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Department of Mathematics

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August 2014

Acknowledge ents

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Declaration

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Abstract

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1 Introduction

The migration of contaminants or reactants through porous media is of critical importance for a wide variety of scientific and industrial processes. This process can include both reaction with and without the porous media.[1] Contamination of soils and sands for example, by liquid pollutants can result in those pollutants entering groundwater systems, thus increasing the chance of harm to crops and ecosystems, and water supplies to the public. It is important therefore to characterise such phenomena, and be able to predict the fate of such species, dependent on porous media and pollutant type, over a wide range of conditions. It is known that the nonlinear diffusion of unreactive pollutants can be classified into slow, fast or superfast diffusion, dependent on the characteristics of the diffusion coefficient in the porous medium equation. The diffusion coefficient is often dependent on the saturation level of the contaminant species in the porous medium, which drives the diffusion. This changes over time as the solution

1.1 Background to di usion through porous media

is the porous medium equation (PME) in 1D Cartesian coordinates. It can be generally stated as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D(u(x(t);t)) \frac{\partial u}{\partial x} \right) + s(x);$$

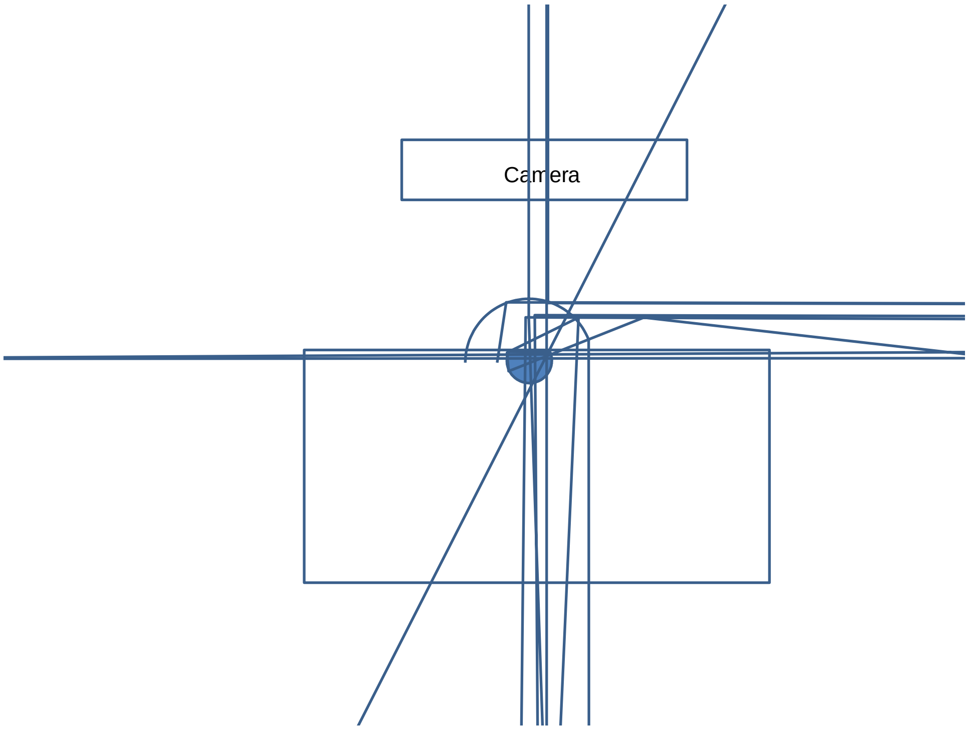
where $D(u(x(t);t))$ is the di usion coe cient (which, when $D(u) = u^m$, is the PME). For radial di usion, the nonlinear PME is given by

$$\frac{\partial u}{\partial t} = \frac{1}{r^{d-1}} \frac{\partial}{\partial r}\left(u^m r^{d-1} \frac{\partial u}{\partial r} \right) + s(r); \tag{1}$$

where d is the number of dimensions. In this study we take $d = 2$. For both radial and cartesian coordinates, $s(x(t))$ and $s(r(t))$ represent a source or sink term, such as ingress of another source of liquid into the system, or evaporation of the liquid out of the system respectively. In the case of this study, we do not consider any source or sink terms, particularly since we assume that our migrating liquid is a non reacting, non volatile liquid.

The transition of transport through porous media is very much dependent on the saturation level of the solvent. This change in di usion behavior can be very sharp as detailed in the study by Lukyanov et al [2]. This transition is seen to occur at about 20% saturation for low saturation levels. The general classication of the "speed" of non linear di usion with respect to the value of m (142 11.9551 Tf 2249375 0 Td [(d)-0.902

for slow ($m = 1$) and superfast ($m = 3/2$) diffusion. Where the concentration $u(r)$ is greater than 20% of $u(r = 0)$, for $m = 1$, we present a self similar/analytic solution for $u(r(t); t)$, which is used to obtain the boundary values of the velocity (dr/dt), rate ($\partial u/\partial t$) and gradient ($\partial u/\partial r$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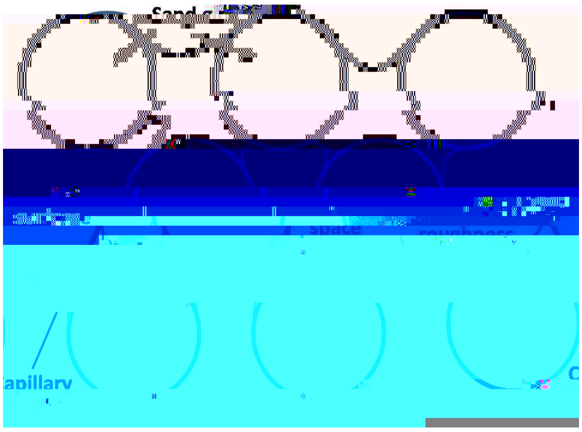

Figure 3: Structure of capillary areas between sand particles at saturation levels greater than 10%

This paper provided evidence, from an environmental pollution perspective, that small concentrations of harmful solvents can travel long distances through packed soils/sands over long periods of time, thus justifying the requirement to investigate the migration of harmful contaminant species.

The transport of liquid through the porous media was deduced to be due to capillary action at the surface-roughness scale of the individual particles, and indeed the total saturation of the sand is the interplay between that within the capillary bridges, and that in the surface roughness around the particles. Generally, the flow obeys a Darcy-like Law, where permeability is related to the geometry of the grooves [4]. In macroscopic modelling of the migration of the nonvolatile liquid, it was assumed that the particles were perfect spheres, and that the wetted grooves within the surface roughness of the particles were completely filled. Modelling results were then compared following non-dimensionalisation, for saturation, distance and time. When comparing to equation (1), $= m + 1$. This is known as *superfast diffusion*. In the standard porous medium equation (which we will use for the slow section of our model), where $m > 1$ in equation (3), the low saturation levels at the edge of the liquid delay the onset of the wetting front, however (and in our model for superfast diffusion), for superfast regimes, the velocity of the wetting front decreases over time as the saturation level/concentration decreases. The results of the model are in agreement with experiment, and for the characterised system with the organic solvent, trioctyl-phosphate (TEHP) at low concentration (deposited on well characterised sand) the superfast diffusion regime is said to hold for $0.5\% < s < 10\%$. It was also shown that the wetted volume, V fell on a power law with time, where $V \propto t^{0.75}$ as shown in Figure 4.

occur where the solution value for concentration (or saturation) is 20% of its maximum value. As expected, the central node will not move, however, the other nodes in the mesh will move with a velocity that changes with time. This is a rst attempt at

In Chapter 4 we present the results of the moving mesh methods for both slow and superfast diffusion in isolation for 2D radial diffusion, followed by the results for the combined numerical method. In the case of the slow diffusion we show the results for constant mass between spatial nodes. We discuss the stability and limitations on the discretisation of time via a Lagrangian type Courant-Friedrichs-Lewy (CFL) condition [7]. This condition prevents the spatial nodes from overtaking one another.

In Chapter 5 we draw some conclusions from the results, comparing the behaviour of the slow and superfast regimes in isolation with the combined diffusion moving mesh scheme.

In Chapter 6 we make recommendations for future study0.13dr315523T19(o)0.13069726(f63(o

2 Generation of an analytic solution for slow diffusion

In this chapter we look for an analytic solution to the porous medium equation where $m = 1$, by generating a self similar solution. We shall firstly describe the scale invariance applied to 2D radial diffusion, and follow this with the generation of the self similar/analytic solution. This method is discussed in reference [8], and adopted in 1D Cartesian coordinates, where $m = 4$, in reference [9].

2.1 Scale invariance for slow diffusion

We consider the case where $m = 1$ and $d = 2$ in the nonlinear porous medium equation (2)

$$\frac{\partial u}{\partial t} = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}u^{m}\frac{\partial u}{\partial r}\right) \qquad (2)$$

where d is the number of dimensions. We now apply a scaling transformation to equation (2),

$$u = \hat{u}; \quad t = \hat{t}; \quad r = \hat{r}; \qquad (3)$$

using the scaling parameter, , where and are constants. This transforms equation (2) into the new non-dimensionalised variables $\hat{u}$, $\hat{t}$, and $\hat{r}$;

$$\frac{\partial \hat{u}}{\partial \hat{t}} = \frac{{}^{2}\;{}^{2}}{\hat{r}^{d-1}}\frac{\partial}{}$$

and

$$u(b) = 0 \quad \text{at the boundary} \quad r = b(t):$$

This results in

$$\frac{@}{@t}\int_0^{b(t)} r^{d\ 1} u\, dr = 0;$$

and so

$$\int_0^{b(t)} r^{d\ 1} u dr = k; \qquad (6)$$

where k is a constant. Upon transformation of equation (6) into $\hat u$, $\hat t$ and $\hat r$ we obtain

$$^{d}\int_0^{b(t)} \hat r^{d\ 1}\hat u d\hat r = {}^{0}k;$$

thus generating a second equation in and ,

$$+ 2 \quad = 0: \qquad (7)$$

Therefore, solving equations (5) and (7)

$$= \frac{1}{4}; \quad \text{and} \quad = \frac{1}{2}:$$

This results in a scale transformation

$$u = {}^{\frac{1}{2}}\hat u; \quad t = \hat t; \quad \text{and} \quad r = {}^{\frac{1}{4}}\hat r;$$

so

$$= \frac{u^{1=}}{\hat u^{1=}} = \frac{t}{\hat t} = \frac{r^{1=}}{\hat r^{1=}}:$$

An in depth text on scaling methods and self similarity can be found in reference [11].

2.2 Generation of a self similar solution for slow, nonlinear di usion

We now introduce two variables, and y, that are independent of , and which are invariant under transformation equation (3). We then make a function of y, and then transform as follows

$$= \frac{u}{t} = \frac{\hat u}{\hat t};$$

$$y = \frac{r}{t} = \frac{\hat{r}}{\hat{t}}:$$

With

where d is a constant of integration. Therefore

$$= A \quad \frac{y^2}{8};$$

where A is a constant. We we can write

$$= A \quad \frac{y^2}{8}; \quad \frac{y^2}{8} \quad A:$$

$$= \quad A \quad \frac{y^2}{8} \quad {}_{+} \quad :$$

Figure 5 illustrates the self similar solution, equation (8) where $A = 2$, for the original non-linear 2D equation (2), where $d = 2$, $m = 1$. Four time steps are shown. It can be seen that the pro le gradually attens over time and is symmetric about $r = 0$. In this study we take the initial time,

$$= \quad ru\frac{@u}{@r}\Big|_{r_A(t)}^{r_B(t))} + \quad ur\frac{dr}{}$$

Algorithm for slow diffusion alone with mass conservation

Initially:

1. Define the initial condition everywhere to be

$$u(r) = 2 - \frac{r^2}{8} \quad \text{at} \quad t_0 = 1$$

2. Discretise the mesh $r_i(t) = r_0(t) + i\Delta r$, where Δr are N uniform discretisations $i = 0, 1, \ldots, N$, across the domain at t_0.
3. Calculate the initial masses between nodes using Simpson's Rule $b_i = \int_{r_0}^{r_i} ur\,dr$, between the origin at r_0, and nodes i, for $i = 0, \ldots, N$.

Then, at each time step,

1. Calculate the velocities from equation (12), approximating the gradient by suitable finite differences, for example, central differences:

$$\left.\frac{\partial u}{\partial r}\right|_i \approx \frac{u_{i+1}(t) - u_{i-1}(t)}{r_{i+1}(t) - r_{i-1}(t)} \quad i = 1, \ldots, N-1.$$

2. Define the velocity of the final node at r_N through linear extrapolation

$$v_N = 2v_{N-1} - v_{N-2}$$

or by using the one-sided difference

$$v_N \approx \frac{u_N - u_{N-1}}{r_N - r_{N-1}}.$$

3. Update the new positions at the next time step using the explicit Euler scheme

$$r_i(t + \Delta t) = r_i(t) + \Delta t \left.\frac{dr}{dr}\right|_i ;$$

for equally spaced time steps Δt.

4. Update the solutions at the next time step using the new positions from step 3,

$$u_i(t + \Delta t) = \frac{b_{i+1} - b_{i-1}}{r_i(t + \Delta t)\,(r_{i+1}(t + \Delta t) - r_{i-1}(t + \Delta t))}; \quad \text{for} \quad i = 1, \ldots, N-1.$$

5. Calculate u at the origin by approximating the integral/mass between the origin and the first mesh point at $t + \Delta t$,

$$\int_0^{r_1(t+\Delta t)} ur\,dr \approx \frac{1}{4}\left(u_0(t+\Delta t) + u_1(t+\Delta t)\right)\left(r_1(t+\Delta t)^2 - r_0(t+\Delta t)^2\right) = b_1.$$

The value of b_1 is constant for all time. Therefore

$$u_0(t+\Delta t) = \frac{4b_1}{r_1(t+\Delta t)^2} - u_1(t+\Delta t)$$

as $r_0(t) = 0 \quad \forall t$.

6. Finally, the value of $u_N(t+\Delta t) = \frac{1}{5}u_0(t+\Delta t)$, from the boundary condition at $r_N \quad \forall t$.

Results are presented in Section 4.1.

At r_l, it is assumed that the boundary values are given by the self-similar solution $u(r;t) = u(0;t)=5$. The initial conditions are also taken to be the solutions to the self similar solution at $r_i(t)$. We start with the initial conditions

$$u_l = \frac{2}{5} \quad \text{at} \quad t_0 = 1$$

therefore from equation (8)

$$r_0 = \frac{8}{\sqrt{5}} \quad \text{at} \quad t_0 = 1:$$

Also, from the self similar solution, substituting $t_0 = 1$ into the appropriate equations

$$\left.\frac{\partial u}{\partial t}\right|_l = \frac{(r_l)^2}{8} - 1 \quad \text{at} \quad t_0;$$

and

$$\left.\frac{\partial u}{\partial r}\right|_l = -\frac{r_l}{4} \quad \text{at} \quad t_0:$$

We can also find an expression for the initial velocity of the slow/fast interface, $r_l(t)$, from

$$\left.\frac{du}{dt}\right|_l = \left.\frac{\partial u}{\partial r}\right|_l \cdot \left.\frac{dr}{dt}\right|_l + \left.\frac{\partial u}{\partial t}\right|_l :$$

Since at the interface

$$\left.\frac{du}{dt}\right|_l = 0 \quad \text{at time} \quad t;$$

then the velocity at the interface node is therefore given by

$$\begin{aligned} \left.\frac{dr}{dt}\right|_l &= -\left.\frac{\partial u}{\partial t}\right|_l \Big/ \left.\frac{\partial u}{\partial r}\right|_l \\ &= \frac{r_l(t)}{2t} - \frac{4}{r_l(t)\sqrt{t}} \quad \text{at} \quad t: \end{aligned} \qquad (13)$$

Due to the flux into the region at $r_l(t)$ (from what is the slow diffusion regime)

$$ru\frac{\partial u}{\partial r} + ru\frac{dr}{dt} = 0 \quad \text{at} \quad r_l(t):$$

There is an additional boundary condition at $r_N(t)$, that maintains/drives migration of the liquid.

$$u_b = 0{:}01 \quad \text{at } r_b; \quad t \quad 0{:}$$

The total mass in the fast domain $({}_{l}(t); r_b(t))$

integrals of u from $r_0(t)$ to $r_i(t)$, where $i = I (= 0); 1; \ldots; N$

at time t, we obtain

$$\left.\frac{dr}{dt}\right|_i = \frac{u_I r_I}{u_i r_i}(1 \quad {}_i)\left.\frac{dr}{dt}\right|_I + \frac{(u_I)^m r_I}{u_i r_i}(1 \quad {}_i)\left.\frac{@u}{@r}\right|_I \quad (u_i)^{m \quad 1}\left.\frac{@u}{@r}\right|_i : \qquad (19)$$

Hence, calculating the velocity of the internal nodes at any time requires knowledge of the boundary values at r_I, i.e, u_I, $\left.\frac{dr}{dt}\right|_I$ and $\left.\frac{@u}{@r}\right|_I$, making use of the self similar solution. Using the previous equation (13), to recap

$$\left.\frac{dr}{dt}\right|_I = \frac{r_I}{2t} \quad \frac{4}{r_I \sqrt{t}} \quad \text{at time } t$$

and from our knowledge at the interface,

$$\left.\frac{@u}{@r}\right|_I = \quad \frac{r_I}{4t} \quad \text{and} \quad \left.\frac{@u}{@t}\right|_I = \frac{(r_I)^2}{8t^2} \quad \frac{1}{t^{3=2}};$$

we can substitute these into expressions into equation (19) to obtain

$$\left.\frac{dr}{dt}\right|_i = \frac{u_I r_I}{u_i r_i}(1 \quad {}_i)\left(\frac{r_I}{2t} \quad \frac{4}{r_I\sqrt{t}}\right) + \frac{(u_I)^m r_I}{u_i r_i}(1 \quad {}_i)\left(\quad \frac{r_I}{4t}\right) \quad (u_i)^{m \quad 1}\left.\frac{@u}{@r}\right|_i :$$

This is how the velocity of the spatial nodes $r_i(t)$ progress in the superfast diffusion domain.

We now seek a suitable time-stepping method for the superfast moving mesh method, and update the position of each spatial node r_i by

$$r_i(t + \quad t) = r_i(t) + \quad t \left.\frac{dr}{dt}\right|_i ;$$

and the total mass ${}_{fast}(t + \quad t)$ by

$${}_{fast}(t + \quad t) = \quad {}_{fast}(t) + \quad t \; {}^0_{fast}(t);$$

where ${}^0_{fast}(t)$ is given by equation (15). We have expressions for both $\left.\frac{@u}{@r}\right|_I$ and $\left.\frac{dr}{dt}\right|_I$ (derived from the self similar solution). So equation (15) becomes

$${}^0_{fast}(t) = \quad \frac{(r_I)^2 u_I}{2t} \quad \frac{(u_I)^{m \quad 1}}{2}\left(1 \quad + \quad \frac{4u_I}{\sqrt{t}}\right) : \qquad (20)$$

We can now recover the solution for $u(r(t + \quad t); t + \quad t)$ at the next time step, approximating equation (16) as

$$u_i(t + \quad t) = {}_{fast}(t + \quad t)\frac{{}_{i+1} \quad {}_{i \quad 1}}{r_i(t + \quad t)\,(r_{i+1}(t + \quad t) \quad r_{i \quad 1}(t + \quad t))} : \qquad (21)$$

The individual mass fractions ($_i$) do not change with time. The self similar solution is used to calculate values of $u(r; t)$ and the velocities of the spatial nodes v_i at time $t = 0$, at what would be the boundary with the slow di usion regime.

In the superfast region we can no longer use the self-similar solution, and t a parabola between the nal node of the self similar solution/slow parabola, and the nal node in the superfast di usion pro le, where we have assigned a nite value of $u(N; t) = 0.01$. We give the solution at the nal node this small value of u in order to ensure the advancement of the uid. In reality, where $u(N; t) = 0$ the capillary bridges in Figure 3 would collapse and no longer exist, and therefore there would be no further

The initial values of $u_i(t_0)$ in the fast di usion regime are given by

$$u_i \;=\; u_N + (u_I \quad u_N) \; \frac{N \quad r + r_I \quad r_i}{(N \quad r)^2} \;: \qquad (22)$$

The total mass in the fast region, at t_0 is therefo

3. Define the boundary conditions at $r_l(t)$ at time t

$$u_l = \frac{2}{\sqrt{t}} \quad \frac{r_l}{8t}; \quad \text{as given by the self similar solution}$$

$$v_l = \frac{r_l}{2t}{}_l \quad \frac{4}{r_l\sqrt{t}}; \quad \text{as given by the self similar solution}$$

4. Define the boundary conditions at $r_b(t)$

$$u_b$$

4. Compute the new total mass from $r_I (t + \quad t)$ to $r_N (t + \quad t)$ using ${}^{0}_{fast} (t)$ calculated from Equation (20), and explicit Euler method.

$$_{fast} (t + \quad t) = \quad _{fast} (t) + (\quad {}^{0}_{fast} (t)) \quad t$$

The boundary conditions at $r_0 = 0$, $t_0 = 1$ are

$$
\begin{aligned}
u_0 &= 2; \\
\frac{\partial u}{\partial r}\Big|_0 &= 0; \\
\frac{dr}{dt}\Big|_0 &= 0;
\end{aligned}
$$

and for all time $t > 0$,

$$\frac{\partial u}{\partial r}\Big|_0 = 0;; \tag{23}$$

$$\frac{dr}{dt}\Big|_0 = 0;; \tag{24}$$

as, intuitively, the origin about which diffusion is symmetric, $r_0(t)=0$.

Let us denote the position and solution at the interface between slow and superfast diffusion as r_I and u_I respectively, at t_0, where I is the spatial nodal identity. We know from previous equations (25), (26), (27), and (28) respectively, that at $t_0 = 1$ for an initial discretisation $\Delta r = \frac{1}{4\sqrt{5}}$,

$$r_I = \frac{8}{\sqrt{5}}; \tag{25}$$

$$\frac{\partial u}{\partial r}\Big|_I = \frac{2}{\sqrt{5}}; \tag{26}$$

$$\frac{dr}{dt}\Big|_I = \frac{3\sqrt{5}}{10}; \tag{27}$$

$$\frac{\partial u}{\partial t}\Big|_I = \frac{3}{5}. \tag{28}$$

HowI.8)

Therefore, since

$$\frac{dr}{dt}_{\;0} = 0 \quad \text{and} \quad \frac{@u}{@r_0} = 0 \quad \text{at} \quad r_0(t) = 0$$

$$\frac{dr}{dt}_{\;i} = \frac{u_l\, r_l \quad {}_i}{u_i\, r_i} \quad @u$$

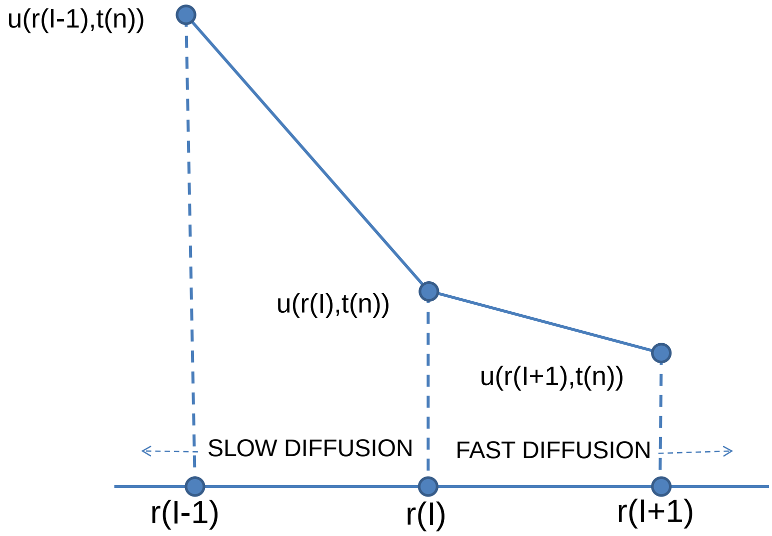

Figure 6: Identi cation of velocity of interface node between slow and fast di usion regimes

There will be zero rate of change of mass as mass ows at the same rate into the superfast regime as ows out of the slow regime, by the application of the continuity equation at the boundary;

$$\frac{d}{dt}\left(\frac{1}{{}_{slow(1)}}\int_{r_{l-1}(t)}^{r_l(t)} ur\,dr + \frac{1}{{}_{fast(1)}}\int_{r_l(t)}^{r_{l+1}(t)} ur\,dr\right) = 0: \qquad (31)$$

where ${}_{slow(1)}$ and ${}_{fast(1)}$ represent the total mass in the slow and fast regimes respectively at time t_0. The integral to the left of the boundary is given by

$$\frac{d}{dt}\frac{1}{{}_{slow(1)}}\int_{r_{l-1}}^{r_l} ur\,dr = \frac{1}{{}_{slow(1)}}\left(\int_{r_{l-1}}^{r_l}\frac{@u}{@t}r\,dr + \left[ur\frac{dr}{dt}\right]_{l-1}^{l}\right) \frac{1}{({}_{slow(1)})^2}\frac{d\,{}_{slow(1)}}{dt}\int_{r_{l-1}}^{r_l} ur\,dr; \qquad (32)$$

an to t r t t s

$$\frac{}{t} \quad 1$$

w r

$$A = \frac{u_l r_l}{\text{slow}(1)} \quad 1 \quad l \quad 1 \quad \frac{(r_l^2 \quad r_{l\ \ 1}^2)(u_l + u_{l\ \ 1})}{4 \ \text{slow}(1)}$$

$$+ \frac{u_l^m r_l}{\text{fast}(1)} \quad \frac{(r_{l+1}^2 \quad r_l^2)(u_{l+1} + u_l)}{4 \ \text{fast}(1)} \quad l+1$$

an

$$B = \frac{u_l r_l}{\text{slow}(1)} \quad 1 \quad l \quad 1 \quad \frac{(r_l^2 \quad r_{l\ \ 1}^2)(u_l + u_{l\ \ 1})}{4 \ \text{slow}(1)}$$

$$+ \frac{u_l r_l}{\text{fast}(1)} \quad \frac{(r_{l+1}^2 \quad r_l^2)(u_{l+1} + u_l)}{4 \ \text{fast}(1)} \quad l+1$$

7. Co bining the initial pro le for slow and fast di usive regi es at initial ti e t_0.

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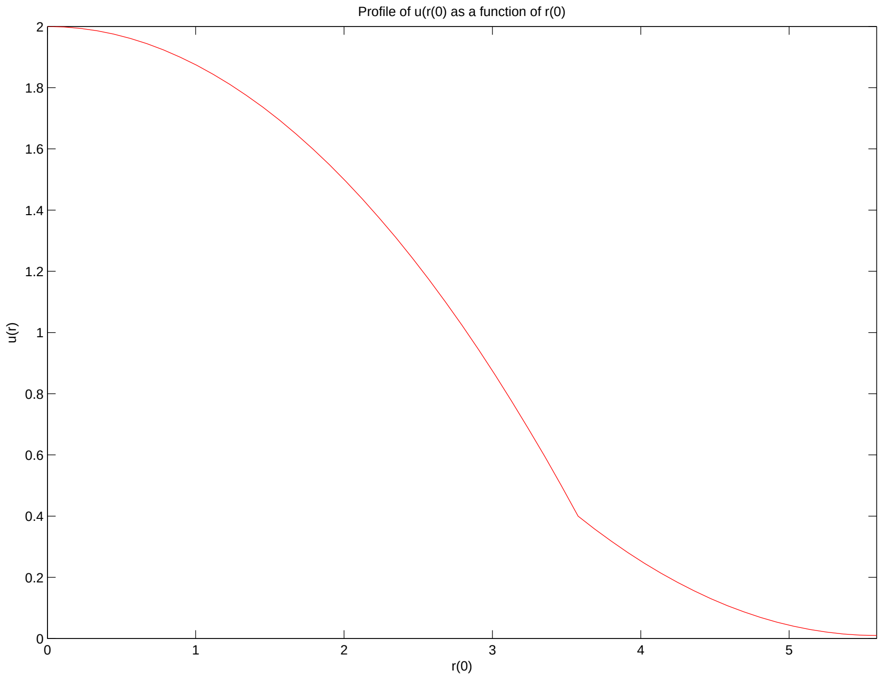

Figure 7: Combined slow and fast di usion profiles at t_0.

7. Generating an algorith for the co bined di usion.

now n rat an a or t to a van t n t a pro s r n t on

Combining slow and superfast di usion

For a s r_i $i = 0; ::: ; N$

$$0 = r_0(t) < r_i(t) < ::: < r_I < ::: < r_{N\ 1}(t) < r_N(t)$$

w s un or at t_0 w t N no s a stan r apart $r_N(t)$ n a ov n oun ar nt r a no $r_I(t)$ s t no at w t s ow us on r an s to t sup r ast r

In t a

At t_0 an v n t oun ar on t on

$$u_N(t) = 0{:}01;$$

Ca u at t v o t

Fro t oun ar on t on t so ut on or t na spat a no s a onstant

$$u_N(t + \quad t) = 0{:}01:$$

[redacted]

$$u_0(t + \quad t) = \frac{4 \; _{slow}(t + \quad t) \quad _1}{(r_2(t + \quad t)^2} \quad u_1(t + \quad t);$$

as s r n t on

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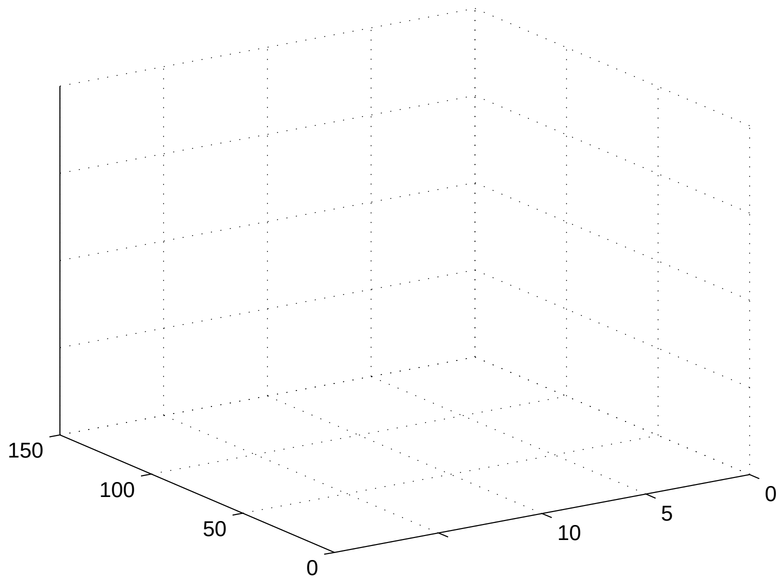
$$u_I(t + \quad t) = \; ^{u \quad t}$$

4 Results



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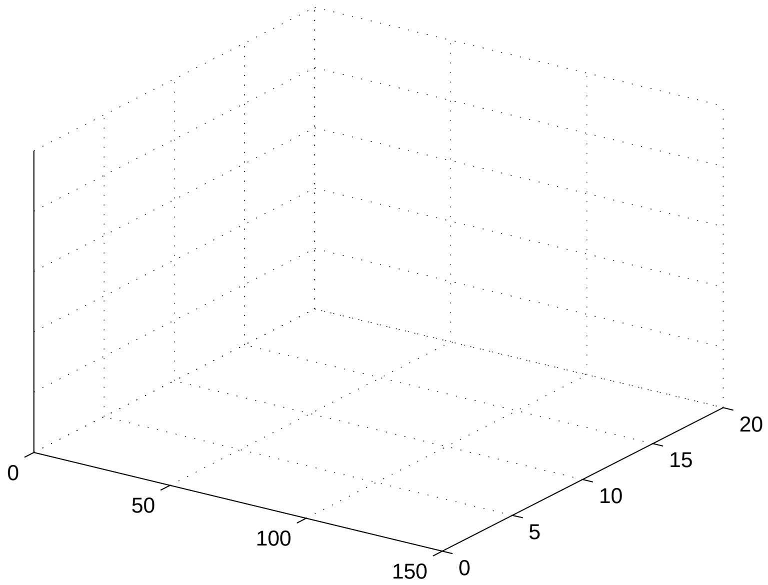
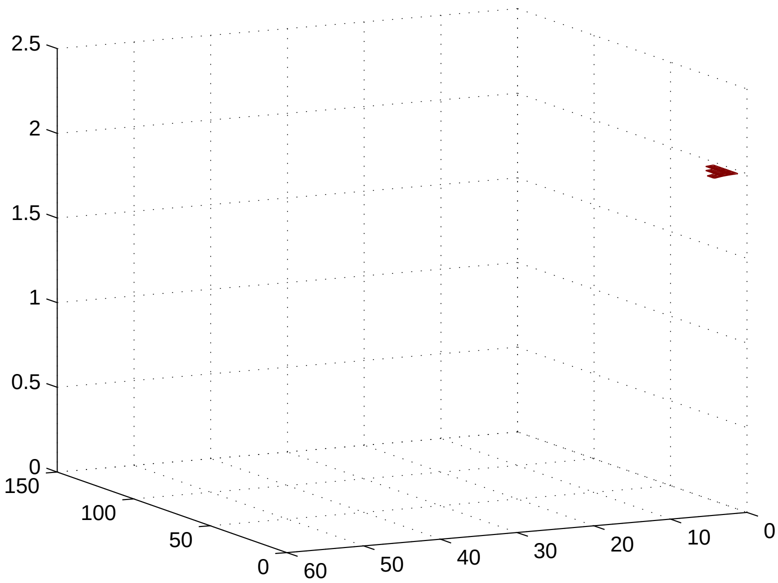


F ur v o t s o t no s r as w t n r as n t an t v o t spa t pro att ns out s s n p n w t t p r nta o s rvat ons t at as t pro r ss s t qu urt st ro t or n $r_0(t)$ w v ntua stop w n t r s no on r an r v n pr ssur ro qu n t ap ar r s s an a on p r o o t as s r n r r n



on t r s o t nt r a

$$\frac{@u}{@r_l} \quad \frac{u_l \quad u_{l \quad 1}}{r_l \quad r_{l \quad 1}}$$ on slow us on s o nt r a

an

$$\frac{@u}{@r}$$

5 Discussion and Conclusions

In t s s t on w s uss t p at ons an on us ons ro t r su ts

s pro t as oo at t nu r a o n o ot s ow an sup r ast us on t rou a porous u n a two ns ona ra a o a n t o as ons r on non vo at non r a t n sp s an p t n t r n s s to appro at part a r vat v s $\frac{@u}{@r}$ pro t as oo at sup r ast us on o a ov n s s as on onstant ass ra t ons w r ass nt rs t sup r ast o a n ro t t an oun ar on t on w r a s s ar so ut on s us n t s ow o a n to t r n t va u s at t nt r a

ov n s n t r n t o as prov su ss u n o n t pro o on ntrat on o t sp s a a nst spa an t ow v r w t tat ons on t t st p p n nt on t CF on t on

t o or o n s ow us on w t onstant ass ov r t o a n as prov su ss u prov n t at $t < 0{:}011$ or t sp $r = \frac{1}{4 p \overline{5}}$ at t_0 s avo s spat a no s ov rta n on anot r t us nsur n str n nt t r qu r nt t an t s p as o s ow us on w t ontant ass For sta t n t s r t s r qu r t at $t < 5 \quad 10^{5}$ s tat on s to a r to t CF on t on

In t s t o t va u o $u_l(t)$ at t t an oun ar o t o a n was

6 Recommendations for future work

rou out t ours o t s stu w av a a nu r o assu pt ons on o t a n on s n t at t s poss to o a trans t on tw n t porous u quat on or t s ow r w r $m = 1$ an or t sup r ast r w r $m = 3=2$ s an tw n $m = 1$ an $m = 3=2$ s n r a t to nvo v a ra ua r u t on o m as t saturat on v o t so v nt on ntrat on o t so v nt r as s Fro a o n p rsp t v t a or r a st to n u a number o nt r a s sa $i = I_1; I_2; : : : ; I_R$ or R an s n m As n t a or t or t o n us on n t s stu t a t at a nu r o nt r a no a v o t s t n n to a u at pr or to t v o t s o t no s n t nt r at pos t ons tw n t nt r a s

It as n s n t rou out t r su ts s t on t at t us o p t n t r n t o to appro at v o t s $v_i(t)$ an so ut on va u s $u_l(t)$ as r su t n tat ons on t va u o t t st ps t t at an us n t nu r a t o It as n s own t at a CF on t on ust a r to n or r to nsur t at no ov rta n o s not o ur at w po nt t t o w a It s t r or su st t at an p t t o us or t t st pp n n or r to avo t tr a an rror t o o t r n n t ar st t st p t at an us or no ov rta n o s an ssu Furt r ta s o p t t o s or ov n oun ar s an oun n r r n s wou a so a ow us to ta a ar r or pra t a t st p to r u t o putat ona t t at wou n or v r o p o pro s an avo n t n to o ow t a ran an t p CF on t on In or r to s oot out t so ut on pro at t nt r a an prov a ura t a n ssar to r u r n t s spa n s r n t r on aroun t nt r a In a urt r v op nt a to nv st at a ov n s n t nt t o s s a o on us t o part u ar or o p o tr s st s an urt r ta s an oun n an w r t att r r r n n u s n t nt t o s or o o n ous an n o o nous so v nts n porous at r a s

In a t on pra t a t r ar a ar nu r o p s a r t t at wou n u ra ua vaporat on o to nvo v a t p ratur an pr ssur p n n A pro r ss on or t s stu ou

B o r p

Du n F A Porous media. Fluid transport and pore structures on A a r ss an D o

u anov A us Ba n s J an o anous G Superfast nonlinear di usion: capillary transport in particulate porous media s v tt [redacted]

a qu J The porous medium equation - Mathematical theory or at at a ono rap s or n v rs t r ss or [redacted]

Yost F G an oo E J Capillary ow in irregular surface grooves an u r

E Ba n s J an on an n a J A moving mesh approach for modelling avascular tumour growth App u r a at at s [redacted]

Ba n s J Hu ar E an J a A moving mesh nite element algorithm for the adaptive solution of time dependent partial di erential equations with moving boundaries Co un Co put s [redacted]

orton an a rs D F Numerical solution of partial di erential equations s on Ca r n v rs t r ss Ca r [redacted]

o an J D An introduction to non-linear partial di erential equations s on w J rs [redacted]

s an MSc Thesis: Numerical schemes for a non-linear di usion problem n v rs t o a n [redacted]

r n J [redacted] v rs t o a
n [redacted]

Bar n att G I Scaling Ca r n v rs t r ss Ca r [redacted]

Ba n s J an E A large time-step implicit moving mesh scheme for moving boundary problems u t o s or art a D r nt a Equat ons
[redacted]

n r G F an Gra G Finite element simulation in surface and subsurface hydrology A a r ss w Yor