The Valuation of Weather Derivatives using Partial Differential Equations

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Abstract

In this dissertation we derive and solve numerically a partial differential equation (PDE) for the value of a weather derivative. We use historical data to suggest a stochastic process that describes the evolution of temperature and cumulative heating degree days, and then use this process to derive a convection-diffusion PDE for

Acknowledgements

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I confirm that this is my own work and that the use of all material from other sources has been properly and fully acknowledged.

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List of Abbreviations

CDD	Cooling Degree Day(s)
EUR	Euro(s)
EWMA	Exponentially Weighted Moving Average
HDD	Heating Degree Day(s)
PDE	Partial Differential Equation
SDE	Stochastic Differential Equation
WMO	World Meteorological Organisation

Glossary of Notation

K	Strike
cap	Payment cap
$tick$	Tick size
t	Time from start of contract period
T	Length of contract period
τ	$T - t$
r	Risk-free interest rate
$e^{-r(T-t)}$	Discount factor applicable from time T to time t
S	Cumulative Heating Degree Days (HDD)
X	Temperature
P	Option payoff
V	Option value
E	Expected value
dW	Standard Wiener process
μ	Drift rate
σ	Volatility
m	Mean
s	Standard deviation
$N(m, s)$	Normal distribution with mean m, standard deviation s

Chapter 1

Introduction

1.1 Background

A derivative is a financial instrument whose value depends on the value of other, more basic underlying variables. For a weather derivative, the underlying variables are measures of the weather, for example precipitation or snowfall levels, wind speed or, most commonly, temperature.

Weather derivatives are used to control the risks of naturally-arising exposures to weather. Businesses subject to weather risk, and therefore likely to benefit from weather derivatives, include energy producers and consumers, supermarket chains, the leisure industry and agricultural industries.

The first transaction in the weather derivatives market took place in the US in 1997. Many companies then decided to hedge their seasonal weather risk after experiencing a serious loss of earnings during the very severe El Niño winter of 1997-98. Since then the market for weather derivatives has expanded rapidly, largely driven by companies in the energy sector. Although the market is still in its early stages, and is currently not very active, the number of players and volume of trades continues to increase.

The most common type of weather derivative is a ‘Heating Degree Day’ (HDD) or ‘Cooling Degree Day’ (CDD) option. This contract provides the holder with a payoff at the end of the contract period (at ‘expiry’) dependent on the excess of the period’s cumulative Degree Days (HDD or CDD) over the ‘strike’ (for a ‘call’ option), or the excess of the strike over the cumulative Degree Days (for a ‘put’ option). We define these Degree Days and set out the precise form of the payoff for each type of contract in the next section.

1.2 Definitions

The Heating Degree Days (HDD) on day i are defined by

$$HDD_i = \max(18 - X_i, 0),$$

where $X_i = \frac{X_i^{max} + X_i^{min}}{2}$ is the average temperature measured on day i in degrees Celrespectively on day i.

Similarly, we define the Cooling Degree Days (CDD) on day i by

$$CDD_i = \max(X_i - 18, 0).$$

Suppose we have a contract period, $0 \leq t \leq T$, consisting of N days. Then the cumulative number of HDD and CDD for that period are

$$\begin{aligned} H_N &= \sum_{i=1}^{N} HDD_i \quad \text{and} \\ C_N &= \sum_{i=1}^{N} CDD_i \quad \text{respectively.} \end{aligned}$$

If we denote the strike level by K and the 'tick size' (the monetary value paid out per degree Celcius) by $tick$, then the payoff for an uncapped HDD call or put option is

$$\begin{aligned} P_{call} &= \max(H_N - K, 0) \times tick \quad \text{or} \\ P_{put} &= \max(K - H_N, 0) \times tick \quad \text{respectively,} \end{aligned}$$

and similarly for an uncapped CDD call or put option.

We will work with a purchased HDD put option as an example. We will now use S to represent cumulative HDD, to be consistent with stock options, where the underlying share price is usually represented by S. In addition, we will introduce a 'payment cap', cap

1.3 An Example Contract

We illustrate the previous definitions by setting out the indicative terms and conditions of a real-life HDD put option contract. The contract below was prepared by ABN Amro, but not actually traded. We shall attempt to value this example contract using various methods in later chapters.

Contract Period:	1 November 2002 up to and including 31 March 2003.
Payment Cap:	EUR 1,000,000.
Weather Unit:	On each day during the Contract Period, HDD rounded to the nearest 0.1 degrees Celcius calculated as follows: The Base Temperature for calculation of HDD is 18 degrees Celcius. If the Daily Average Temperature is

with the payoff satisfying (1.1).

1.4 Possible Valuation Methods

In order to determine a reasonable price which should be paid to acquire the weather derivative, we need to be able to value the contract at time $t = 0$ (the contract start date).

The traditional method for valuing options is via the Black-Scholes model. Unfortunately, this model is based on certain assumptions that do not apply realistically to weather derivatives, the most fundamental of these being the assumption of a tradeable underlying commodity. The underlying variables for weather derivatives, for example temperature, are not themselves tradeable in a market, and so the theory applied in the derivation of the Black-Scholes formula (see Black and Scholes [3]) is not valid.

Degree Day weather options tend therefore to be valued by developing models for either temperature or cumulative Degree Days, and then running simulations based on Monte Carlo methods or on historical data (Burn Analysis).

In this dissertation, we will develop new valuation methods based on the numerical solution of partial differential equations (PDE's) and use these to value the example contract in the previous section. We start in Chapter 2 by analysing the relevant historical temperature data, and from this, make some conclusions on the distribution of the HDD and temperature data. In Chapter 3, we use these conclusions together with expectation theory to derive a formula for our contract value. In Chapter 4, we again use the results from Chapter 2, this time to derive a partial differential equation (PDE) satisfied by the option value. Chapters 5, 6, 7 and 8 are concerned with the numerical solution of this PDE. In Chaper 9, we demonstrate the results of Monte Carlo simulations and Burn Analysis, to enable a final comparison with our previous methods.

Chapter 2

Analysis of Historical Temperature Data for the Example Contract

2.1 Introduction

Although an active or 'liquid' market does not yet exist for weather derivatives, we do have access to extensive historical weather data. This means that weather derivative models tend to be calibrated to past data. To be able to value our example contract described in section 1.3, we therefore need to analyse the relevant historical temperature data.

We have accessed the last fifty years' daily maximum and minimum temperatures for Vlissingen, The Netherlands, (the weather station of the example contract in section 1.3) from the Royal Netherlands Meteorological Institute website[1]. From this data we have calculated the daily average temperatures and Heating Degree Days (HDD) for each day in our contract period (1 November to 31 March inclusive), for each of the last fifty years.

We have then developed and tested various hypotheses about the distribution of the average temperature and HDD data, the results of which we will use in later chapters when we develop models for valuing our weather derivatives contract.

2.2 Hypotheses and Results

Hypothesis 1: *The cumulative HDD for the contract period are normally distributed.*

This hypothesis is claimed to be valid in McIntyre [11]. In our case, we have calculated the cumulative HDD for the contract period for the last fifty years, prorating the values by one day for leap years. We have then tested our hypothesis by performing a χ^2 test at the 5% level. The result of this test is that the observed distribution of

[1] www.knmi.nl

cumulative HDD is consistent with the normal distribution. We can see the closeness of fit of the observed distribution to the normal distribution from Figure 2.1.

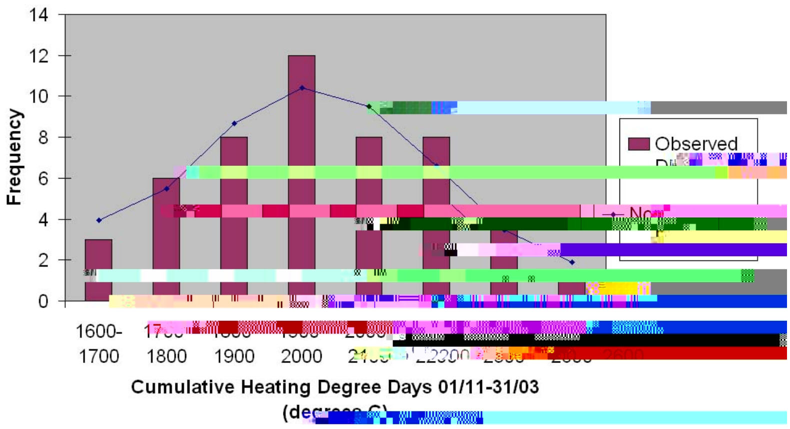

Figure 2.1: Cumulative HDD distribution for Vlissingen

We have calculated the mean (m) and standard deviation (s) of this cumulative HDD distribution for use in later chapters. We obtain

$$m = 1966.4\ (^\circ\mathrm{C}) \quad \text{and}$$
$$s = 188.5\ (^\circ\mathrm{C}).$$

Hypothesis 2: *The daily increments in average temperature are normally distributed.*

We have tested this hypothesis by performing Jarque-Bera tests (at the 5% level) for goodness of fit to a normal distribution, using Matlab. (This is computationally more e□cient than χ^2 tests since we are performing the test for each of 151 daily increments). The results of these tests are that our hypothesis can not be rejected for 88% of the days in the contract period. Since such a high proportion of the daily increments have distributions consistent with the normal distribution, we will make the approximation that all daily increments in average temperature are normally distributed, that is, we assume that our hypothesis is true.

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We calculate the standard deviation of the $(N+1)$th daily increment by

$$
\begin{aligned}
s_{N+1} &= \sqrt{\frac{1}{\sum_{i=1}^{N} \lambda^{N-i}} \sum_{i=1}^{N} \lambda^{N-i}(x_i - m)^2} \\
&\simeq \sqrt{(1-\lambda) \sum_{i=1}^{N} \lambda^{N-i}(x_i - m)^2}
\end{aligned}
$$

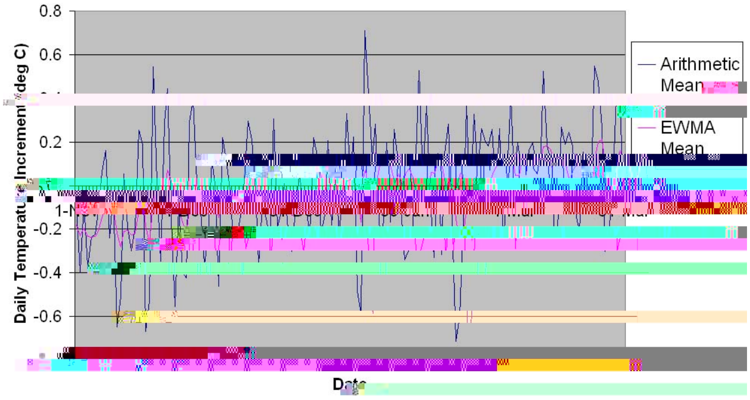


Figure 2.2: Mean daily increments in average temperature for Vlissingen

Hypothesis 3: *The daily increments in cumulative HDD are normally distributed.*

As for Hypothesis 2, we have performed Jarque-Bera tests (at the 5% level) for goodness of fit to a normal distribution, using Matlab. In this case, we find that our hypothesis can not be rejected for 78% of the days in the contract period. Although this percentage is not as high as for the temperature increment test, we still consider this to be sufficiently high a proportion that we can assume all daily increments in

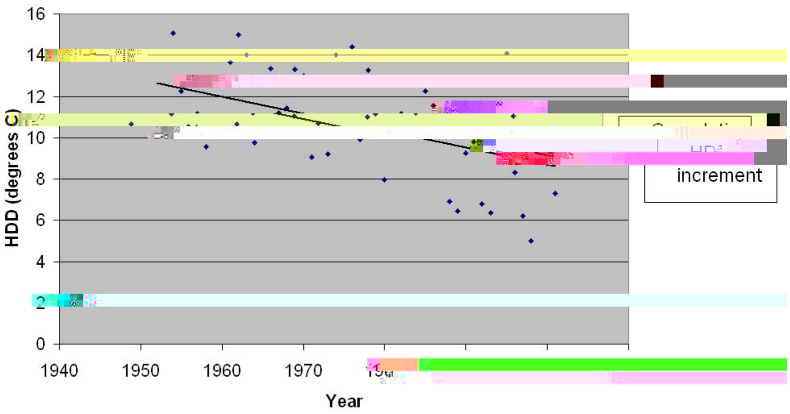


Figure 2.3: Scattergram of cumulative HDD increments for Vlissingen: 30-31 March

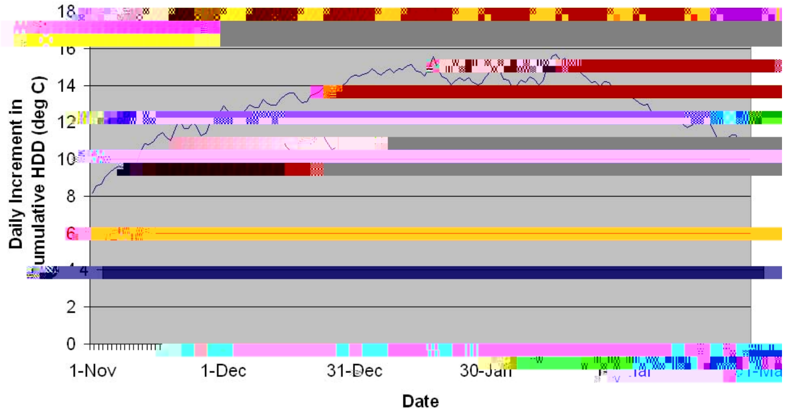


Figure 2.4: Arithmetic mean daily increments in cumulative HDD for Vlissingen

2.3 Problems with the Analysis of Historical Temperature Data

We note here some general issues with the collection and analysis of historical temper-

Chapter 3

An Expectation-Based Formula

3.1 Introduction

In this chapter, we derive a similar result to that quoted in McIntyre [11], for the value of an HDD put option. The derivation makes the assumption, consistent with the results of the temperature analysis in the previous chapter, that the cumulative Heating Degree Days (HDD) over the life of a contract are normally distributed. The value of the option is then calculated as being the expected payoff of the option, appropriately discounted to account for the time value of money.

After proving this result, we apply the formula to the example contract in section 1.3, to obtain a value for this option at the contract start date.

3.2 Derivation of the Formula

We consider a general HDD put option, and use the notation set out on page vi. In addition, we define

S_T = the cumulative number of HDD at time $t = T$ (the end of the contract),

$P(S_T)$ = the option payoff at time $t = T$, neglecting the tick size and payment cap, that is,

$$P(S_T) = \max(K - S_T, 0), \tag{3.1}$$

and

V = the option value at time $t = 0$ (the contract start date), neglecting the tick size and payment cap.

(We will use $P\ (S_T)$ and V to represent the option payoff and value when the tick size and payment cap are taken into consideration.)

We assume that S_T

$x)$ N where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$ is the cumulative standard normal distribution function, and $f(x)$ is the normal probability density function defined in (3.2).

We now recall that the actual payoff of the option, including the tick size and payment cap is

$$P(S_T) = \min\{\max(K - S_T, 0) \times tick, cap\},$$

and so the actual value is

$$\begin{aligned} V &= e^{-rT} E[P(S_T)] \\ &= e^{-rT} E[\min\{\max(K - S_T, 0) \times tick, cap\}] \\ &= \min\left\{e^{-rT} E[\max(K - S_T, 0)] \times tick, e^{-rT} cap\right\} \\ &= \min\{V \times tick, e^{-rT} cap\}. \end{aligned}$$

We can therefore incorporate the tick size and payment cap in (3.4) to give a final value for the HDD put option at time $t = 0$:

81][T/FF106.974Tf1658-4.114Td][T/F119.963CTf3.8750Td[N]T/FT/F76

$$V = e^{-rT} \min\left\{\left((K - m)N\left(\frac{K - m}{s}\right) + s^2 f(K)\right) \times tick, cap\right\}. \qquad (3.5)$$

Therefore, using formula (3.5), we obtain

$$
\begin{aligned}
V &= e^{-0.05\times\frac{151}{365}} \min\left(((1750 - 1966.4) \times 0.1255) + (188.5^2 \times 0.0011)) \times 5000, 1000000\right) \\
&= 58415.
\end{aligned}
$$

This shows that our expectation-based formula values the example contract in section 1.3 at 58,415 Euros at the start of the contract.

Here we follow the same approach, but for an HDD put option, with the underlying cumulative HDD / temperature assumed to evolve according to the SDE (4.1). This derivation is more general than that given in Brody *et al* [4], in particular with regard to the inclusion of a generalised payoff function.

Theorem 1 *Suppose that* $\omega : \mathbb{R} \times [0, T] \to \mathbb{R}$ *is a twice continuously differentiable function with bounded derivatives, satisfying the parabolic PDE*

$$-\omega_\tau(z, \tau) - r\omega(z, \tau) + \mu_{T-\tau}\omega_z(z, \tau) + \frac{1}{2}\sigma^2_{T-\tau}\omega_{zz}(z, \tau) = 0, \tag{4.3}$$

with the initial condition

$$\omega(z, 0) = P(z), \tag{4.4}$$

where $P(z)$ *is the payoff of the option. Then*

$$V(\bar{Z}, T) = \omega(\bar{Z}, T). \tag{4.5}$$

Proof First we integrate SDE (4.1) from 0 to t to give

$$Z_t = \bar{Z} + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s. \tag{4.6}$$

We now define the process ; t).

$$\eta_t = \int_0^t \sigma_s dW_s \tag{4.7}$$

and the function

$$f(\eta, t) = \bar{Z} + \int_0^t \mu_s ds + \eta. \tag{4.8}$$

Then (4.6),(4.7)and (4.8) imply that

$$Z_t = f(\eta_t, t). \tag{4.9}$$

Let us consider the function g of two variables, defined by

$$g(\eta, t) = e^{-rt}\omega(f(\eta, t), T - t), \tag{4.10}$$

where $f(\eta, t)$ is given by (4.8) and $\omega(z, \tau)$ is the function introduced in Theorem 1.

We first note that partial differentiation of (4.10) gives the following results:

$$g_\eta(\eta, t) = e^{-rt}\omega_z(f(\eta, t), T - t), \tag{4.11}$$

$$g_{\eta\eta}(\eta, t) = e^{-rt}\omega_{zz}(f(\eta, t), T - t), \text{ and} \tag{4.12}$$

$$\begin{aligned} g_t(\eta, t) = {} & e^{-rt}[-r\omega(f(\eta, t), T - t) + \omega_z(f(\eta, t), T - t)\mu_t \\ & -\omega_\tau(f(\eta, t), T - t)] \;\; \omega f \; t \end{aligned}$$

At time $t = T$ this becomes

$$g(\eta_T, T) = g(0,0) + \int_0^T \left(g_t(\eta_t, t) + \frac{1}{2} g_{\eta\eta}(\eta_t, t)\sigma_t^2 \right) dt + \int_0^T g_\eta(\eta_t, t)\sigma_t dW_t.$$

Substituting for g, g_η, $g_{\eta\eta}$ and g_t from (4.10),(4.11),(4.12) and (4.13) gives

$$\begin{aligned} e^{-rT}\omega(f(\eta_T, T), 0) \;=\; & \omega(f(0,0), T) + \int_0^T e^{-rt} \Big(- r\omega(f(\eta_t, t), T - t) \\ & + \omega_z(f(\eta_t, t), T - t)\mu_t - \omega_\tau(f(\eta_t, t), T - t) \Big) dt \\ & + \frac{1}{2}\int_0^T e^{-rt}\omega_{zz}(f(\eta_t, t), T - t)\sigma_t^2 dt \\ & + \int_0^T e^{-rt}\omega_z(f(\eta_t, t), T - t)\sigma_t dW_t. \end{aligned}$$

From the definition of f in (4.8) and the fact that $Z_t = f(\eta_t, t)$ from (4.9), we have

$$\begin{aligned} e^{-rT}\omega(Z_T, 0) \;=\; & \omega(\bar{Z}, T) + \int_0^T e^{-rt} \Big(- r\omega(Z_t, T - t) + \omega_z(Z_t, T - t)\mu_t \\ & - \omega_\tau(Z_t, T - t) + \frac{1}{2}\omega_{zz}(Z_t, T - t)\sigma_t^2 \Big) dt \\ & + \int_0^T e^{-rt}\omega_z(Z_t, T - t)\sigma_t dW_t. \end{aligned} \tag{4.14}$$

Taking expectations of both sides of (4.14), and using the fact that

$$\mathrm{E}\left[\int_0^T \varphi(t) dW_t\right] = 0$$

for a Wiener process dW_t and any bounded, suitably measurable function $\varphi(t)$ (see Jäckel [10]), we deduce that

$$\begin{aligned} e^{-rT}\mathrm{E}\left[\omega(Z_T, 0)\right] \;=\; & \omega(\bar{Z}, T) + \mathrm{E}\Big[\int_0^T e^{-rt} \Big(- r\omega(Z_t, T - t) + \omega_z(Z_t, T - t)\mu_t \\ & - \omega_\tau(Z_t, T - t) + \frac{1}{2}\omega_{zz}(Z_t, T - t)\sigma_t^2 \Big) dt \Big]. \end{aligned} \tag{4.15}$$

Now, we know that the value of our HDD put option at time $t = 0$ must be equal to the expected value of the payoff from the option, discounted back in time, that is,

$$V(\bar{Z}, T) = \mathrm{E}\left[e^{-rT}P(Z_T)\right]. \tag{4.16}$$

We now add (4.16) ‘side by side’ to (4.15), to give

$$V(Z, T) + e^{-rT}\mathrm{E}[\omega(Z_T, 0)] = \omega(Z, T) + \mathrm{E}\left[\int_0^T e^{-rt}\left(-r\omega(Z_t, T-t)\right.\right.$$

$$\left.\left. + \omega_z(Z_t, T-t)\mu_t - \omega_\tau(Z_t, T-t) + \frac{1}{2}\omega_{zz}(Z_t, T-t)\sigma_t^2\right) dt\right]$$

$$+ \mathrm{E}\left[e^{-rT} P(Z_T)\right].$$

Rearranging, we get

$$V(Z, T) + e^{-rT}\mathrm{E}[\omega(Z_T, 0) - P(Z_T)]$$

$$= \omega(Z, T) + \mathrm{E}\left[\int_0^T e^{-rt}\left(-r\omega(Z_t, T-t) + \omega_z(Z_t, T-t)\mu_t - \omega_\tau(Z_t, T-t) + 1\right.\right.$$

4.4 Transformation of the PDE and Initial/Boundary Conditions

Chapter 5

Accuracy and Stability of Numerical Schemes

5.1 Possible Numerical Schemes

Here we set out various finite difference schemes which may be proposed to provide a numerical solution to our PDE (4.20):

$$u_\tau = \frac{1}{2}\sigma^2_{T-\tau}u_{zz} + \mu_{T-\tau}u_z.$$

We make the approximation

$$u^n$$

3. Crank-Nicolson:

$$
\begin{aligned}
\frac{u_j^{n+1}-u_j^n}{\Delta\tau} = \frac{1}{2}\Big[& \frac{1}{2}\sigma^2_{T-\tau_n}\frac{1}{(\Delta z)^2}(u^n_{j+1}-2u^n_j+u^n_{j-1}) \\
& + \frac{1}{2}\sigma^2_{T-\tau_{n+1}}\frac{1}{(\Delta z)^2}(u^{n+1}_{j+1}-2u^{n+1}_j+u^{n+1}_{j-1})\Big] \\
& + \frac{1}{2}\Big[\mu_{T-\tau_n}\frac{1}{2\Delta z}(u^n_{j+1}-u^n_{j-1}) \\
& + \mu_{T-\tau_{n+1}}\frac{1}{2\Delta z}(u^{n+1}_{j+1}-u^{n+1}_{j-1})\Big] .
\end{aligned} \tag{5.3}
$$

4. Crank-Nicolson with Downwind/Upwind Convection:

$$
\begin{aligned}
\frac{u_j^{n+1}-u_j^n}{\Delta\tau} = \frac{1}{2}\Big[& \frac{1}{2}\sigma^2_{T-\tau_n}\frac{1}{(\Delta z)^2}(u^n_{j+1}-2u^n_j+u^n_{j-1}) \\
& + \frac{1}{2}\sigma^2_{T-\tau_{n+1}}\frac{1}{(\Delta z)^2}(u^{n+1}_{j+1}-2u^{n+1}_j+u^{n+1}_{j-1})\Big] \\
& + \begin{cases} \mu_{T-\tau_n}\frac{1}{\Delta z}\left(u^n_{j+1}-u^n_j\right) & \text{if } \mu_{T-\tau_n} > 0 \\ \mu_{T-\tau_n}\frac{1}{\Delta z}\left(u^n_j\right. \end{cases}
\end{aligned}
$$

We can see from these results that the Crank-Nicolson scheme (5.3) has the greatest level of accuracy whilst also being unconditionally Fourier stable. This is therefore the preferred scheme out of those listed in the previous table. In the next section, we detail the accuracy and stability calculations for this chosen scheme, and in Chapter 6, we demonstrate how this scheme is implemented to approximately solve our PDE.

5.3 Accuracy and Stability Calculations for the Crank-Nicolson Scheme

5.3.1 Accuracy

We define the discrete linear operator, L_h, by

$$\begin{aligned} L_h u_j^n \equiv \frac{u_j^{n+1} - u_j^n}{\Delta\tau} &- \frac{1}{2}\left[\frac{1}{2}\sigma^2_{T-\tau_n}\frac{1}{(\Delta z)^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)\right. \\ &\left. + \frac{1}{2}\sigma^2_{T-\tau_{n+1}}\frac{1}{(\Delta z)^2}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})\right] \\ &- \frac{1}{2}\left[\mu_{T-\tau_n}\frac{1}{2\Delta z}(u_{j+1}^n - u_{j-1}^n)\right. \\ &\left. + \mu_{T-\tau_{n+1}}\frac{1}{2\Delta z}(u_{j+1}^{n+1} - u_{j-1}^{n+1})\right]. \end{aligned} \tag{5.5}$$

Then the truncation error, ${}_j^n$, is defined to be

$$\begin{aligned} {}_j^n &= L_h u(z_j, \tau_n) - L_h u_j^n \\ &= L_h u(z_j, \tau_n), \end{aligned} \tag{5.6}$$

since $L_h u_j^n = 0$ by definition.

Combining (5.5) and (5.6), we obtain

$$\begin{aligned} {}_j^n = &\frac{1}{\Delta\tau}(u(j\Delta z, (n+1)\Delta\tau) - u(j\Delta z, n\Delta\tau)) \\ &- \frac{1}{4(\Delta z)^2}\sigma^2_{T-\tau_n}(u((j+1)\Delta z, n\Delta\tau) - 2u(j\Delta z, n\Delta\tau) + u((j-1)\Delta z, n\Delta\tau)) \\ &- \frac{1}{4(\Delta z)^2}\sigma^2_{T-\tau_n-\Delta\tau}(u((j+1)\Delta z, (n+1)\Delta\tau) \\ &- 2u(j\Delta z, (n+1)\Delta\tau) + u((j-1)\Delta z, (n+1)\Delta\tau)) \\ &- \frac{1}{4\Delta z}\mu_{T-\tau_n}(u((j+1)\Delta z, n\Delta\tau) - u((j-1)\Delta z, n\Delta\tau)) \\ &- \frac{1}{4\Delta z}\mu_{T-\tau_n-\Delta\tau}(u((j+1)\Delta z, (n+1)\Delta\tau) - u((j-1)\Delta z, (n+1)\Delta\tau)). \end{aligned} \tag{5.7}$$

We expand (5.7) about $(j\Delta z, n\Delta\tau)$ using Taylor series, and collect coefficients of powers of Δz and $\Delta\tau$ to give

$$
\begin{aligned}
{}^n_j \;=\; & (u_\tau - \tfrac{1}{2}\sigma^2_{T-\tau}u_{zz} - \mu_{T-\tau}u_z) + \frac{\Delta\tau}{2}\Big[u_{\tau\tau} - \frac{1}{2}\sigma^2_{T-\tau}u_{zz\tau} + \frac{1}{2}\sigma^2_{T-\tau}u_{zz} - \mu_{T-\tau}u_{z\tau} \\
& + \mu_{T-\tau}u_z\Big] + (\Delta\tau)^2\Big[\frac{1}{6}u_{\tau\tau\tau} - \frac{1}{8}\sigma^2_{T-\tau}u_{zz\tau\tau} + \frac{1}{4}\sigma^2_{T-\tau}u_{zz\tau} - \frac{1}{8}\sigma^2_{T-\tau}u_{zz} \\
& - \frac{1}{4}\mu_{T-\tau}u_{z\tau\tau} + \frac{1}{2}\mu_{T-\tau}u_{z\tau} - \frac{1}{4}\mu_{T-\tau}u_z\Big] \\
& + (\Delta z)^2\Big[-\frac{1}{24}\sigma^2_{T-\tau}u_{zzzz} - \frac{1}{6}\mu_{T-\tau}u_{zzz}\Big] + \cdots
\end{aligned}
\tag{5.8}
$$

(using u and τ to represent $u(j\Delta z, n\Delta\tau)$ a… respectively, and σ^2, σ^2, μ and μ to represent the first and second derivatives of σ^2 and μ with respect to τ).

583 Td[( )]TJ/F3 6.974 Tf 5.778 0 Td[( )]TJ/F10 6.974 Tf 6.227 0 Td[( )]TJ/F11 9.963 T

Chapter 6

Solution of the PDE for Cumulative HDD

6.1 Introduction

We consider the general PDE set out in equation (4.20) of Chapter 4, and now assume that the independent variable z represents cumulative HDD. We therefore replace z by S, to be consistent with our previous notation, and the PDE (4.20) becomes

$$u_\tau = \frac{1}{2}\sigma^2_{T-\tau}u_{SS} + \mu_{T-\tau}u_S, \tag{6.1}$$

with initial and boundary conditions

$$\begin{aligned} u(S,0) &= P(S), && (6.2)\\ u(S_1,\tau) &= e^{r\tau}B_1, \quad \text{and} && (6.3)\\ u(S_2,\tau) &= e^{r\tau}B_2. && (6.4) \end{aligned}$$

(Here $P(S)$ is the option payoff, and B_1, B_2 will be taken to be the option value at $S = S_1$, $S = S_2$ respectively.)

By the definition of the option, we have

$$P(S) = \min\{\max(K - S, 0) \times tick, cap\}$$

(see equation (1.1)), which completes our initial condition.

Since HDD are positive (by definition), we know that $S \geq 0$ at all points in time. Also since S is cumulative, if $S \geq K$ at any time during the contract, we will have $S \geq K$ at expiry, and hence a zero payoff. This means that we are only required to solve for u in the region $0 \leq S \leq K$.

We therefore have

$$\begin{aligned} S_1 &= 0,\\ S_2 &= K, \quad B_2 = 0. \end{aligned}$$

The value of B_1 is less obvious. If $S = 0$ at time $t = T - \tau$, this simply tells us that the temperature has been greater than or equal to 18 $^{\circ}$C (no HDD have occurred) up to time t. The option payo is greater for smaller values of S at expiry. This implies that if $S = 0$ at $t = t_2$

may be assumed to represent the value of the option at time $t = T - \tau$, when the cumulative HDD equals S. Therefore by solving the PDE for $u(S, \tau)$ in the region $0 \leq S \leq K$, $0 \leq \tau \leq T$, we hope to gain information about the evolution of the option value during the contract period, rather than achieving a value for the option at time $t = 0$ only.

This is beneficial since in reality weather derivatives such as our HDD put option tend to be traded during the contract period, and hence may need to be valued at any time t, $0 \leq t \leq T$. Mid-contract valuation is also necessary for a company to establish the value of its option portfolio at a point in time.

6.2 Numerical Solution

We will show how we implement the Crank-Nicolson scheme analysed in Chapter 5 to solve PDE (6.1) with initial and boundary conditions (6.5), (6.6) and (6.7).

The scheme is

$$\begin{aligned}
\frac{u_j^{n+1} - u_j^n}{\Delta\tau} &= \frac{1}{2}\left[\frac{1}{2}\sigma^2_{T-\tau_n}\frac{1}{(\Delta S)^2}(u^n_{j+1} - 2u^n_j + u^n_{j-1})\right. \\
&\quad \left. + \frac{1}{2}\sigma^2_{T-\tau_{n+1}}\frac{1}{(\Delta S)^2}(u^{n+1}_{j+1} - 2u^{n+1}_j + u^{n+1}_{j-1})\right] \\
&\quad + \frac{1}{2}\left[\mu_{T-\tau_n}\frac{1}{2\Delta S}(u^n_{j+1} - u^n_{j-1})\right. \\
&\quad \left. + \mu_{T-\tau_{n+1}}\frac{1}{2\Delta S}(u^{n+1}_{j+1} - u^{n+1}_{j-1})\right],
\end{aligned}$$

where $u_j^n \simeq u(S_j, \tau_n)$, $S_j = j\Delta S$, $\tau_n = n\Delta\tau$.

If we set

$$\begin{aligned}
\lambda_n &= \frac{1}{4}\sigma^2_{T-\tau_n}\frac{\Delta\tau}{(\Delta S)^2}, \quad \text{and} \\
\nu_n &= \frac{1}{4}\mu_{T-\tau_n}\frac{\Delta\tau}{\Delta S},
\end{aligned}$$

the scheme becomes

$$\begin{aligned}
u_j^{n+1} - u_j^n &= \lambda_n(u^n_{j+1} - 2u^n_j + u^n_{j-1}) + \lambda_{n+1}(u^{n+1}_{j+1} - 2u^{n+1}_j + u^{n+1}_{j-1}) \\
&\quad + \nu_n(u^n_{j+1} - u^n_{j-1}) + \nu_{n+1}(u^{n+1}_{j+1} - u^{n+1}_{j-1}).
\end{aligned}$$

This rearranges to

$$\begin{aligned}
&(-\lambda_{n+1} - \nu_{n+1})u^{n+1}_{j+1} + (1 + 2\lambda_{n+1})u^{n+1}_j + (-\lambda_{n+1} + \nu_{n+1})u^{n+1}_{j-1} \\
&= (\lambda_n + \nu_n)u^n_{j+1} + (1 - 2\lambda_n)u^n_j + (\lambda_n - \nu_n)u^n_{j-1}
\end{aligned}$$

We let j run from 0 to J, where $J\Delta S = K$. Then we can write the problem as the $(J-1)$-dimensional tridiagonal matrix system

$$\mathrm{A}\mathbf{u}^{n+1} = \mathrm{B}\mathbf{u}^{n} + \mathbf{c}^{n},$$

where

$$\mathrm{A} = \begin{pmatrix} 1+2\lambda_{n+1} & -\lambda_{n+1}-\nu_{n+1} & 0 & \cdots & 0 \\ -\lambda_{n+1}+\nu_{n+1} & 1+2\lambda_{n+1} & -\lambda_{n+1}-\nu_{n+1} & & \vdots \\ 0 & -\lambda_{n+1}+\nu_{n+1} & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\lambda_{n+1}-\nu_{n+1} \\ 0 & 0 & & -\lambda_{n+1}+\nu_{n+1} & 1+2\lambda_{n+1} \end{pmatrix},$$

$$\mathrm{B} = \begin{pmatrix} 1-2\lambda_n & \lambda_n+\nu_n & 0 & \cdots & 0 \\ \lambda_n-\nu_n & 1-2\lambda_n & \lambda_n+\nu_n & & \vdots \\ 0 & \lambda_n-\nu_n & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \lambda_n+\nu_n \\ 0 & 0 & & \lambda_n-\nu_n & 1-2\lambda_n \end{pmatrix},$$

and the column vectors $\mathbf{u}^n$, $\mathbf{u}^{n+1}$and $\mathbf{c}^n$ are given by

$$\mathbf{u}^n = \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ \vdots \\ u_{J-1}^n \end{pmatrix}, \mathbf{u}^{n+1} = \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ \vdots \\ u_{J-1}^{n+1} \end{pmatrix}, \mathbf{c}^n = \begin{pmatrix} (\lambda_{n+1}-\nu_{n+1})u_0^{n+1} + (\lambda_n-\nu_n)u_0^n \\ 0 \\ \vdots \\ 0 \\ (\lambda_{n+1}+\nu_{n+1})u_J^{n+1} + (\lambda_n+\nu_n)u_J^n \end{pmatrix}.$$

Given $\mathbf{u}^n$, we use the boundary conditions (6.6) and (6.7) to compute $\mathbf{c}^n$ and hence the right hand side of the system, $\mathbf{d}^n = \mathrm{B}\mathbf{u}^n + \mathbf{c}^n$. We then solve the system $\mathrm{A}\mathbf{u}^{n+1} = \mathbf{d}^n$ using an LU tridiagonal matrix solver. We start with $\mathbf{u}^0$, as given by the initial condition (6.5), and step forward in increments of $\Delta\tau$, until we reach $\mathbf{u}^N$, where $T = N\Delta\tau$.

Note that, if $\lambda_{n+1} \geq |\nu_{n+1}|$, then

$$\begin{aligned} -\lambda_{n+1}+\nu_{n+1} &\leq 0, \text{ and} \\ -\lambda_{n+1}-\nu_{n+1} &\leq 0, \end{aligned}$$

so

$$|-\lambda_{n+1}+\nu_{n+1}| + |-\lambda_{n+1}-\nu_{n+1}| = 2\lambda_{n+1} < |1+2\lambda_{n+1}|,$$

and therefore the matrix A is strictly diagonally dominant and hence non-singular, implying that the system has a unique solution.

The condition $\lambda_{n+1} \geq |\nu_{n+1}|$ is equivalent to

$$\frac{1}{4}\sigma^2_{T-\tau_{n+1}}\frac{\Delta\tau}{(\Delta S)^2} \geq \frac{1}{4}|\mu_{T-\tau_{n+1}}|\frac{\Delta\tau}{\Delta S},$$

or

$$\Delta S \leq \frac{\sigma^2_{T-\tau_{n+1}}}{|\mu_{T-\tau_{n+1}}|}, \tag{6.8}$$

for each value of n, $0 \leq n \leq N-1$. This is a sufficient condition for the system to have a unique solution.

In fact it can be shown using eigenstructure analysis that the matrix A is non-singular for all real values of λ_n and ν_n (see Nichols [15]) and hence that the system will always have a unique solution.

6.3 Results

We have applied the previous methodology to solve the PDE (6.1) with initial and boundary conditions (6.5), (6.6) and (6.7) for the example contract detailed in section 1.3. We used an HDD step-length of $\Delta S = 17.5$ (100 HDD steps, since we have $0 \leq S \leq 1750$), and a time-step of 1 day (151 time-steps, since $T = 151$ days).

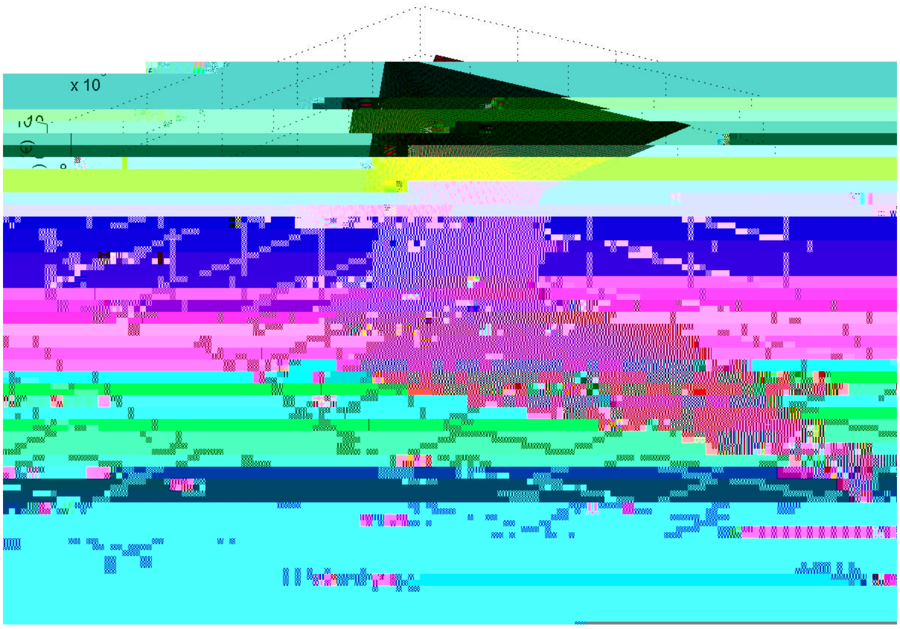


Figure 6.1: Numerical solution of PDE for Cumulative HDD using the Crank-Nicolson method

Two numerical issues are apparent from Figure 6.1:

1. It can be seen that our numerical solution contains 'spurious oscillations'. This is consistent with Morton [12], which shows that discretising the convection term using central differences for PDE's with low diffusion relative to convection can produce solutions containing spurious oscillations.

   In the case of our PDE:

$$u_\tau = \frac{1}{2}\sigma$$

Anderson [2] concludes that when this condition is violated (as in our case), the numerical solution produced by the standard central-difference approximation will be oscillatory. This is consistent with the behaviour of our solution apparent from Figure 6.1. To satisfy the cell-Peclet condition for this scheme, that is to enforce $\Delta S \leq 0.82$, is not feasible in practice due to limited computer resources. We will examine alternative methods of dealing with this problem in the next chapter.

2. Another feature of our numerical solution as shown in Figure 6.1 is that a discontinuity exists between the boundary condition $S = 0$ and the solution for $S > 0$. This is due to the fact that the boundary condition $u(0, \tau)$ is not smooth

Chapter 7

Resolution of Numerical Issues for Cumulative HDD PDE

7.1 Spurious Oscillations

7.1.1 Downwind/Upwind Scheme

As per Morton [12], it is the discretisation of the convection term using central differences that has produced the spurious oscillations in Figure 6.1. This suggests that we should be able to resolve this numerical issue by discretising the convection term using a downwind/upwind scheme. We will therefore implement the Crank-Nicolson scheme with downwind/upwind convection, analysed in Chapter 5. We note from section 5.2 that this scheme is only first order accurate in τ and S. We shall therefore need to use more grid-points for each variable to achieve the same significant figures of accuracy as that in the previous chapter. In addition, this scheme is not unconditionally stable. From section 5.2, we require $\max_{\tau_n} |\mu_{T-\tau}$

Following the same approach as in section 6.2, if we set

$$\begin{aligned}\lambda_n &= \frac{1}{4}\sigma^2_{T-\tau_n}\frac{\Delta\tau}{(\Delta S)^2}, \quad \text{and}\\ \nu_n &= \mu_{T-\tau_n}\frac{\Delta\tau}{\Delta S},\end{aligned}$$

our problem becomes

$$\mathbf{A}\mathbf{u}^{n+1} = \mathbf{B}\mathbf{u}^n + \mathbf{c}^n,$$

where

$$\mathbf{A} = \begin{pmatrix} 1+2\lambda_{n+1} & -\lambda_{n+1} & 0 & \dots & 0 \\ -\lambda_{n+1} & 1+2\lambda_{n+1} & -\lambda_{n+1} & & \vdots \\ 0 & -\lambda_{n+1} & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\lambda_{n+1} \\ 0 & 0 & & -\lambda_{n+1} & 1+2\lambda_{n+1} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 1-2\lambda_n-\nu_n & \lambda_n+\nu_n & 0 & \dots & 0 \\ \lambda_n & 1-2\lambda_n-\nu_n & \lambda_n+\nu_n & & \vdots \\ 0 & \lambda_n & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \lambda_n+\nu_n \\ 0 & 0 & & \lambda_n & 1-2\lambda_n-\nu_n \end{pmatrix},$$

and

$$\mathbf{c}^n = \begin{pmatrix} \lambda_{n+1}u_0^{n+1} + \lambda_n u_0^n \\ 0 \\ \vdots \\ 0 \\ \lambda_{n+1}u_J^{n+1} + (\lambda_n+\nu_n)u_J^n \end{pmatrix},$$

with $\mathbf{u}^n$ and $\mathbf{u}^{n+1}$ defined as before.

Note that in this case we have

$$|-\lambda_{n+1}| + |-\lambda_{n+1}| = 2\lambda_{n+1} < |1+2\lambda_{n+1}|,$$

and therefore the matrix $\mathbf{A}$ is always strictly diagonally dominant and hence non-singular, implying that the system has a unique solution.

We have used the same method as in section 6.2 to solve this tridiagonal system. We used the same HDD step-length ($\Delta S = 17.5$) and time-step (1 day) as in section 6.3.

We can see from Figure 7.1 that the spurious oscillations have been eliminated, as desired.

We use a grid, as before, with $u_j^n \simeq u(S_j, \tau_n)$, where $S_j = j\Delta S$, $\tau_n = n\Delta\tau$. To construct a semi-Lagrangian scheme, we begin by assuming that the solution is known at τ

$$c_j^n = \alpha_j^n u_{i+1}^n + \beta_j^n u_i^n,$$

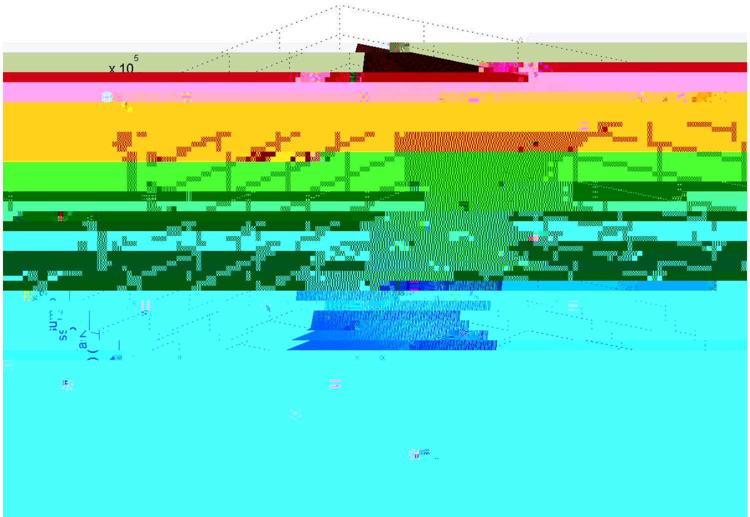
and

$$\mathbf{d}^n = \begin{bmatrix} \lambda_{n+1} u_0^{n+1} + \lambda_n u_0^n \\ 0 \\ \vdots \\ 0 \\ \lambda_{n+1} u_J^{n+1} + \lambda_n u_J^n \end{bmatrix},$$

with $\mathbf{u}^n$ and $\mathbf{u}^{n+1}$ defined as before.

Note that, as for the downwind scheme, the matrix $\mathbf{A}$ is always strictly diagonally dominant and hence non-singular, implying that the system has a unique solution.

We have used the same method as in sections 6.2 and 7.1.1 to solve this tridiagonal system, with the same HDD step-length (ΔS = 17.5) and time-step (1 day).



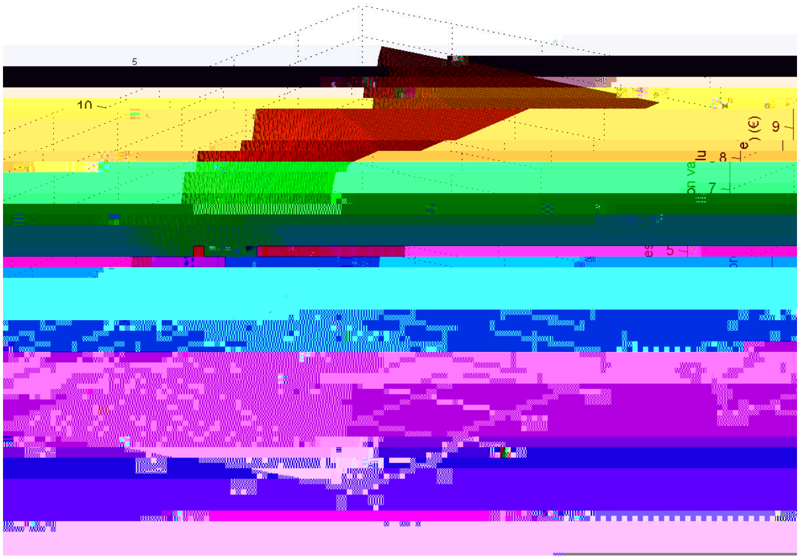


Figure 7.4: Numerical solution of PDE for Cumulative HDD using the semi-Lagrangian method with monotone cubic interpolation

7.1.3 Accuracy Testing

Since the order of accuracy of the downwind and semi-Lagrangian (with linear/cubic interpolation) schemes is not apparent from the numerical solution graphs in Figures 7.1, 7.3 and 7.4, we have used the numerical results to perform some accuracy testing. We have chosen four representative points on the $S-t$ grid, and for each scheme, calculated the average value of ω at these points when the number of cumulative HDD steps $J = \frac{K}{\Delta S} = 8$, 16, 32 and 64 (using $\Delta\tau = 1$ day). From the previous theory, the semi-Lagrangian scheme with monotone cubic interpolation should be the most accurate. We have therefore taken the average value of ω produced by this scheme for $J = 64$ to be the 'exact' solution, and hence calculated a representative absolute error of the solution produced by each scheme for $J = 8$, 16, 32 and 64.

Figure 7.5 shows the natural logarithm of the representative absolute error   plotted against the natural logarithm of the cumulative HDD step-length ΔS for each scheme. For a scheme that is pth order accurate in S, we should find that

$$\propto (\Delta S)^p,$$

and hence that

$$\ln(\ ) = p \ln(\Delta S) + c,$$

where c is a constant. This implies that the gradients of the lines in Figure 7.5 should approximate the order of accuracy in S of the respective schemes.

Since order of accuracy is an asymptotic result, we are really interested in the gradient of the lines for very small ΔS. However, in practice, we are unable to obtain the numerical solution of the PDE for small ΔS, as we are restricted by the availability of

may expect $u_{\tau\tau\tau}$ and higher derivatives to be large in the same region. This means that the Taylor series expansion of the truncation error for the Crank-Nicolson scheme (see equation (5.9)) does not decrease term by term, and so we cannot neglect higher order terms. This is also the case for the Taylor series expansions of the truncation error for the downwind and semi-Lagrangian schemes. It is therefore possible that all of these schemes will lose accuracy in this region.

One possible approach to resolving this issue would be to refine our grid (decrease the cumulative HDD step-length and the time-step) around the point $(S = 0, t = 32)$ and also around the ridge that runs from $(S = 0, t = 32)$ to $(S = 1750, t = 151)$. However since this ridge is not fixed in S or t, an irregular grid of this form would be very difficult to construct in practice. (See section 10.3.) Also, as mentioned in the previous section, decreasing the time-step beyond one day would require interpolation of the historical temperature data, which is not within the scope of this dissertation.

We could also consider using an irregular grid where the ridge running from $(S = 0, t = 32)$ to $(S = 1750, t = 151)$ was taken to be one of the grid lines. However this is complicated by the fact that the drift μ_t and volatility σ_t change on each time-step, so the ridge itself is not a straight line. (See section 10.3.)

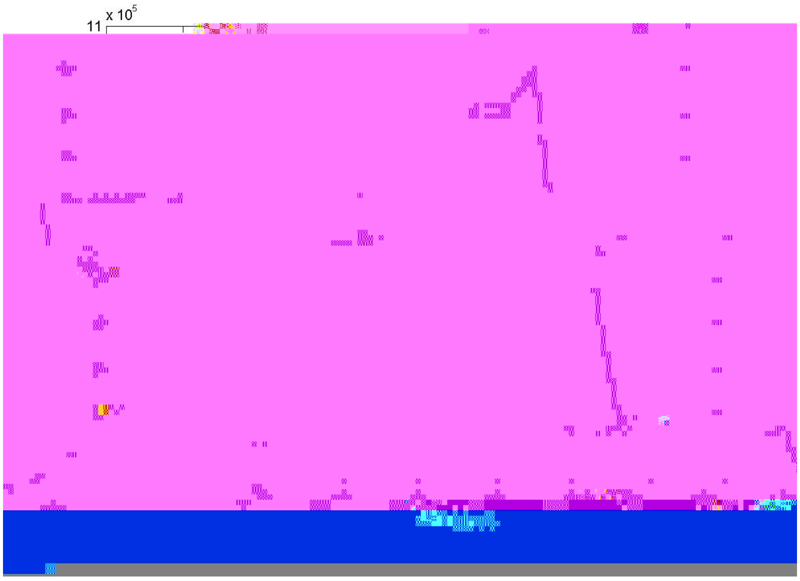


Figure 7.6: Boundary condition $u(0, \tau)$ for Cumulative HDD PDE

Chapter 8

Solution of the PDE for Temperature

8.1 Introduction

We again consider the general PDE set out in equation (4.20) of Chapter 4, but this time assume that the independent variable z represents temperature. We therefore replace z by X, and the PDE becomes

$$u_\tau = \frac{1}{2}\sigma^2_{T-\tau} u_{XX} + \mu_{T-\tau} u_X. \tag{8.1}$$

To be consistent with the PDE for cumulative HDD, we would like our initial condition to take the form

$$u(X,0) = P(X),$$

where $P(X)$ is the option payo□. However, the option payo□ at expiry does not depend on the temperature X at the expiry date; instead it depends on the cumulative HDD at expiry. We therefore need to include the variable S (cumulative HDD) in our solution for u, and the initial condition then4Tf5.704-1.495Td[()-78()]TJ/F89.963TrTd[(at)-250(th)1(213(p)1(a)oTd[(u)

2. As $X \rightarrow +\infty$, the contract period is assumed to be so warm, and hence the cumulative number of HDD, S, so small that $(K - S) \times tick > cap$, and so the payoff from the option will be the payment cap. This implies that the value of the option at time $t = T - \tau$ must be $e^{-r\tau} cap$.
Therefore we have X

2. At the current daily sampling point, the value of S is updated. If S_n is the cumulative number of HDD at day n, from definition (8.5), we have

$$S_n = S_{n-1}$$

$$u(X, S + \max(18 - X, 0), \tau) = u(X, S_i, \tau) + \left[\left(\frac{u(X, S_{i+1}, \tau) - u(X, S_i, \tau)}{S_{i+1} - S_i} \right) \times (S + \max(18 - X, 0) - S_i) \right].$$

This linear interpolation can be shown to be accurate to $O((\Delta S)^2)$ for smooth u. This is consistent with the second order accuracy in X and τ achieved by the Crank-Nicolson scheme (see Chapter 5).

4. We repeat this process as necessary to arrive at the value of our option at time $t = 0$ ($\tau = T$). Since we are solving the PDE on a three-dimensional grid ($X \times S \times \tau$), we can not use this method to provide graphical information about the evolution of the option value (we would require a four-dimensional surface). However, since we know that at time $t = 0$ we must have $S = 0$ (S is cumulative), by setting $S = 0$, we can obtain the value of the option at $t = 0$ in terms of the initial temperature X_0.

In solving this PDE numerically, we calculate the temperature volatility and drift, $\sigma_{T-\tau}$ and $\mu_{T-\tau}$ respectively, by applying the same method as that used to calculate the cumulative HDD volatility and drift in Chapter 6. We use the approximation of equation (4.2):

$$X_{t+\Delta t} - X_t \sim N(\mu_t \Delta t, \sigma_t \sqrt{\Delta t}), \tag{8.8}$$

combined with our analysis of historical daily temperature increments in Chapter 2.

For $\Delta t = 1$ day, equation (8.8) becomes

$$X_{t+1} - X_t \sim N(\mu_t, \sigma_t).$$

This tells us that

$\mu_t = \mu_{T-\tau} =$ −t)20Xn3(Chapten)28(a3((833(hi)1(e)-331(op)4aily)-333(temp)c)-1(e)-25rWime)-334(8(ter9J/F119.9

the next section). As remarked in section 6.3, condition (6.8) is equivalent to the cell-Peclet condition, which when violated, causes the numerical solution produced by the standard central-di erence approximation to be oscillatory. Therefore, by satisfying this condition, we should be able to prevent spurious oscillations from occurring in our solution.

8.3 Results

We have applied the previous methodology to solve PDE (8.1) with initial and boundary conditions (8.2), (8.3) and (8.4) for the example contract in section 1.3. We used a temperature step-length of $\Delta X = 0.2$ (500 temperature steps, since we are assuming $-50 \leq X \leq 50$), a cumulative HDD step-length of $\Delta S = 4$ (2567 HDD steps, since we have $0 \leq S \leq 10268$) and a time-step of 1 day (151 time-steps, since $T = 151$ days).

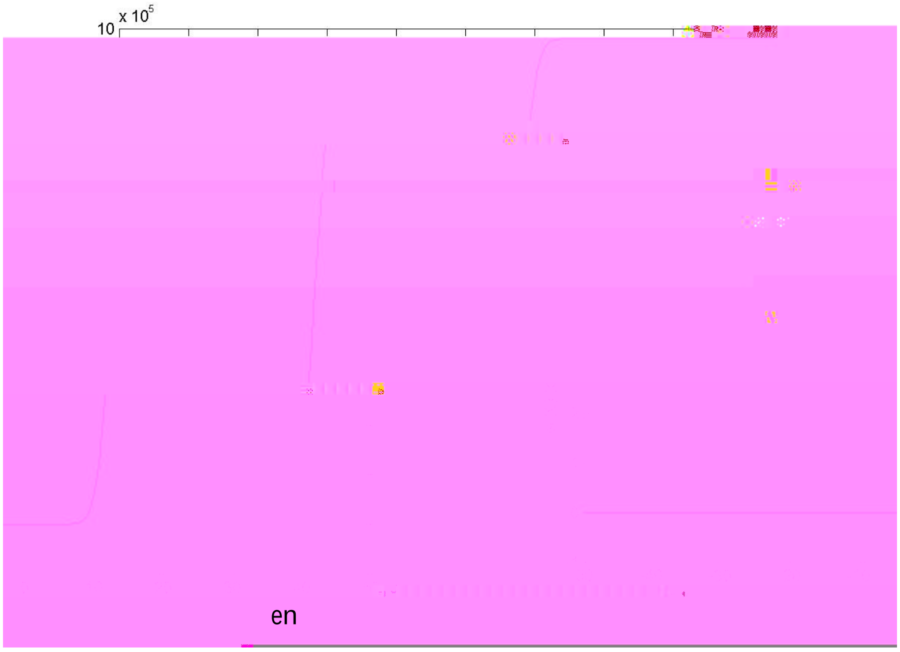


Figure 8.1: Numerical solution of PDE for temperature: $t = 0, S = 0$

From our historical temperature data, the mean (over the last fifty years) of the aver-

The following table shows the effect of refining the temperature and cumulative HDD step-length (for a time-step of 1 day):

ΔX ($^\circ$C)	ΔS ($^\circ$C)	Option value (Euros)
0.2	4	57,524
0.2	2	57,189
0.2	1	57,096
0.1	2	57,566
0.1	1	57,475

For a fixed number of daily sampling points, we know that the combination of our Crank-Nicolson method and linear interpolation should result in a global discretisation error of

$$O((\Delta\tau)^2) + O((\Delta X)^2) + O((\Delta S)^2).$$

We can see from the previous table that the option value only changes by 49 Euros in going from the coarsest to the finest grid. This implies that the absolute value of our global discretisation error is not unduly significant for a fixed time-step $\Delta\tau$. Investigation of refining the time-step is beyond the scope of this dissertation. (See section 10.3.)

Chapter 9

Monte Carlo Simulations and Other Valuation Methods

9.1 Monte Carlo Simulations

'Monte Carlo' is a computer-based technique for generating random numbers, which forward in time increments of $\Delta t = 1$ day using equation (9.1). Since (9.1) implies that

$$X_{t+1} - X_t \sim N(\mu_t, \sigma_t),$$

we calculate μ_t as being the (EWMA) mean of the daily temperature increment $X_{t+1} - X_t$ and σ_t as being the (EWMA) standard deviation (using the historical temperature data). We compute by taking a random drawing from the standard normal distribution. Using these inputs we construct a temperature path for the length of the contract.

We have run $50,000$ such simulations, for each one calculating the daily HDD and hence the cumulative HDD and payoff at expiry. We have averaged these payoffs and discounted the average to give a value for our option at time $t = 0$. Figure 9.1 shows five of these simulations.

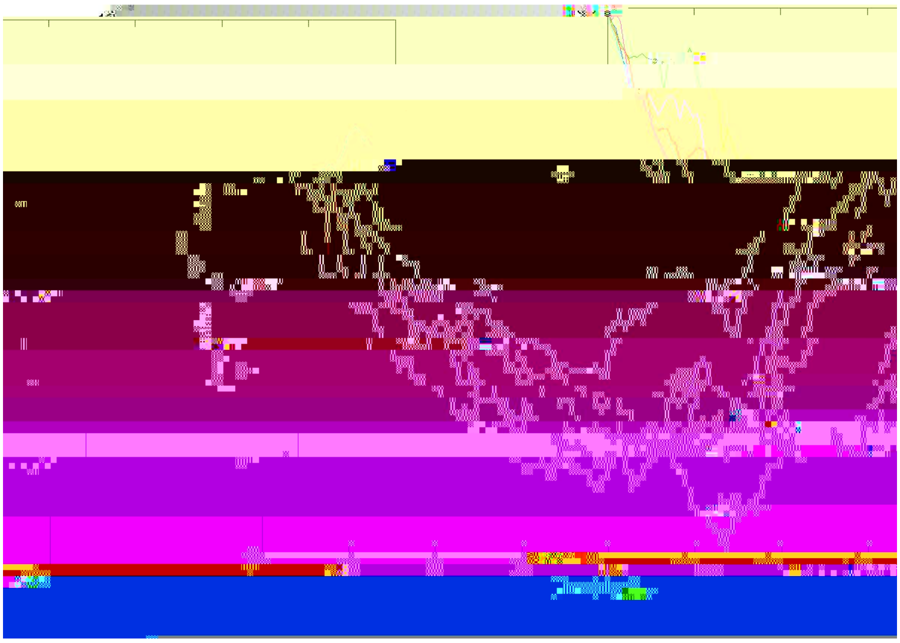


Figure 9.1: Monte Carlo temperature simulations

The above method gives the option value at time $t = 0$ to be $55,630$ Euros. This supports the results obtained using the expectation and PDE approaches in Chapters 3 and 8 respectively.

9.2 Other Valuation Methods

9.2.1 Burn Analysis

This method values the weather derivative based on the payoff that would have been obtained if the contract had been held in the past. (See Nelken [14]). After collection

Chapter 10

Conclusions and Further Research

10.1 Summary of Results

We have shown that, if the daily increments in cumulative HDD and average temperature are assumed to be normally distributed (which appears reasonable from our analysis of historical temperature data), we can formulate an SDE and use this to derive a convection-diffusion PDE with time-dependent coefficients for the value of an HDD put option. When the underlying process is cumulative HDD, we have found this PDE to be convection-dominated. In this case our preferred numerical solution technique is the semi-Lagrangian method with monotone cubic interpolation. Also in this case, we have found that, if we assume cumulative HDD themselves to be normally distributed (consistent with our historical data analysis), we can use expectation theory to derive a valuation result which can be applied as a boundary condition for the PDE. When the underlying process is temperature, we have found the PDE to have convection and diffusion terms of similar magnitude, and discovered that we can solve this numerically as a discretely-sampled Asian option, using the Crank-Nicolson scheme between sampling points.

The table below summarises the results obtained for the value of our example contract in section 1.3, from the numerical solution of our PDE as well as from more traditional methods.

Method	Option value (Euros) at the contract start date
PDE	
-for cumulative HDD [1]	58,415
-for temperature [2]	57,475

With the exception of Burn Analysis, which we know is very simplistic and likely to be inaccurate, the above methods give very similar results for the value of our HDD put option at the contract start date. This demonstrates that the numerical solution of our PDE can be used to give reasonably accurate results for the value of our weather derivative.

10.2 Benefits and Limitations of our PDE Method

The numerical schemes used to solve the PDE all introduce a degree of error (although so too does the discretisation used in the Monte Carlo simulation). The accuracy of our numerical solutions has also been restricted by the fact that our historical temperature data and therefore our drift and volatility parameters are only defined on a daily basis, and so, without performing interpolation of the daily historical data, we have not been able to use a time-step of less than one day. In addition, the value of the option at the contract start date using the PDE for cumulative HDD is just that given by expectation theory, and the numerical solution of the PDE for temperature is extremely sensitive to the initial temperature of the contract period.

However, we have hypothesised that the solution of our PDE for cumulative HDD represents the evolution of the option value. Hence it appears that this method may be used to gain information about the value of the option during the contract period. This is a distinct advantage of the PDE method over traditional methods which tend to value the derivative at one point in time only.

10.3 Further Research

Further work would be required to establish that the solution of our cumulative HDD PDE does indeed represent the evolution of our option value. We could also improve this solution by local grid refinement or the use of an irregular grid, to resolve the numerical issue of the discontinuity discussed in section 7.2.

It would be beneficial to examine methods of interpolating the daily historical temperature data, to enable time-steps of less than one day to be used in the numerical schemes. Assuming that an interpolation method of a sufficiently high order of accuracy could be used, this would increase the accuracy of our numerical solutions.

In addition we could consider developing, analysing and solving numerically PDE's for more physically realistic stochastic temperature processes, such as the mean-reverting processes proposed in Brody *et al* [4] and Alaton *et al* [1].

Appendix A

Itô's Lemma in Integral Form

We quote the version of Itô's Lemma given in Neftci [13]. This is applied in the derivation of the PDE in section 4.2.

Let $F(S_t, t)$ be a twice-differentiable function of t and of the random process S_t:

$$dS_t = \mu_t dt + \sigma_t dW_t, \qquad t \geq 0, \tag{A.1}$$

where dW is a standard Wiener process and μ_t, σ_t are well-behaved drift and diffusion parameters.

Alternatively, in integral form, the random process can be written as

$$S_t = S_0 + \int_0^t \mu_u du + \int_0^t \sigma_u dW_u.$$

Then Itô's Lemma states that

$$dF = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} \sigma_t^2 dt. \tag{A.2}$$

Substituting for dS_t from equation (A.1), equation (A.2) becomes

$$dF = \left(\frac{\partial F}{\partial S_t} \mu_t + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} \sigma_t^2 \right) dt + \frac{\partial F}{\partial S_t} \sigma_t dW_t. \tag{A.3}$$

Integrating both sides of (A.3), we obtain

$$F(S_t, t) = F(S_0, 0) + \int_0^t \left(F_s \mu_u + F_u + \frac{1}{2} F_{ss} \sigma_u^2 \right) du + \int_0^t F_s \sigma_u dW_u. \tag{A.4}$$

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