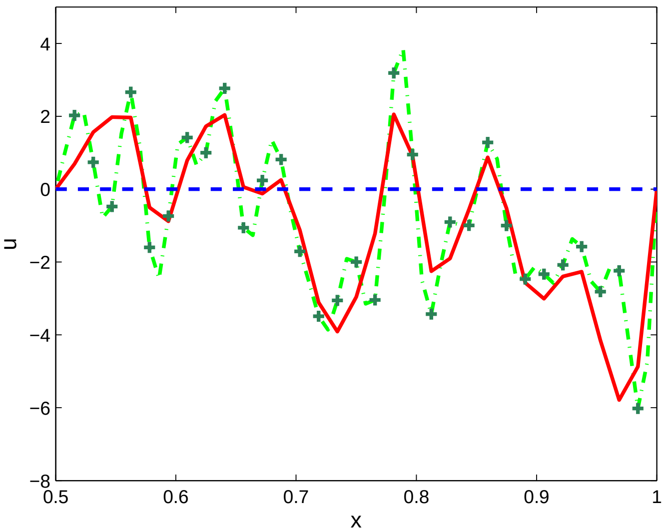
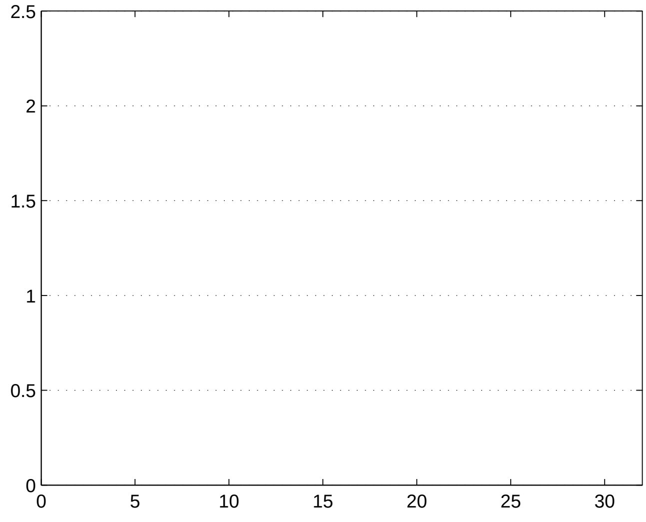
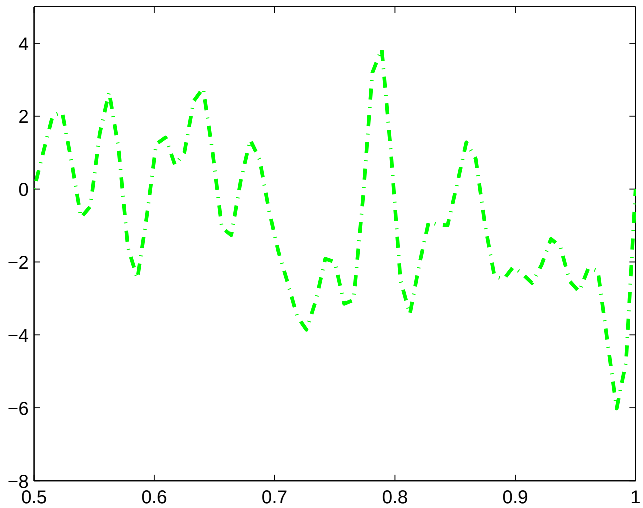


(a) Model outputs for the case with Σ_{α}^{1}.





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−4
−6
−8
0.5
0.6
0.7
0.8
0.9
1

c ed y d e ence n e o on co p ed o o $_0^1$ c e e

no c o nd e n co nc on o ze o p de o c o nd

ec o y no ec n po e pec o n y e cep o LBC

o e e n e e ee o ^{1} c e p de n n y

c o e n o n een n enced y ze o c o nd

e o e on

o c e p nce $_\alpha^1$ nd $_\beta^1$ e o d ypo ze po e

pec d e e e o po e pec o $_0^1$ c e nd o one

o $_1^1$ 1
1

9.3 More realistic examples with a non-zero background error covariance matrix Σ

n de on ed n ec on po en o e o n ence d e en en e e no n o y y n n e e c c e

e e o e e ence ec o y o e

$$\mathbf{u^r}\ \mathbf{x_j^r} \quad n\ \pi\mathbf{x_j^r} \quad n\ \pi\mathbf{x_j^r} \quad n\ \pi\mathbf{x_j^r}\ .$$

co ne one on e nd o e c e con ned n po e pec o LAM one o en e nd one

e e ence ec o y ed o p o de o e on o e e e n C p e e ed pe ec o e on e no dd ndo no e o o e on ndo no e nce σ_0 . on en e nd o e on e o co nce R σ_0 I e I den y e e e σ_0 . o o o e on e o nce ec e yp c e o e o e nd n o d ec on d on de de e e pe e e e o e on e e y LAM dpo n nd ep

A n C p e e e p en de o e N dpo n nd e e p en ep n on ndo LAM co e nd de o p en d n dd e dpo n LAM o p e o on o p en de nd e e LAM ep o e e y p en ep e e ence ec o y ce p e o on o LAM nd o e po e o on d on con n σ . no d ec on peed c

9.3.1 A background $\mathbf{x}^b$ with no random noise

en p e p on on n e p

c o nd ec o y n ene ed y nn n p en de o d o e n cond on c o nd n n e po ed o LAM d nd e n LAM e o on n o nd y cond on p o ded o p en e o on n ene n c o nd ec o y n y ee e p e c o nd p o ded y p en de co e e o on e do n ze n LAM

y n c ce o e n d on pec c o e on c o nd nd c o nd ec o y o co p e c ce o ec d e en c e n n y

e e $_0$[1] no c o nd e n co nc on **A** . **I** e nce on en e **α** d α_i nd **β** d β_i e

$$\alpha_i \; . \quad i \quad ,\dots, \qquad . \quad i \quad ,\dots, M \; ,$$

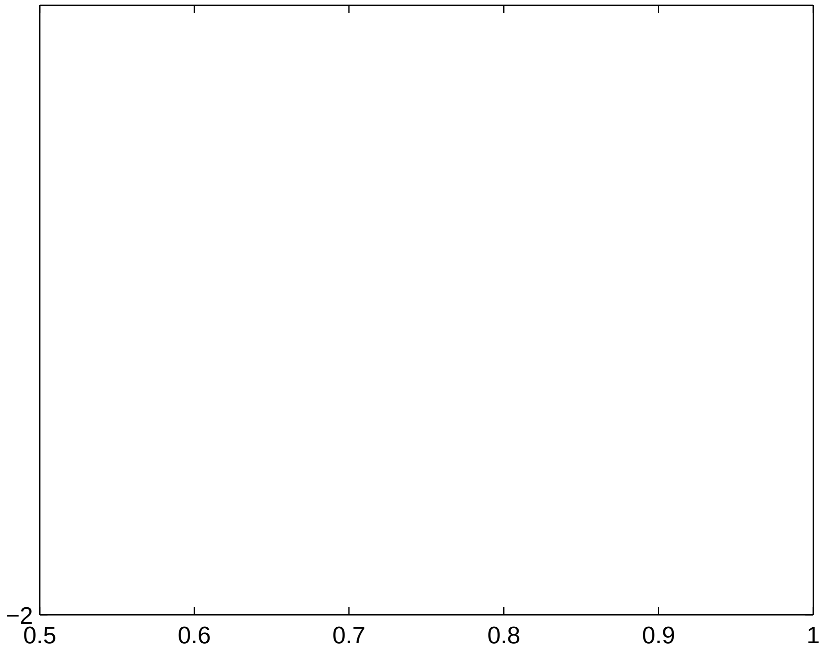
nd

$$\beta_i \; . \quad i \qquad . \quad i \quad ,\dots, \qquad . \quad i \quad ,\dots, M \; .$$

β e y nce on en e **k** nce on o o e en e nd nce e o on o e on on en e **α** e nce **β** on en e n e e y nce on o e en e n e d o **k**

e c o e **α** nd **β** e ec e d c ed n ec on LAM o e e e o en e d e o LBC o en e o con n ed on e n o on n n ec on By p c n e nce on o en e e o con n e c e o c c o nd c ed o cc e y c p e e c e en e e en e nce o e n o e on e e e n ence on c e

e **7** de o p o o c ce o o n y e e p o ed o e o e c o on o en e o e co p ed e e y e e ence ec o y n n een c o nd n e


−2
0.5
0.6
0.7
0.8
0.9
1

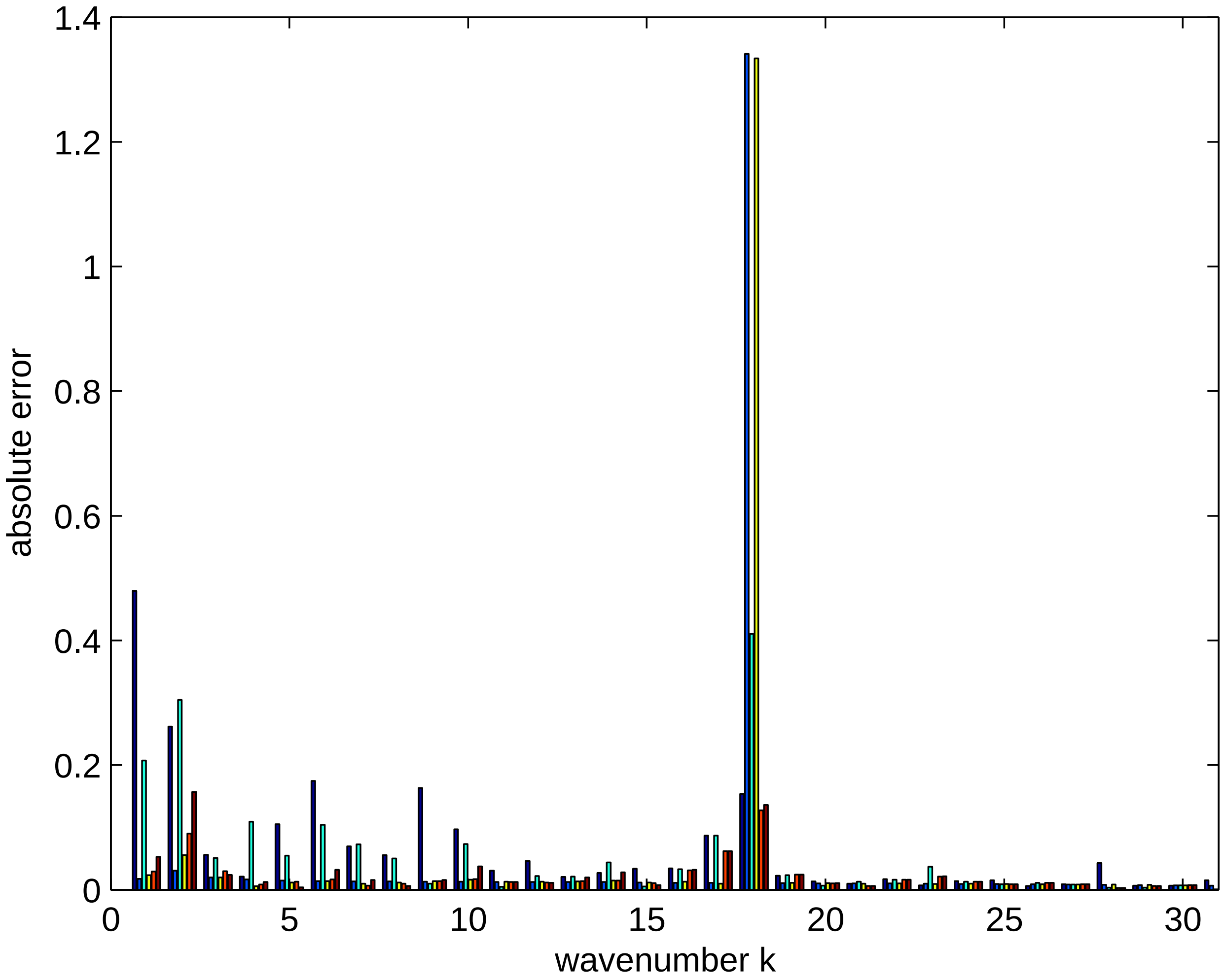
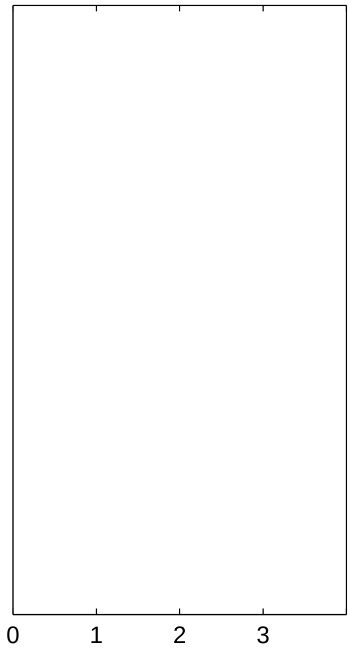


Figure 9.8: Errors in the power spectrum at t


0
1
2
3

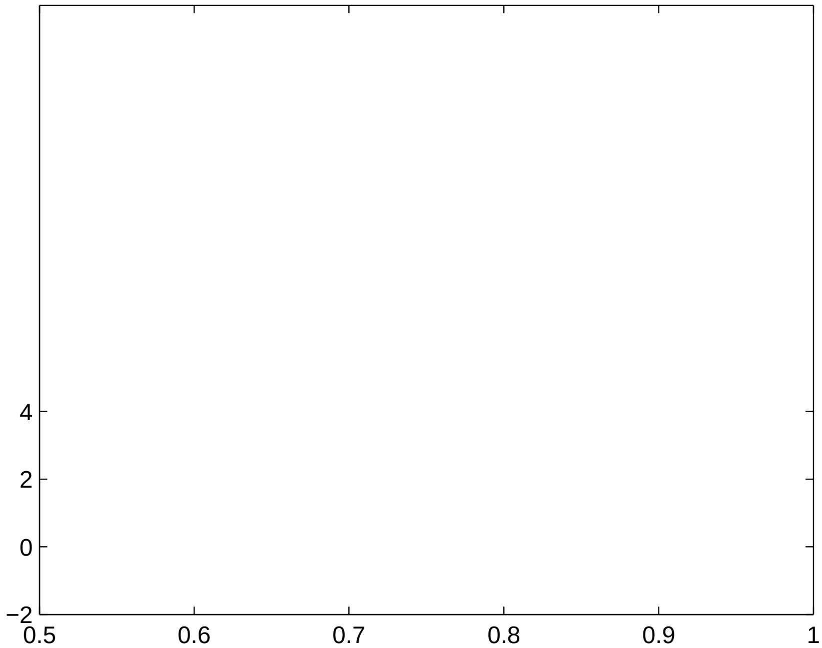
y c o e o on n o on n c o nd e p o e e o

e e o n y e ec e e c n e e e o o nde nd e o

n n y e.

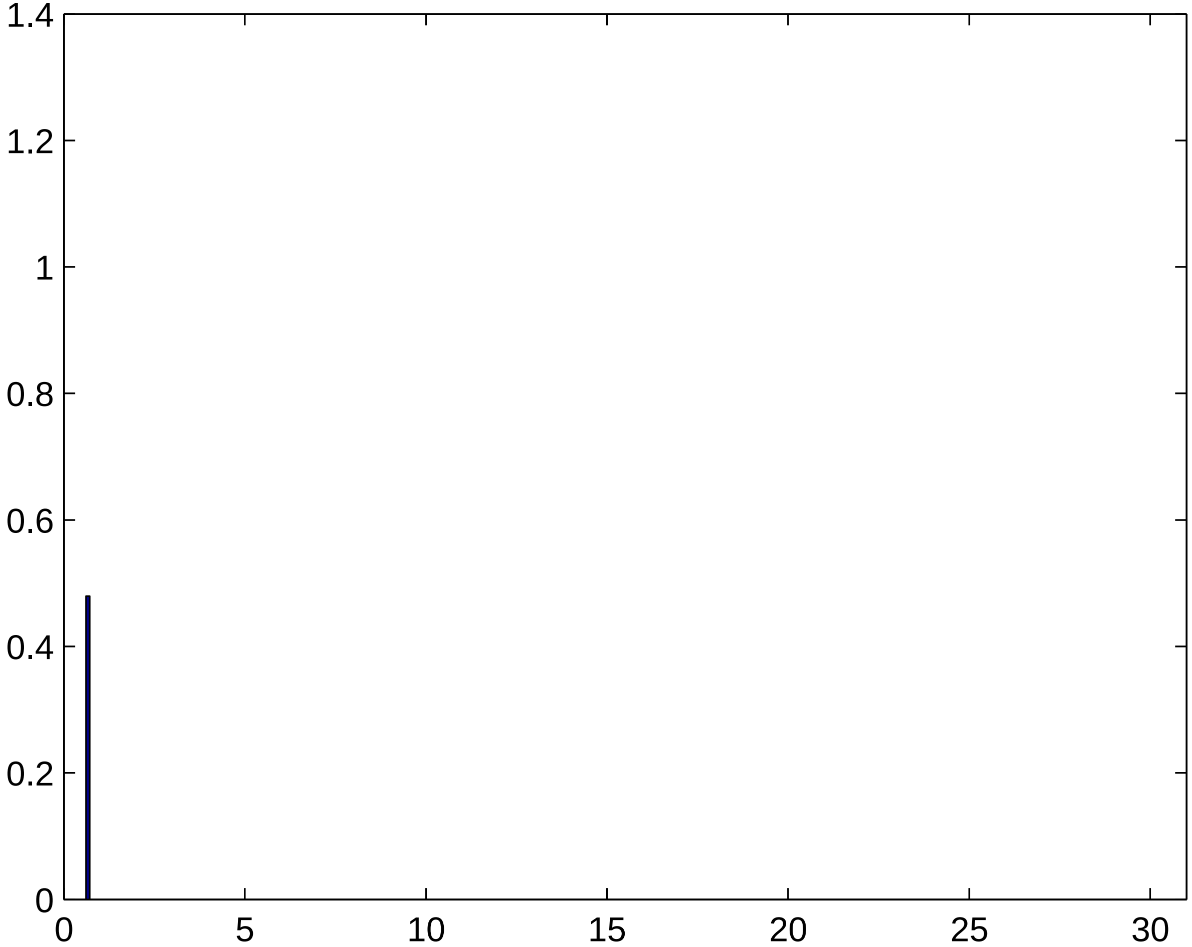
e con de $_0^{\,1}$ c e n ^{1} e n e o ze o e n

n pec o p en ode e ne n cond on e c o nd
o LAM c o nd ec o y n ene ed y nn n c o nd



4
2
0
−2
0.5
0.6
0.7
0.8
0.9
1

A e o de pec n y ene ed 0 1 n c o e y o e on d e o n no c o nd e n co nc on n co p on n y ene ed A


1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30

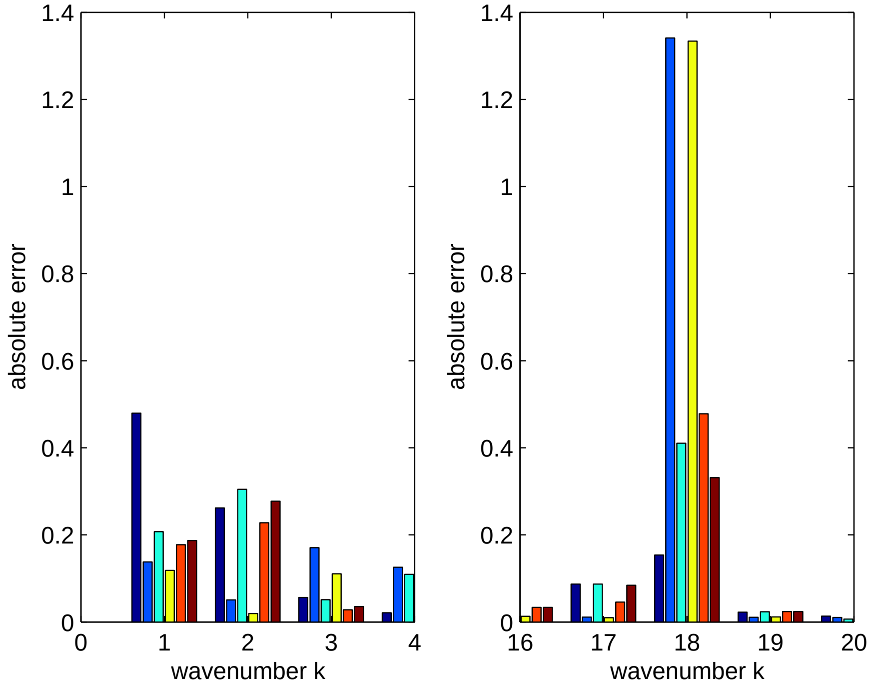


Figure 9.12: Errors in the power spectrum at $t = 0$, a close up of $k = 0, \ldots, 4$ and $k = 16, \ldots, 20$ for the observations (dark blue), the background (mid blue), the analyses generated with $\Sigma_0{}^{1}$ (light blue), Σ_A (yellow), Σ_B (orange) and Σ_γ (red). A close up of $k = 0, \ldots, 4$ and $k = 16, \ldots, 20$.

c y y o e n o c o nd de on e po y o p o n ep e en on o o en e on LAM e en co p ed o c o nd y o e con n n o en e n ne

e c e e cc e y ep e en ed n c ond c e e no
Co p n n y B c e c e nce n y on
en e e ee e o y e o k
ene y o e e o o en e de on e need
o nce e e c ond nce en e n y e
o ed o e on o n ence k n no c y e e n y
c e co p ed o B c e e nce e o o ed e o n
o e on o ne e y n ence n y o en e
e c ond doe no cc e y ep e en c e
po n o no e o e y e nce n

By o e con n n o en e A n y o e cc e n
c o nd o en e By p n c e n y e c n con n
o en e o c c o nd e o n o e on o n ence
en e e e po y o pec y n n c
y o en e LAM n y o cc e y c p e c e p e en n
e o e e no y e c ce o e n nce on d e en
en e

Co paring choices of with di erent variances on di erent wavenu ers

e o c o e nce n n c y o ed ce e o n
LAM n y p c y o en e o ec on e no
LBC c e LAM o o e e e o en e o ec on e
no on e n o on o ed o o en e By p c n
e nce on o en e e o con n e c e o c
c o nd c ed o cc e y c p e e c e
en e e en e nce o e n o e on e
e e n ence on e c e e no co p e d e en c ce o
A e nce p e een d e en c e c nce ed
d e en c e e e een

e e μ d μ_i ν d ν_i ω d ω_i nd η d η_i e

$$\mu_i \quad . \quad i \quad ,\ldots,7 \qquad . \quad i \quad ,\ldots,M\ ,$$

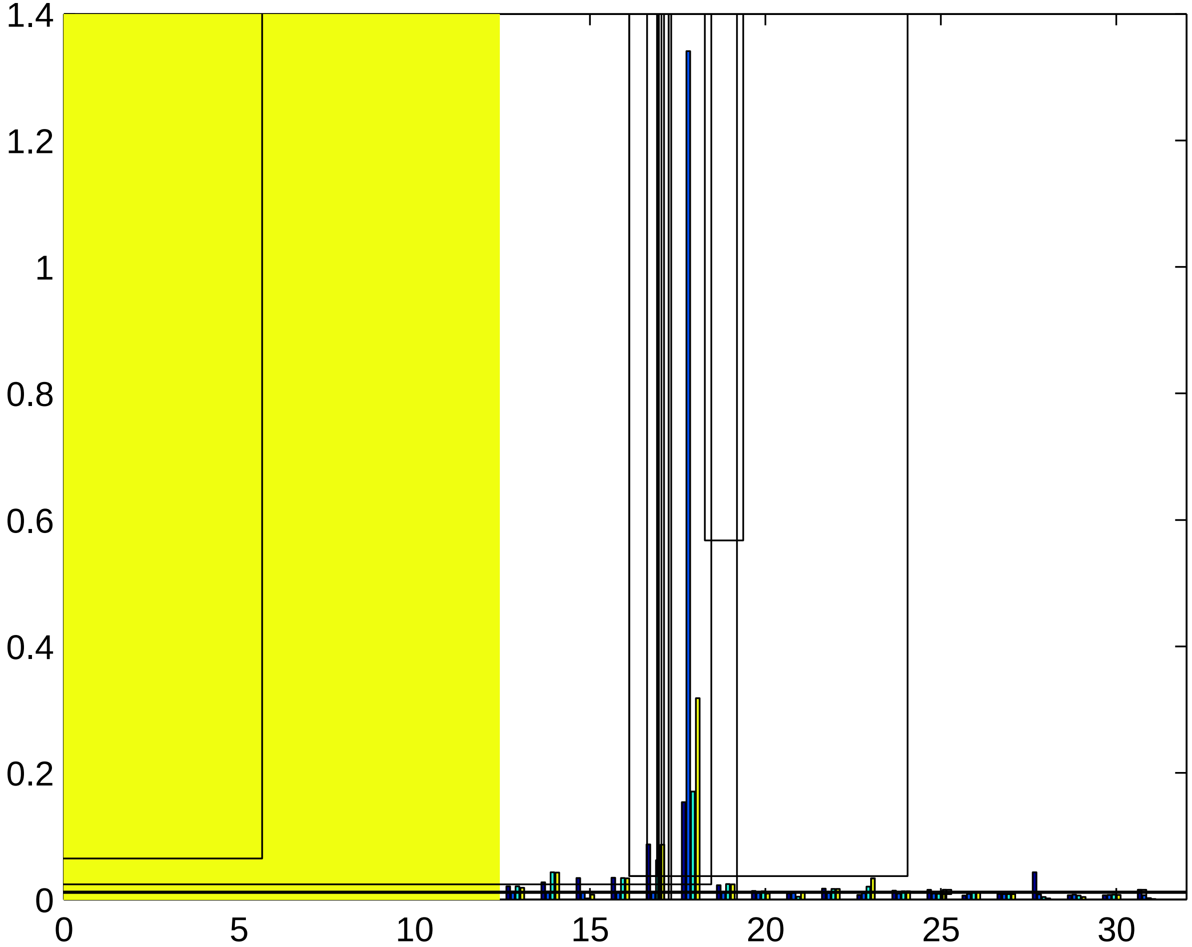
$$\nu_i \quad . \quad i \quad ,\ldots,7 \qquad . \quad i \quad ,\ldots,M\ ,$$

$$\omega_i \quad . \quad i \quad ,\ldots,7 \qquad . \quad i \quad ,\ldots,M\ ,$$

$$\eta_i \quad . \quad i \quad , \qquad . \quad i \quad ,\ldots,7 \qquad . \quad i \quad ,\ldots,M\ .$$

A e n e e e e een d e en n y e e d c o o e e n p y c p ce e e o e do no p o $_1$ de o p n p y c p ce nd con de o e e o n po e pec

e o e e o n po e pec A e o e nce co $_1$ n e e y p c ed nd d c o ee c o e p o en $_1$ e con ned n



1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30

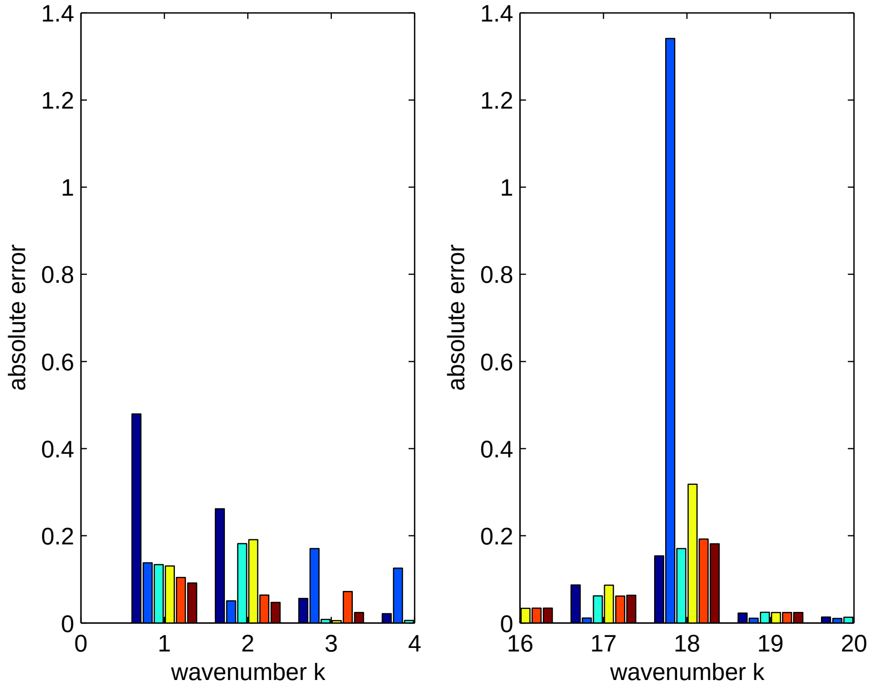


Figure 9.14: Errors in the power spectrum at t = 0, a close up of k = 0, . . . , 4 and k = 16, . . . , 20 for the observations (dark blue), the background (mid blue), the analyses generated with Σ_μ (light blue), Σ_ν (yellow), Σ_ω (orange) and Σ_η (red). A close up of k

po en y c e e oo n on e een e nd c e o e e o e n c e do no ene o n o on n p e o LAM n c o nd p o ded y p en ode

d d nd c con n e c e o e o ode p o d n o nd y cond on y e n o n e e J_k n co nc on Co p n o

e d e o LBC nd on e n o on o ed o o en e o n p en n y c n e o e cc e e c e e e e ed y p c n e nce on o e en e e c n no on y con n e c e o o no de de n LAM n y e c n c y p o e co p ed o c o nd A e e e c n cc e y ep e en c e y nc n nce on en e n c o nd on o e on

e o co p ed o e d o o done y d d nd c nd B d e o d ppe e on n n pec on ee e d e n e e c e n o on o c o nd o ene LAM n y e d c n o on o ed d e co p on o e d n dy

Chapter 10

Conclusions and future work

z do o e e e e o en c ed y con ec e c e e e need o p ed c e e en cc e y c e ed o n need n N o p o e o y o o ec on con ec e c e o cc e y p ed c con ec e c e e need e o on o ec de n o de o e o e do n n o on co ec y

c e de c nno e e o ed y p en de o e e o c e e

p o e en n c e e e e o on o e on

e o LAM d on

en pe ec o e on nd no c o nd e n co nc on

LAM n y c nno co ec y c p e d e o n cc

e od n p o e e nd n dy o e e
e ene ed o e o e on

By n ne d ec on d on ode e e e e o con de d e en

10.2 Further work

A d c ed n ec on e o e o e n co d e pp ed ope on y y need o e e ed n non ne c e A e ne ep o d e o e e o nco po e o e non ne y n o o ode One po y o d e o de e op o d ec on d on ode n o ode o o n y e on e on con n d ec on nd d on e e e dy e

on M Z L Le n N o e nd B d p c

o nd C n Lo n M Co p on d d on e o n
op n n e y e

o on nd c o nd e o con n LAM ec n c epo
ec A e p o L cce ed
de Co e M nd A C Lo enc n ed no on o d on
Ope on e en nd on **J. Met. Soc. Japan** 7

e C d nd O nd On e end n o on on n non ne c o c y e **Tellus** A

e A e o y e n nd B nne y N e c ec pe n O AN o c en c co p n nd d on C d e n e y e

o B Be ey nd en A co p on e o o e z ono d on n y n c e y e **Journal of Computational Physics**

e Co e e O nd nd e c A co p on e een o d en on on on nd p ed e en on e y n on ee d en on on n y **Quarterly Journal of the Royal Meteorological Society**

e nen n e M o nd A on CM ope on p e en on o d en on on on pe en e p ed p c **Quarterly Journal of the Royal Meteorological Society**

ez nd en nd Concep en ce

e d On e d o con e d en o on o e y e o ne e on L e p e e o ne e on oceed n o O o d con e ence n e o M c nd Ap c on d n Ap Ac de c e

o e N Me eo o o c co ponen n o ec o e

7 Zo X nd Y o n on o n op con o o n nd
o nd y cond on n ed e e o c e ode **Monthly Weather Review**
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