Approximations to linear wave scattering by topography using an integral equation approach

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Abstract

This dissertation considers approximations to the scattering of a train of small-amplitude harmonic surface waves on water by topography, using the mild-slope equation and the modified mild-slope equation. The associated boundary value problem is converted into two real-valued integral equations, the solutions of which are approximated by variational techniques. The reproduction of existing results over different shaped taluds are considered and show that this integral equation method is an equally effective solution technique as existing approximate numerical differential equation techniques. Finally, a ripple bed example is considered and it is reaffirmed that the modified mild-slope equation is capable of producing more accurate approximations over a wider range of topographies than the mild-slope equation.

This dissertation is based on the work of Chamberlain (1993).

Acknowledgements

I wish to thank Doctors Peter Chamberlain and Nick Biggs for their support during this dissertation. Their inexhaustible supply of help, advice and most importantly patience has been invaluable in the completion of this dissertation. In addition I wish to thank the EPRSC for their grant, without which I would not have been able to study this course, and my classmates who have provided many light-hearted moments(a)0.23.310405(e)0.k64(m)-0.6657(m)-0.0878119(e)0.04901

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Chapter 1

Introduction

A long-standing problem in water wave theory is the determination of the influence of bed topography on an incident wave train. This is of considerable importance in coastal engineering where the shape of the seabed, and in some instances man-made obstacles, can dramatically effect wave behaviour (e.g.predicting wave heights in harbours). These problems involve a combination of the scattering, diffraction and refraction of waves and are difficult problems to solve.

This dissertation is concerned with the effect on waves of bed topography. The situation considered is where two regions of constant depth (not necessarily equal) are joined by a hump occupying a finite region in the seabed. The linearised equations for modelling such a flow are widely known but unfortunately there are rarely any existing analytic solutions except in very simple cases where vertical and/or horizontal boundaries are used. As a result of this, the mild slope approximation is used in order to generate the an integral equation to be solved.

Chapter 2

Integral equations

Integral equations occur widely in many areas of applied mathematics. An in-

- whether the equation is homogeneous or inhomogeneous.

The 'kind' of an integral equation refers to where the unknown function appears in the equation. If the unknown function only appears under the integral sign then the equation is said to be **first-kind**, whereas if it appears outside the integral sign as well then it is said to be **second kind**.

The interval of integration determines whether the integral equation is a **Volterra** equation or a **Fredholm** equation. If the interval of integration is definite then the equation is said to be a Fredholm equation. If however the interval of integration is indefinite then the equation is said to be a Volterra equation.

The equation is said to be **singular** if the interval of integration is indefinite or if the integrand is unbounded at any one or more points in the interval

2.2 Integral operators

An example of a second kind, Fredholm integral equation is

$$(\ ) = \ (\ ) + \ \int_a^b \ (\quad) \ (\ )\mathrm{dt}$$

where is a constant, ( ) is a **forcing term** and the function ( ) is called the kernel of the function and can be real or complex-valued. Thise740.251.5uin8474.145(i)0.21

Suppose we are trying to find an approximation to the integral equation

$$A \quad =$$

where A is defined as

$$A = (\; - \quad)$$

which is equivalent to

$$\sum_{n=1}^{N} (\ _n A \ _n \quad _m) = (\quad _m) \qquad (\ = 1 \quad N) \tag{2.4}$$

This gives an $N \times N$ matrix system which is solved for the coefficients $\ _n$. This solution technique is called Galerkin's method.

2.4 The Petrov-Galerkin method

The standard three-dimensional equations for fluid flow involve the use of Laplace's equation in three dimensions in order to solve the problem. However the modified mild-slope equation seeks to reduce the dimension of the problem by approximating the dependence of on z.

3.2 Derivation of the modified mild-slope equation

It was stated in the previous section that the fluid we are considering is deemed incompressible and irrotational. This means that a velocity potential exists and satisfies Laplace's equation in three space dimensions. As we are only attempting to approximate the solution, we can seek a weak solution $\simeq$ of Laplace's equation in the sense that ∇^2 is orthogonal to a given function . Hence

$$\int \int_D \left(\int_{-h}^{0} \nabla^2 \ \mathrm{dz} \right)$$

where the function $h(x, y)$ is the still water depth at the location (x, y) and the function $\eta_0(x, y$

$$
{}_2(\quad y) = \frac{\mathrm{sech}^2(\quad)}{12(\quad + \sinh(\quad))^3}(\quad{}^4 + 4\quad{}^3 \sinh(\quad) - 9\sinh(\quad)\sinh(2\quad)
$$
$$
+ 3\quad(\quad + 2\sinh(\quad))(\cosh^2(\quad) - 2\cosh(\quad) + 3))
$$

The abbreviation $\quad = 2$ has been used above.

Equation (3.1) is known as the modified mild-slope equation.

3.3 The mild-slope equation

The mild-slope equation is an alternate approximation to the modified mild-slope equation. It can be easily obtained from (3.1) above by making the assumption that the second derivative of and the square of its first derivative are negligibly small. This process results with

$$
\nabla \cdot (\quad \nabla \quad {}_0) + \quad{}^2 \quad = 0 \tag{3.2}
$$

which is the mild-slope equation.

The mild slope equation was the initial approximation used and was derived by Berkhoff (1973,1976). The paper that my dissertation has been based upon was written at a time when the mild-slope equation was a standard approximation to the function $(\quad y\ z)$ in the expression for the velocity potential $\Phi(\quad y\ z\quad)$. However since this paper has been published, the modified mild-slope equation has been derived. This was initially derived because many authors had commented that the mild-slope equation was failing to produce adequate approximations for certain types of topography such as ripple beds (where a finite interval of varying depth consists of small-amplitude sinusoidal ripples). This lack of accuracy led to many authors having to model ripple bed problems by alternate means. In 1995, Chamberlain and Porter developed the modified mild-slope equation as an alternate approximation technique. It has been found that this approximation can more accurately predict behaviour over a wider range of topographies.

The methods of solution discussed in the ensuing chapters are equally applicable to the modified mild-slope equation and the mild slope equation, the only difference being the inclusion of the extra two terms in the initial equation.

Chapter 4

Formulation of the wave scattering problem

$$[illegible](\ ) = \begin{cases} [illegible]_0 & \forall \ \le 0 \\ [illegible]_1 & \forall \ \ge [illegible] \end{cases}$$

where $[illegible]_0$ and $[illegible]_1$ are constant for a given problem and may or may not be equal. We further assume that $[illegible](\ )$ is continuous. The final assumption we make is that the wave motion is such that the crests are parall and $_0$ is the wavenumber corresponding to $_0$. The primes here denote differentiation with respect to .

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$$(\ ) = \begin{cases} A^{-} e^{ik \ x} + B^{-} e^{-ik \ x} & \forall \ \leq 0 \\ (& \end{cases}$$

$$'(\ell+) = \ '(\ell-) - \frac{'(\ell-)\ \ (\ell)}{2\ \ (\ell)}$$

This pair of equations tells us that although is continuous at $= 0$ and $=\ell$, its first derivative is not continuous at these locations. Furth

- if $A^- = 0$ then $_2 = \frac{B}{A}$ and

As a result of this, we must choose $\hat{\ }$ to satisfy the inhomogeneous equation subject to the homogeneous forms of (4.5) and (4.6). Hence

$$\hat{\ }'' + {}_0^2\,\hat{\ } =$$

These conditions would result in the integral equation having a complicated kernel. However, if we rewrite the boundary conditions as

$$'(0) + {}_0\,(0) = 2\,{}_0(c_1 + c_2\,(0) + c_3\,(1)) \tag{4.9}$$

and

$$'(l) - {}_0\,(l) = -2\,{}_0 e^{-ik\,l}(c_4 + c_5\,(0) + c_6\,(l)) \tag{4.10}$$

the resulting integral equation is of a lot simpler form. Note that if A^+ is set to 0 as it is for the subsequent numerical results, the coefficients c_1 c_6 are given by

$$c_1 = 1 \qquad c_2 = -\frac{'(0)}{4\,{}_0\,(0)} \qquad c_3 = 0$$

$$c_4 = 0 \qquad c_5 = 0 \qquad c_6 = \left(\frac{'(l)}{4\,{}_0\,(l)} - (\ {}_1\ {}_0\right.$$

were to set $A^- = 0$ then the coefficients could be calculated in the same way but they would be different to the coefficients above.

If we regard the right-hand sides of (4.9) and (4.10) as known then they only contribute to the function $_0$ and not to $\hat{}$

and

$$_2(\ ) = \sin(\ _0\ ) + \frac{1}{2\ _0}\int_0^l \sin(\ _0|\ \ -\ \ |)\ (\ )\ _2(\ )\mathrm{dt} \tag{4.16}$$

These equations may be recast as equations on the real, infinite-dimensional Hilbert space X consisting of the real elements of $L_2(0, l)$ by introducing two self-adjoint operators L and , which map X into itself, according to

$$(L\ \ )(\ ) = \frac{1}{2\ _0}\int_0^l \sin(\ _0|\ \ -\ \ |)\ (\ )d$$

and

$$(\ \ \ )(\ ) = \ (\ )\ (\ )$$

If we also define $_1$ and $_2$ by $_1(\ ) = \cos(\ _0\ )$ and $_2(\ ) = \sin(\ _0\ )$ then $_1$ and $_2$ are solutions of the two operator equations

$$A\ _1 = \ _1 \tag{4.17}$$

and

$$A\ _2 = \ _2 \tag{4.18}$$

in X where $A = \quad -L \quad$. In order to determine the reflection and transmission coefficients, we need to find approximations to $\;_1$ and $\;_2$ and also to $\;(0)$ and $\;(\ell)$.

Chapter 5

Methods of solution

As already stated, approximations to $_1$, $_2$, (0) and (ζ) are required. The functions $_1$ and $_2$ can be solved by numerous approximation methods for inhomogeneous integral equations whereas the approximations to (0) and (ζ) are not as straightforward. The following two results are useful in what follows.

Lemma: Suppose that $-L$ is invertible and that $_1$ and $_2$ are the solutions of (4.15) and (4.16) where L and are self-adjoint. Then $(\ _1 \quad _2) = (\ _2 \quad _1)$.

Proof:

$(\ _1 \quad _2) = (\quad _1 (\ -L\ )\ _2) = ((\ -L\ )\ _2)$ 4020Td[(1)-0.263136]TJ/R5711.9551Tf8.756250

$$_3 = \frac{1}{2}(c_3 + c_6) \qquad _4 = \frac{1}{2}(c_1 + c_4 e^{-2ik\ l})$$

$$_5 = \frac{1}{2}(c_2 + c_5 e^{-2ik\ l}) \qquad _6 = \frac{1}{2}(c_3 + c_6 e^{-2ik\ l} - e^{-ik\ l})$$

Also,

$$^+ = _1 + _2\ (0) + _3\ ()$$

and

$$^- = _4 + _5\ (0) + _6\ ()$$

These equations are not difficult to obtain. The expression for $^+$ arises from putting $= 0$ into (4.11) whereas putting $=$ into (4.11) yields the expression for $^-$. Substituting (4.14) into the plus option of (4.13) gives

$$^+ = (c_1 + c_2\ (0) + c_3\ () - \ ^-)B_2 + (c_4 + c_5\ (0) + c_6\ () - \ ^+)B_1 \quad (5.1)$$

whereas putting (4.14) into the minus option of (4.13) gives

$$^- = (c_1 + c_2\ (0) + c_3\ () - \ ^-)B_1 + (c_4 + c_5\ (0) + c_6\ () - \ ^+)\overline{B}_2 \quad (5.2)$$

The two simultaneous equations come from substituting the already known expressions for $^{\pm}$ into (5.1) and (5.2).

This last result tells us that a knowledge of the inner products A_{11} A_{12} and $A_2 2$ will allow us to determine the values (0) and (ℓ) via (4.17) and (4.18). This in turn will allow us to calculate the reflection and transmission coefficients. However, before we can solve for (0) and (ℓ), we need to determine approximations to $_1$ and $_2$.

5.1 Inner product approximations

We attempt to find an approximation to the three inner products A_{jk}, ($=$ 1 2) by using variational calculus.

Consider the functional $_{jk} : X^2 \to \Re$ defined by

$$_{jk}(p\ q) = (\ _j\ q) + (p\ \ _k) - (Ap\ q) \qquad = 1\ 2$$

If we let $p = \ _j + \delta\ _j$ and $q = \ _k + \delta\ _k$ where $\delta\ _j$ represents the variation in the approximation to $_j$ and $\delta\ _k$ represents the variation in the approximation to

mation to

$$
\begin{aligned}
&= \quad {}_{jk}(\quad {}_j \quad {}_k) + (\quad {}_j - A \quad {}_j \quad {}_k) + (\quad {}_j \quad\quad {}_k - A^* \quad {}_k) \\
&+ \quad O(|| \quad {}_j|||| \quad {}_k||)
\end{aligned}
$$

and hence we deduce that ${}_{jk}(p \; q$

Putting these expressions into the expression for $_{jk}$ yields

$$
\begin{aligned}
{}_{jk}(\;{}_j\,,\;{}_k) \;\simeq\; & \sum_{m=1}^{N} {}^{(k)}_{m}(\;{}_j\;\;{}^{(k)}_{m}) + \sum_{n=1}^{N} {}^{(j)}_{n}(\;{}^{(j)}_{n}\;\;{}_k) \\
& -\; {}^{N}
\end{aligned}
$$

finite-dimensional trial spaces from which the approximations are selected.

We seek an approximation to which is the solution of the integral equation $= \ + L$. The Neumann series for this equation is given by

$$= \ + \sum_{n=1}^{\infty} (L \ )$$

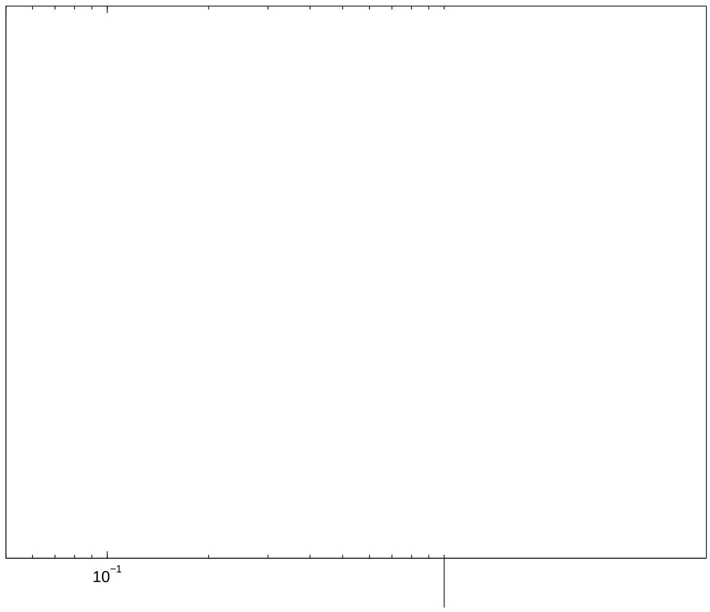
Chapter 6

Numerical results

In this section, we attempt to reproduce various results that h which he produced a graph of $| \quad |$ against $\quad_s$, a dimensionless parameter. The paper presented by Chamberlain (1993) considered the reproduction of Booij's graph using a very similar method to the one set out in this dissertation. He produced results that to the naked eye seemed to be identical to the original set produced by Booij. To confirm the accuracy of our methods we are here again interested in reproducing Booij's results for the MSE.

In Booij's original paper, all length scaling is conducted with respect to the deep-water wavenumber $\frac{\sigma^2}{g}$. Chamberlain chose to non-dimensionalise all length values with respect to [illegible] and create a non-dimensional ( ) instead of the [illegible] (

The program was run using $\frac{\sigma^2}{g} = 1$ and for values of $\;_s$ between 0.05 and 6. The number of discrete points in each interval used for the numerical computation of the integrals was 200. This ensured that the step size was small which is crucial when performing numerical integratio



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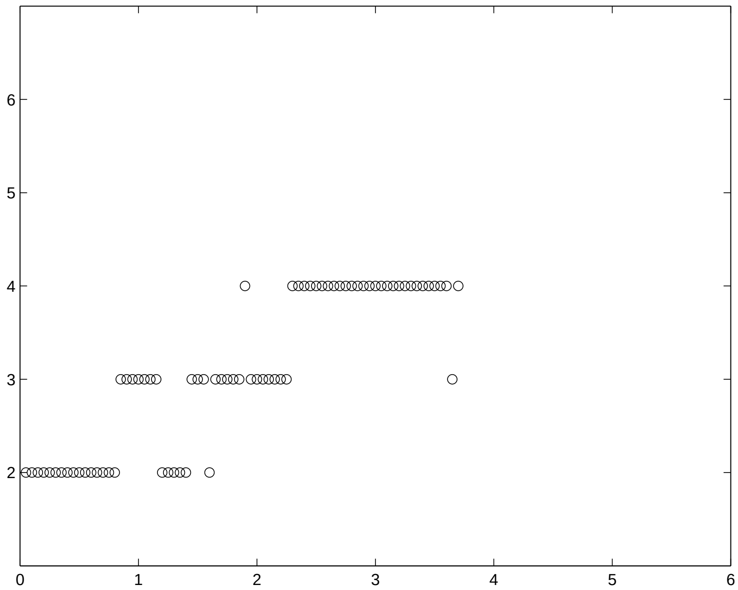

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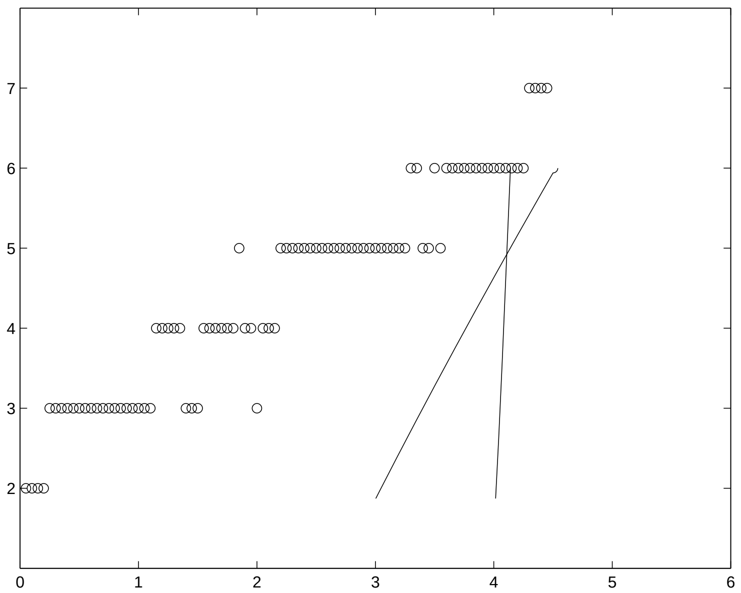

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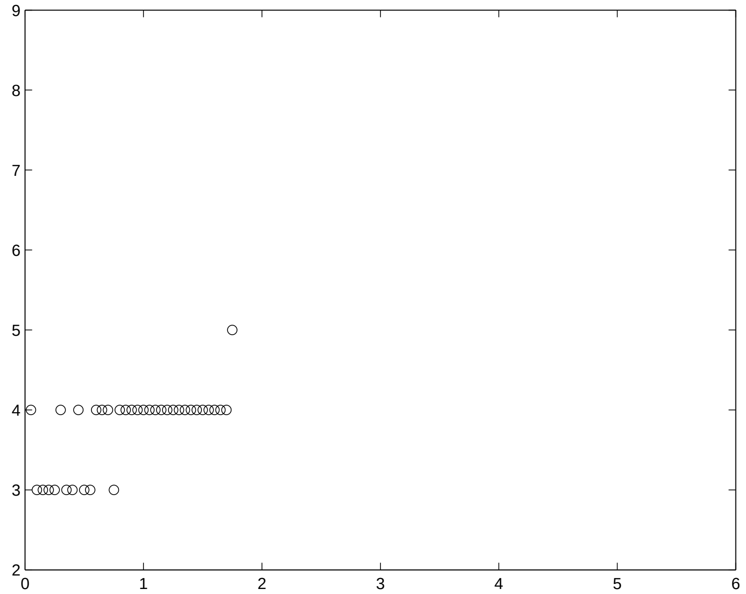

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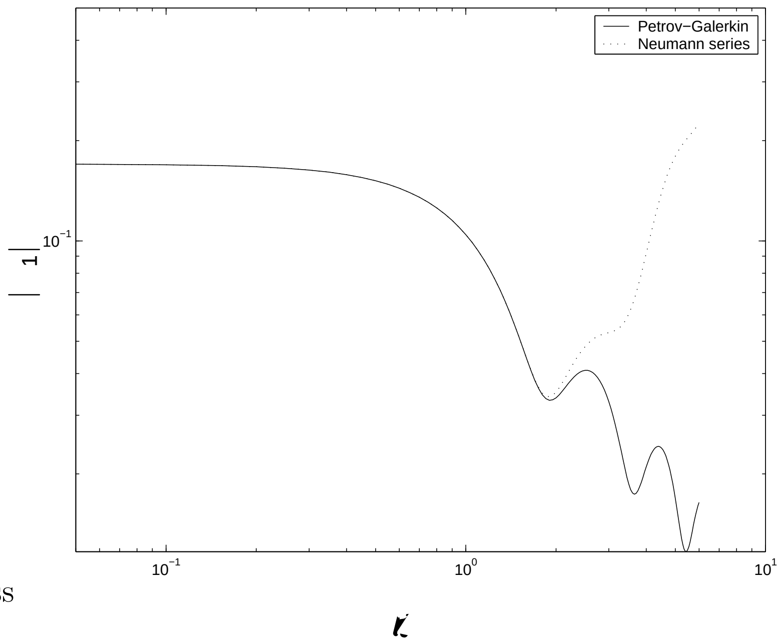

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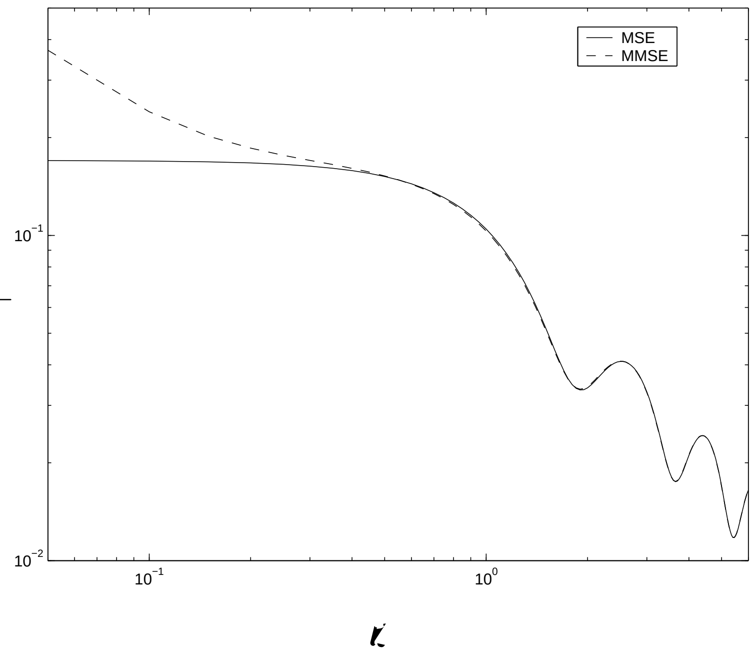
discontinuous. If ′( ) is continuous then our current set of boundary conditions are correct, whereas if ′( ) is discontinuous at = 0 and/or =, then new boundary conditions are required.

Booij's test problem involves two slope discontinuities in ′ at = 0 and =. This means that our existing boundary conditions are incorrect. The correct set of boundary conditions for this problem are

$$'(0) + {}_0 (0) = 2 {}_0 \left(1 + \frac{{}_1(0)}{2 {}_0 \sqrt{\ (0)}}\right) - 2 {}_0 \quad '$$



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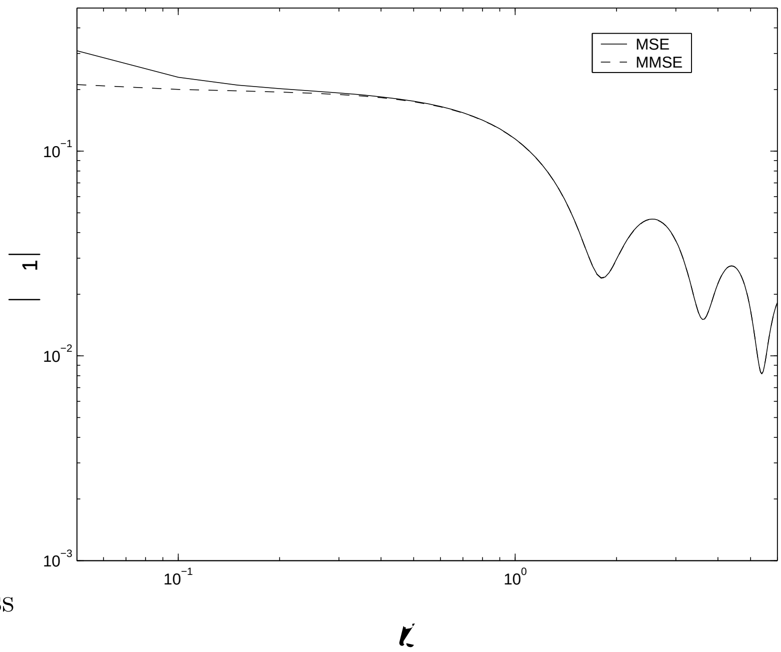

Figure 6.7: Comparison of the MSE and MMSE subject to discontinuous boundary conditions applied to Booij's test problem.

and Staziker.

6.2 The effects of different types of talud on the reflection coefficients

This example is again taken from the paper by Chamberlain (1993). In his paper, Chamberlain endeavoured to determine the effect of different types of talud on the reflection coefficient. He considered three different types of talud: a concave talud, a convex talud and a linear talud. The non-dimensional fluid depths to the left and right of each talud are 1 and 0 5 respectively. The equations for each talud are defined to be:

$$_{1}(\quad) = \frac{1}{2} + \frac{1}{2}(\quad - 1)^2$$

for the convex talud,

$$_{2}(\ ) = 1 - \frac{1}{2}$$

for the linear talud, and

$$_{3}(\ ) = 1 - \frac{^{2}}{2}$$

for the concave talud. As discussed in the previous example, Chamberlain employed a non-dimensional scaling on his variables with respect to . Thus his above is defined to be equal to the used in our notation divided through by . Also, the above equations defining the shape of each talud need to be divided through by $_0$. Therefore in our notation the shapes of the taluds become

$$_{1}(\ ) = \frac{_{0}}{2}\left(1 + \frac{(\ -\ )^{2}}{^{2}}\right)$$

$$_{2}(\ ) = _{0}\left(1 - -\right.$$

$$_{2}(\ ) = _{0}\left(\right.\quad ^{2}$$

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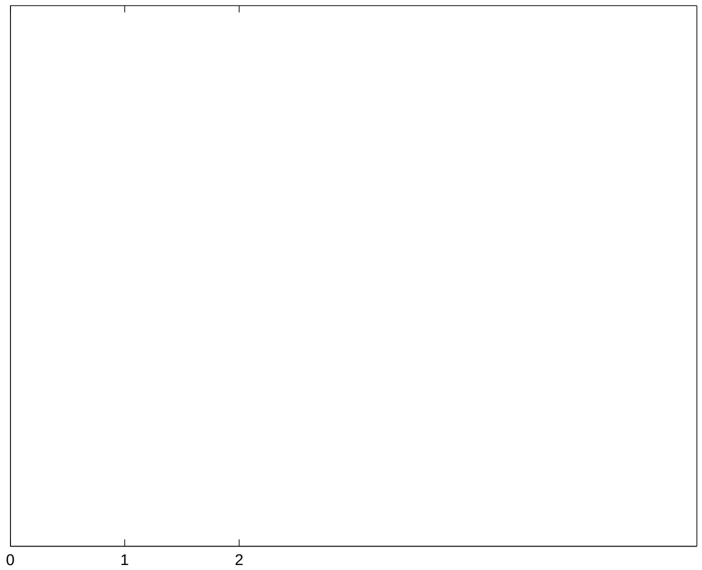

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Chamberlain and Porter (1995) considered some examples of wave scattering over ripple beds to emphasise the difference in approximations produced by the MSE and the MMSE. They used some experimental data provided by Davies and Heathershaw (1984) and compared their solutions to these results in order to draw conclusions about accuracy. We now attempt to reproduce this here.

The function $h(\ )$ in this problem is now taken to be

$$h(\ ) = h_0 - d \sin\left(\frac{2n}{L}\right) \qquad 0 \qquad L$$

where n is the number of ripples. The real constant d will be defined shortly. This bedform therefore consists of a sequence of n sinusoidal ripples about the mean depth $z = -h_0$.

The examples considered by Chamberlain and Porter (1995) involved the non-dimensionalisation of the parameters in the problem. They plotted a graph of $|\ _1|$ against a parameter in the interval (0 5 2

$$ = \frac{d}{\ _0}$$

and fix a particular $\in (0\ 5\ \ 2\ 5)$ we obtain

$$ _0 = \frac{}{20}$$

$$ = n \quad \tanh(\ _0)$$

$$ _0 = \ _0 \tanh(\ _0)$$

so provided that and n are specified at the start of every problem, we can calculate every necessary quantity.

Figure (6.9) shows the approximations produced by the MSE and the MM54(t)-0.147047(u)-0.310

very different. The MSE picks out that there should be a peak at $= 1$ but it does not predict the correct value at the peak. It also completely misses the peak at $= 2$. The MMSE does however pick out the correct values at the peak at $= 1$ and $= 2$. This graph is again identical to the naked eye to the graph produced by Chamberlain and Porter.

Chapter 7

Conclusions

The scattering of small-amplitude waves by variations in a one-dimensional topography has been examined. Rather than attempting to solve the original boundary value problem that was formulated as an ordinary differential equation, the problem has been converted to a complex-valued integral equation and then split up into two further real-valued integral equations. This method is very similar to the method used by Chamberlain (1993) except that the integral equation formed is not self-adjoint and the function is not forced to be entirely one-signed. This method does not have the advantage of providing an integral equation for which we can easily derive error bounds for but the implementation of the approximation is made considerably more simple.

We have seen that this method has proved just as effective as the original integral equation method used by Chamberlain, and that it can easily reproduce the results that other people have obtained using numerically solving differential equation techniques. In addition, some calculations have been performed using the modified mild-slope equation which has never been done

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