

On Approach to Meshless Methods

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N₂, H₂, CO, CH₄, C₂H₆

$$N = p \quad f_{ij} = g_{ij}$$

[illegible]

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2)

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N

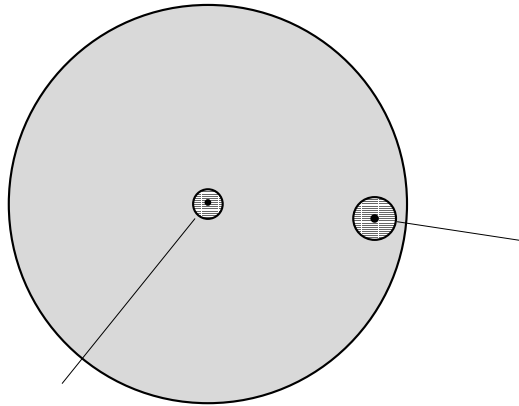
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Let $\mathbf{P}_k$ be the space of polynomials of degree k in d variables, $\mathbf{P}_k = \{p \in \mathcal{C}^0(\mathbb{R}^d) : p(\mathbf{x}) = \sum_{|\alpha| \leq k} c_\alpha \mathbf{x}^\alpha\}$, where $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$, and $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$. Let $\mathbf{P}_k$ be the space of polynomials of degree k in d variables, $\mathbf{P}_k = \{p \in \mathcal{C}^0(\mathbb{R}^d) : p(\mathbf{x}) = \sum_{|\alpha| \leq k} c_\alpha \mathbf{x}^\alpha\}$, where $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$, and $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$.

Remark 1.4. Let $\mathbf{P}_k$ be the space of polynomials of degree k in d variables, $\mathbf{P}_k = \{p \in \mathcal{C}^0(\mathbb{R}^d) : p(\mathbf{x}) = \sum_{|\alpha| \leq k} c_\alpha \mathbf{x}^\alpha\}$, where $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$, and $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$. Let $\mathbf{P}_k$ be the space of polynomials of degree k in d variables, $\mathbf{P}_k = \{p \in \mathcal{C}^0(\mathbb{R}^d) : p(\mathbf{x}) = \sum_{|\alpha| \leq k} c_\alpha \mathbf{x}^\alpha\}$, where $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$, and $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$. $\blacksquare$

2 Polynomial Reproducing Systems

Let $\mathbf{P}_k$ be the space of polynomials of degree k in d variables, $\mathbf{P}_k = \{p \in \mathcal{C}^0(\mathbb{R}^d) : p(\mathbf{x}) = \sum_{|\alpha| \leq k} c_\alpha \mathbf{x}^\alpha\}$, where $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$, and $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$.



Proof of Theorem 2.6. Let $\mathbf{f} = (f_1, \dots, f_n)^T \in \mathbb{R}^n$ and $\mathbf{g} = (g_1, \dots, g_n)^T \in \mathbb{R}^n$. Then, for $j = 1, \dots, n$, we have

$$\sum_{j=1}^n \left(\frac{1}{2} \mathbf{f}_j^T \mathbf{g}_j \right) = \frac{1}{2} \mathbf{f}^T \mathbf{g}.$$

Let $\mathbf{f} = (f_1, \dots, f_n)^T \in \mathbb{R}^n$ and $\mathbf{g} = (g_1, \dots, g_n)^T \in \mathbb{R}^n$. Then, for $j = 1, \dots, n$, we have

$$\mathbf{f}_j^T \mathbf{g}_j = \mathbf{f}^T \mathbf{g}.$$

Let $\mathbf{f} = (f_1, \dots, f_n)^T \in \mathbb{R}^n$ and $\mathbf{g} = (g_1, \dots, g_n)^T \in \mathbb{R}^n$. Then, for $j = 1, \dots, n$, we have

$$\mathbf{f}_j^T \mathbf{g}_j = \mathbf{f}^T \mathbf{g}.$$

Let $\mathbf{f} = (f_1, \dots, f_n)^T \in \mathbb{R}^n$ and $\mathbf{g} = (g_1, \dots, g_n)^T \in \mathbb{R}^n$. Then, for $j = 1, \dots, n$, we have

Let $\mathbf{P}_i$ be a d -th order polynomial in $\mathbb{R}^d$. Assume that the basis $\mathbf{e}_i$ of Theorem 2.6 satisfies condition (2.10), i.e., for some $\mathbf{P}_i \in \mathcal{P}_d$ we have

Corollary 2.7 Let $\mathbf{P}_i$ be a d -th order polynomial in $\mathbb{R}^d$. Assume that the basis $\mathbf{e}_i$ of Theorem 2.6 satisfies condition (2.10), i.e., for some $\mathbf{P}_i \in \mathcal{P}_d$ we have

$$\mathbf{P}_i(\mathbf{x}) = \sum_{j=0}^d \mathbf{e}_j(\mathbf{x}) \mathbf{g}_j(\mathbf{x})$$

Then there exists a constant C depending on $\mathbf{P}_i$ such that for each $\mathbf{x} \in \mathbb{R}^d$ there exists

$$\mathbf{P}_i(\mathbf{x}) = \sum_{j=0}^d \mathbf{e}_j(\mathbf{x}) \mathbf{g}_j(\mathbf{x})$$

[illegible]

$$0 \leq \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right)^{1-1} = 1$$

[illegible]

$$v_{\text{SP}} = \frac{k_1}{k_1 + k_2} \frac{1}{[P]} - \frac{k_H}{k_2} \quad (2)$$

[illegible]
$$e_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for } i = 1, 2, 3, 4$$

- 2) $P_{\text{min}} = A \cdot p_{\text{min}}$, $p_{\text{min}} = 0.001$,
- 2) $L_{\text{min}} = A \cdot L_{\text{min}}$, $L_{\text{min}} = 0.001$.

$\mathcal{A} = \{A_{ij} \in \mathbb{R}^{n \times n} \mid A_{ij} = \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{i=k\}} \mathbf{1}_{\{j=k\}}\}$

vthe

[illegible]

$$G = \begin{pmatrix} G_{11} & \dots & G_{1N} \\ \vdots & \ddots & \vdots \\ G_{N1} & \dots & G_{NN} \end{pmatrix} \in \mathbb{R}^{N \times N}$$

$$a_{i=1}^{N_1} \text{Ga}^{+} \quad i) \quad a_{k=1}^{N_2} \text{K}^{+} \quad j)$$

$\bullet$ G r i p i r a $2R^Q$ P a 6
 i a Ga $-$ r r e $k=1$ a_k k
 e i $-$ r i 2 i 2 p
 i 2 p i i 2 R^d i e
 r p r
 G p r i i k k

$$A_{ij} = \sum_{l=1}^{\infty} e_{i-l} A_{l,j} = \sum_{i;k;l} e_{i-l} A_{l,j} = \sum_{i;k} e_{i-l} A_{l,j}$$

1. $H^1(X, \mathbb{R}) \cong H^1(X, \mathbb{C}) \oplus H^1(X, \mathbb{R})$ (by the Hodge decomposition theorem).

Exercise 2.14. $\mathbb{R}^n$ is a vector space over $\mathbb{R}$. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$ be linearly independent vectors. Let $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$, and $\mathbf{w}_3 = \mathbf{v}_3 + \mathbf{v}_1$. Are $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ linearly independent?

$$i_{\mathcal{A}} = \frac{p_i}{\sum_{j=1}^n p_j} = \frac{p_i}{\sum_{j=1}^n \frac{p_j}{n}} = \frac{p_i}{\frac{1}{n} \sum_{j=1}^n p_j} = \frac{p_i}{\frac{1}{n}}$$

$$H^1(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^n \quad \text{and} \quad H^1(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^n \quad \text{for } n \geq 1. \quad \blacksquare$$

A $\mathbb{P}^1$ -bundle $\pi: X \rightarrow \mathbb{P}^1$ is called *trivial* if $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathcal{L}$ be a line bundle on $\mathbb{P}^1$. A $\mathbb{P}^1$ -bundle $\pi: X \rightarrow \mathbb{P}^1$ is called *twisted* if $X \cong \mathbb{P}^1 \times_{\mathcal{L}} \mathbb{P}^1$.

Exercise 2.15. Let $\mathcal{A}$ and $\mathcal{B}$ be σ -algebras on Ω . A probability measure $\mathbb{P}$ on $\mathcal{A}$ is said to be *independent of* $\mathcal{B}$ if

$$\sum_{i=1}^N \left(i^{(1)} - i^{(2)} \right) = N(N-1)/2$$

[illegible]

[illegible]

$$i \cdot i = \frac{i}{i}$$

$\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ into function $\bullet \rightarrow \bullet \rightarrow \mathbb{P}^1 \rightarrow \bullet$ (1 p. 1)

$$= \bigotimes_{j=1}^d \mathbb{R}_0 \otimes \mathbb{R}_0$$

[illegible]

$$\begin{array}{ccc} & 8 & \\ & \wedge & \\ A) - & & B) \end{array} \quad \begin{array}{c} \bullet \text{ r} \\ \bullet \text{ r} \\ \bullet \text{ r} \end{array} \quad \left| \begin{array}{c} \\ \\ \\ \end{array} \right.$$

[illegible][illegible]

Assumption 2.18. $\mathbf{P}^k(\mathbf{R}^d) = \mathbf{P}^k(\mathbf{R}^d) \cap \mathbf{P}^k(\mathbf{R}^d) = \mathbf{P}^k(\mathbf{R}^d)$.

$$\begin{aligned} \text{Rem. 2.19.} \quad & \text{Let } \mathbf{r} = (r_1, \dots, r_n) \in \mathbb{R}^n \text{ be a vector. Then } \mathbf{r} \in \mathbb{R}^n \text{ is a vector in } \mathbb{R}^n \text{ if and only if } \mathbf{r} \in \mathbb{R}^n. \\ & \text{If } \mathbf{r} \in \mathbb{R}^n \text{ is a vector, then } \mathbf{r} \in \mathbb{R}^n \text{ is a vector in } \mathbb{R}^n \text{ if and only if } \mathbf{r} \in \mathbb{R}^n. \\ & \text{If } \mathbf{r} \in \mathbb{R}^n \text{ is a vector, then } \mathbf{r} \in \mathbb{R}^n \text{ is a vector in } \mathbb{R}^n \text{ if and only if } \mathbf{r} \in \mathbb{R}^n. \end{aligned}$$
[illegible]

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Theorem 2.20 *Let $\mathcal{T}_h$ satisfy cone condition (2.1) and $\mathcal{D}_h$ be a $\mathcal{D}$ -triangulation. Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ and $\mathbf{f}_{ij} = \mathbf{f} \cdot \mathbf{g}_{ij} \in \mathbf{L}^2(\Gamma)$. Set*

$$\mathbf{u}_h = \mathbf{f} \cdot \mathbf{g}_{ij} \quad (2.2)$$

and assume the covering condition

$$[\mathbf{f}_{ij}]_i \leq \mathbf{f}_{ij} \leq [\mathbf{f}_{ij}]_i + \mathbf{f}_{ij} \quad (2.3)$$

$$\begin{aligned}
 & \left(\frac{1}{2} \left(\frac{\partial \mathbf{u}}{\partial t} \right)^T \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \left(\frac{\partial \mathbf{v}}{\partial t} \right)^T \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2} \left(\frac{\partial \mathbf{w}}{\partial t} \right)^T \frac{\partial \mathbf{w}}{\partial t} + \frac{1}{2} \left(\frac{\partial \mathbf{p}}{\partial t} \right)^T \frac{\partial \mathbf{p}}{\partial t} \right) \\
 & - \frac{1}{2} \left(\mathbf{f} \cdot \mathbf{j}_\omega - \mathbf{g} \cdot \mathbf{j}_\omega \right) \\
 & - \frac{1}{2} \left(\mathbf{j} \cdot \mathbf{G}^{-1} \mathbf{j} \right) \left(\frac{1}{2} \left(\frac{\partial \mathbf{u}}{\partial t} \right)^T \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \left(\frac{\partial \mathbf{v}}{\partial t} \right)^T \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2} \left(\frac{\partial \mathbf{w}}{\partial t} \right)^T \frac{\partial \mathbf{w}}{\partial t} + \frac{1}{2} \left(\frac{\partial \mathbf{p}}{\partial t} \right)^T \frac{\partial \mathbf{p}}{\partial t} \right)
 \end{aligned}$$

$\mathbf{u}_j^{\text{old}}$ is the previous iteration value of $\mathbf{u}_j$ and $\mathbf{u}_j^{\text{new}}$ is the new value of $\mathbf{u}_j$. The new value of $\mathbf{u}_j$ is computed by solving the following system of equations:

$$\mathbf{A}(\mathbf{u}_j^{\text{new}}) = \mathbf{b} + \mathbf{A}(\mathbf{u}_j^{\text{old}}) - \mathbf{A}(\mathbf{u}_j^{\text{old}}) \quad (1)$$
 where $\mathbf{A}(\mathbf{u}_j^{\text{old}})$ is the matrix $\mathbf{A}$ evaluated at $\mathbf{u}_j^{\text{old}}$. The new value of $\mathbf{u}_j$ is then used to compute the new value of $\mathbf{u}_i$ for $i = 1, \dots, n$.

3. step: $\mathbf{u}_i^{\text{new}} = \mathbf{u}_i^{\text{old}} + \mathbf{u}_j^{\text{new}} - \mathbf{u}_j^{\text{old}}$

2. step: $\vdash^r$ L₁ $\nabla \text{ , } \dots \text{ , } \Delta$

$$\|k\|_{L^\infty(B_{1/2}(x))} - \|k\|_{L^\infty(B_1)} \leq \frac{1}{2} \quad (1)$$

[illegible]

$$\mathbf{k} \cdot \mathbf{k}_{L\infty} = \mathbf{j} \cdot \mathbf{j}$$

[illegible]

$$-\frac{1}{x-x_*} - \frac{1}{x-x_*} = -\frac{2}{x-x_*} \quad \text{,} \quad \mathbf{6}$$

[illegible]

3. step: Let $j = \max\{j \in \mathbb{N} : p_j = 1\}$, $r = \lfloor \frac{1}{2} \log_2 p_j \rfloor$, $\ell = \lfloor \frac{1}{2} \log_2 p_j \rfloor$.

$$\frac{1}{k_j} \frac{1}{k} \quad j \quad 1 \quad 6$$

$$\bullet \quad j = 1, \dots, n \quad \text{with} \quad \mathbf{r} \in \mathbb{R}^d \quad \text{with} \quad \mathbf{k} \in \mathbb{R}^d$$

$$f_j = \frac{1}{2} \frac{g}{e}$$

[illegible]

$$k^+ \quad jk \quad \ell^+ \quad k^- \quad k^- \quad \ell^+ \quad , \quad)$$

$$k_j = k_{j-1} + \frac{1}{2} \Delta t \left(\frac{1}{2} \Delta t \right)$$

$$r(r+1) \leq 2e(r+1) \leq \frac{1}{2}r(r+1)$$

$$j = - \quad j = - \quad \frac{1}{k} = \quad j = -$$

1. $\mathcal{P} \leq 1$

$$k \quad -k \quad k \quad -k \quad \frac{k_j - k}{k_j \quad k}$$

4. step:

$$\mathbf{k} - \mathbf{k} \frac{\mathcal{L}^T}{\mathbf{k}_j - \mathbf{k}} = \frac{\mathbf{k}_j - \mathbf{k}}{\mathbf{k}_j - \mathbf{k}} \mathcal{L}^T$$

5. step: $\mathbf{k} - \mathbf{k} \frac{\mathcal{L}^T}{\mathbf{k}_j - \mathbf{k}} = \mathbf{k} - \mathbf{k} \frac{\mathcal{L}^T}{\mathbf{k}_j - \mathbf{k}} \frac{\mathbf{k}_j - \mathbf{k}}{\mathbf{k}_j - \mathbf{k}}$

$$\mathbf{k} - \mathbf{k} \frac{\mathcal{L}^T}{\mathbf{k}_j - \mathbf{k}} = \frac{\mathcal{L}^T}{\mathbf{k}_j - \mathbf{k}} \frac{\mathbf{k}_j - \mathbf{k}}{\mathbf{k}_j - \mathbf{k}} \mathbf{k} - \mathbf{k} \frac{\mathcal{L}^T}{\mathbf{k}_j - \mathbf{k}} \frac{\mathbf{k}_j - \mathbf{k}}{\mathbf{k}_j - \mathbf{k}}$$

6. step: $\mathbf{k} - \mathbf{k} \frac{\mathcal{L}^T}{\mathbf{k}_j - \mathbf{k}} = \frac{\mathcal{L}^T}{\mathbf{k}_j - \mathbf{k}} \frac{\mathbf{k}_j - \mathbf{k}}{\mathbf{k}_j - \mathbf{k}} \mathbf{k} - \mathbf{k} \frac{\mathcal{L}^T}{\mathbf{k}_j - \mathbf{k}} \frac{\mathbf{k}_j - \mathbf{k}}{\mathbf{k}_j - \mathbf{k}}$

7. step: $\mathbf{k} - \mathbf{k} \frac{\mathcal{L}^T}{\mathbf{k}_j - \mathbf{k}} = \frac{\mathcal{L}^T}{\mathbf{k}_j - \mathbf{k}} \frac{\mathbf{k}_j - \mathbf{k}}{\mathbf{k}_j - \mathbf{k}} \mathbf{k} - \mathbf{k} \frac{\mathcal{L}^T}{\mathbf{k}_j - \mathbf{k}} \frac{\mathbf{k}_j - \mathbf{k}}{\mathbf{k}_j - \mathbf{k}}$



2 Bibliographical Remarks

For a survey of the literature on the numerical solution of the Laplace equation, see [1].

Let $\mathbf{f} \in \mathcal{C}^5(\mathbb{R}^d)$ and let $\mathbf{A} = (A_{ij})_{i,j=1}^N$ be the matrix of the scattered interpolation problem (3.1). Then $\mathbf{A}$ is symmetric and positive definite. Moreover, the matrix $\mathbf{B} = (B_{ij})_{i,j=1}^N$ defined by

$$B_{ij} = \frac{1}{h} \int_{\mathbb{R}^d} \frac{1}{|\mathbf{x} - \mathbf{x}_i|} \frac{1}{|\mathbf{x} - \mathbf{x}_j|} d\mathbf{x} = \frac{1}{h} \int_{\mathbb{R}^d} \frac{1}{|\mathbf{x} - \mathbf{x}_i|} \frac{1}{|\mathbf{x} - \mathbf{x}_j|} d\mathbf{x} \quad (3.2)$$

is symmetric and positive definite.

Proposition 3.4 Let $\mathbf{f} \in \mathcal{C}^5(\mathbb{R}^d)$. Then

1. $\mathbf{A}^{-1} \mathbf{B}$ is symmetric and positive definite.
2. $\mathbf{A}^{-1} \mathbf{B}$ is symmetric and positive definite.
3. $\mathbf{A}^{-1} \mathbf{B}$ is symmetric and positive definite.
4. $\mathbf{A}^{-1} \mathbf{B}$ is symmetric and positive definite.

Proof. Let $\mathbf{f} \in \mathcal{C}^5(\mathbb{R}^d)$. Then $\mathbf{A}$ is symmetric and positive definite. Moreover, the matrix $\mathbf{B}$ is symmetric and positive definite. $\square$

Theorem 3.5 Let Assumption 3.5 be satisfied. Then for distinct points $\mathbf{x}_1, \dots, \mathbf{x}_N \in \mathbb{R}^d$ and $\mathbf{f} \in \mathcal{C}^5(\mathbb{R}^d)$ the scattered interpolation problem

$$\text{Find } \mathbf{c} \in \mathbb{R}^N \text{ such that } \mathbf{f}(\mathbf{x}_i) = \sum_{j=1}^N c_j \phi_j(\mathbf{x}_i) = g_i \quad (3.3)$$

has a unique solution, which satisfies

$$\mathbf{c} = \mathbf{A}^{-1} \mathbf{B} \mathbf{g} \quad (3.4)$$

and

$$\mathbf{f}(\mathbf{x}) = \sum_{j=1}^N c_j \phi_j(\mathbf{x}) = \mathbf{g}^T \mathbf{B} \mathbf{A}^{-1} \mathbf{A} \mathbf{c} = \mathbf{g}^T \mathbf{B} \mathbf{c} \quad (3.5)$$

Proof. Let $\mathbf{f} \in \mathcal{C}^5(\mathbb{R}^d)$. Then $\mathbf{A}$ is symmetric and positive definite. Moreover, the matrix $\mathbf{B}$ is symmetric and positive definite. Let $\mathbf{c} \in \mathbb{R}^N$ be the solution of the scattered interpolation problem (3.3). Then $\mathbf{c} = \mathbf{A}^{-1} \mathbf{B} \mathbf{g}$. Moreover, the matrix $\mathbf{B}$ is symmetric and positive definite. $\square$

Let $\mathbf{f} \in \mathcal{C}^5(\mathbb{R}^d)$ and let $\mathbf{A} = (A_{ij})_{i,j=1}^N$ be the matrix of the scattered interpolation problem (3.1). Then $\mathbf{A}$ is symmetric and positive definite. Moreover, the matrix $\mathbf{B} = (B_{ij})_{i,j=1}^N$ defined by

$$B_{ij} = \frac{1}{h} \int_{\mathbb{R}^d} \frac{1}{|\mathbf{x} - \mathbf{x}_i|} \frac{1}{|\mathbf{x} - \mathbf{x}_j|} d\mathbf{x} = \frac{1}{h} \int_{\mathbb{R}^d} \frac{1}{|\mathbf{x} - \mathbf{x}_i|} \frac{1}{|\mathbf{x} - \mathbf{x}_j|} d\mathbf{x} \quad (3.6)$$

is symmetric and positive definite.

Corollary 3.6 (stability of scattered data interpolation) Let $\mathbf{f} \in \mathcal{C}^5(\mathbb{R}^d)$ be Lipschitz continuous in (or $\mathbb{R}^d$). Let $\mathbf{x}_1, \dots, \mathbf{x}_N \in \mathbb{R}^d$ and suppose Assumption 3.5. Then for $\mathbf{f} \in \mathcal{C}^5(\mathbb{R}^d)$

$$\mathbf{c} = \mathbf{A}^{-1} \mathbf{B} \mathbf{g} \quad (3.7)$$

Proof.

Let $\mathbb{R}^d$ be a domain A and let $\mathbb{R}^d$ be Lipschitz domain. Define the

$$k_{H^1(\Omega)} = \left(\sum_{i=1}^N \int_{\Omega} |\nabla u_i|^2 dx \right)^{1/2}$$

Then there exists such that for $\epsilon > 0$ there holds

$$k_{H^1(\Omega)} \leq \epsilon k_{H^1(\Omega)} + \frac{1}{\epsilon} k_{L^2(\Omega)}$$

Proof.

1. step: $\forall \epsilon > 0$ there exists $\delta > 0$ such that for $\epsilon > 0$ there holds

$$k_{H^1(\Omega)} \leq \epsilon k_{H^1(\Omega)} + \frac{1}{\epsilon} k_{L^2(\Omega)}$$

2. step: $\forall \epsilon > 0$ there exists $\delta > 0$ such that for $\epsilon > 0$ there holds

$$k_{H^1(\Omega)} \leq \epsilon k_{H^1(\Omega)} + \frac{1}{\epsilon} k_{L^2(\Omega)}$$

Corollary 10 Let Assumption 3.5 be satisfied and let $\mathbb{R}^d$ be Lipschitz domain. Define the

$$k_{H^1(\Omega)} = \left(\sum_{i=1}^N \int_{\Omega} |\nabla u_i|^2 dx \right)^{1/2}$$

Then there exists such that for $\epsilon > 0$ there holds

$$k_{H^1(\Omega)} \leq \epsilon k_{H^1(\Omega)} + \frac{1}{\epsilon} k_{L^2(\Omega)}$$

Proof.

1. step: $\forall \epsilon > 0$ there exists $\delta > 0$ such that for $\epsilon > 0$ there holds

$$k_{H^1(\Omega)} \leq \epsilon k_{H^1(\Omega)} + \frac{1}{\epsilon} k_{L^2(\Omega)}$$

2. step: $\forall \epsilon > 0$ there exists $\delta > 0$ such that for $\epsilon > 0$ there holds

$$k_{H^1(\Omega)} \leq \epsilon k_{H^1(\Omega)} + \frac{1}{\epsilon} k_{L^2(\Omega)}$$

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$$k_{H^1(\Omega)} = \left(\sum_{i=1}^N \int_{\Omega} |\nabla u_i|^2 dx \right)^{1/2}$$

Then there exists such that for $\epsilon > 0$ there holds

$$k_{H^1(\Omega)} \leq \epsilon k_{H^1(\Omega)} + \frac{1}{\epsilon} k_{L^2(\Omega)}$$

Figure 10.10: **B** Γ Γ

1 Approximation Theory

Theorem 1 *Let $\mathbb{R}^d$ be Lipschitz domain and let $\mathbf{f}, \mathbf{j} \in \mathbf{L}^2(\mathbb{R}^d)$ be a collection of $2d$ functions. Set $\mathbf{i} = \mathcal{PP}(\mathbf{j})$, $\mathbf{i} = \mathcal{PP}(\mathbf{f})$ and assume*

$$\sum_{i=1}^n \left\| \mathbf{k}_i \right\|_{L^\infty(\mathbb{R}^d)} \leq \frac{G}{i} \quad \text{on } \mathcal{P}_r \text{ for } \mathbf{f} \in \mathbb{R}^d, \mathbf{j} \in \mathbb{R}^d, \mathbf{g} \in \mathbb{R}^d.$$

Assume that $\chi_i, -$, is Lipschitz continuous in s . For each $2f$ get i^{1j} be given and set

$$-\sum_{i=1}^n \frac{(\lambda_i - \mu)}{\lambda_i} = -\sum_{i=1}^n \left(1 - \frac{\mu}{\lambda_i} \right) = -n + \mu \sum_{i=1}^n \frac{1}{\lambda_i}$$

Then
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$$\|p\|_{H^1(\Omega)} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2}$$

$$\|p\|_{H^1(\Omega)} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2}$$

Exercise 4.8. Let $\Omega \subset \mathbb{R}^d$ be a domain, $\mathcal{T}_h$ a triangulation of Ω with N triangles, $\mathcal{P}_p$ the space of piecewise polynomials of degree p on $\mathcal{T}_h$. Let $\mathcal{P}_p(\Omega) = \{v \in \mathcal{P}_p : v|_{\Omega_i} \in \mathcal{P}_p(\Omega_i) \text{ for all } \Omega_i \in \mathcal{T}_h\}$. Let $\mathcal{P}_p(\Omega) = \{v \in \mathcal{P}_p : v|_{\Omega_i} \in \mathcal{P}_p(\Omega_i) \text{ for all } \Omega_i \in \mathcal{T}_h\}$. Let $\mathcal{P}_p(\Omega) = \{v \in \mathcal{P}_p : v|_{\Omega_i} \in \mathcal{P}_p(\Omega_i) \text{ for all } \Omega_i \in \mathcal{T}_h\}$.

$$\|v\|_{H^1(\Omega)} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2}$$

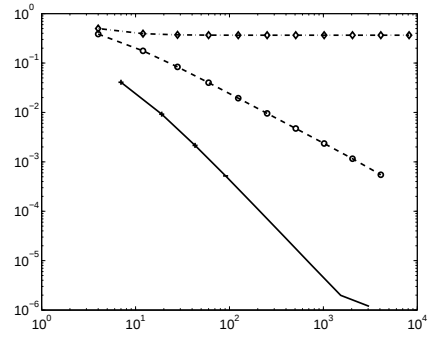
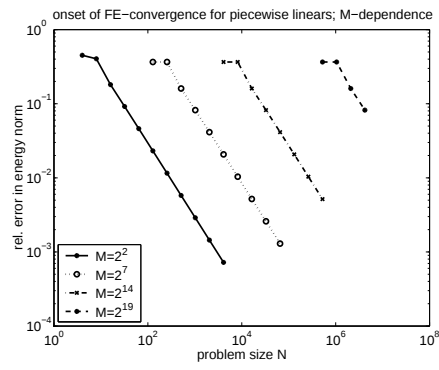
$$\|v\|_{H^1(\Omega)} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2}$$

5 Examples of operator adapted approximation spaces

Let $\Omega \subset \mathbb{R}^d$ be a domain, $\mathcal{T}_h$ a triangulation of Ω with N triangles, $\mathcal{P}_p$ the space of piecewise polynomials of degree p on $\mathcal{T}_h$. Let $\mathcal{P}_p(\Omega) = \{v \in \mathcal{P}_p : v|_{\Omega_i} \in \mathcal{P}_p(\Omega_i) \text{ for all } \Omega_i \in \mathcal{T}_h\}$. Let $\mathcal{P}_p(\Omega) = \{v \in \mathcal{P}_p : v|_{\Omega_i} \in \mathcal{P}_p(\Omega_i) \text{ for all } \Omega_i \in \mathcal{T}_h\}$. Let $\mathcal{P}_p(\Omega) = \{v \in \mathcal{P}_p : v|_{\Omega_i} \in \mathcal{P}_p(\Omega_i) \text{ for all } \Omega_i \in \mathcal{T}_h\}$.

1 A one dimensional example

$$\|v\|_{H^1(\Omega)} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2} = \left(\sum_{i=1}^N \left(\int_{\Omega_i} |\nabla v_i|^2 dx + \int_{\Omega_i} |v_i|^2 dx \right) \right)^{1/2}$$



independent.

2 Laplace's Equation

Let $\Omega \subset \mathbb{R}^d$ be a domain with boundary Γ . We consider the Laplace equation

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma. \quad (2.1)$$

Let $\mathcal{H}^1(\Omega)$ be the Sobolev space of order 1. We define the bilinear form

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad u, v \in \mathcal{H}^1(\Omega).$$

The weak formulation of (2.1) is: Find $u \in \mathcal{H}^1(\Omega)$ such that

$$a(u, v) = \int_{\Omega} f v \, dx \quad \text{for all } v \in \mathcal{H}^1(\Omega).$$

Let $\mathcal{H}^1_0(\Omega)$ be the subspace of $\mathcal{H}^1(\Omega)$ with zero boundary values. We define the bilinear form

$$a_0(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad u, v \in \mathcal{H}^1_0(\Omega).$$

The weak formulation of (2.1) is: Find $u \in \mathcal{H}^1_0(\Omega)$ such that

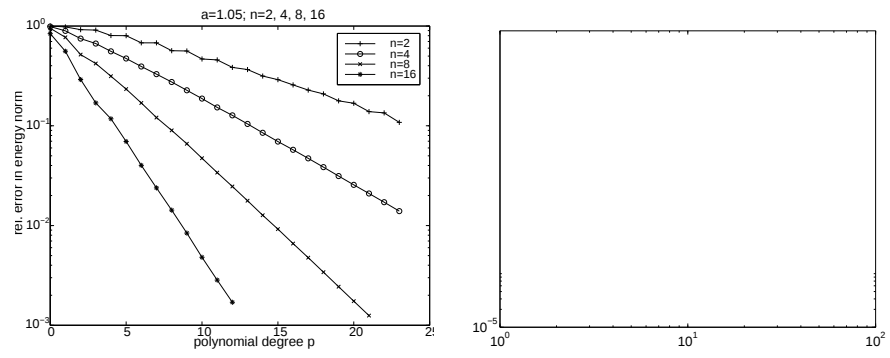
$$a_0(u, v) = \int_{\Omega} f v \, dx \quad \text{for all } v \in \mathcal{H}^1_0(\Omega).$$

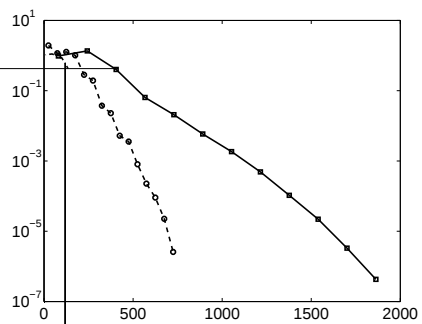
Let $\mathcal{H}^1(\Omega)$ be the Sobolev space of order 1. We define the bilinear form

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad u, v \in \mathcal{H}^1(\Omega).$$

The weak formulation of (2.1) is: Find $u \in \mathcal{H}^1(\Omega)$ such that

$$a(u, v) = \int_{\Omega} f v \, dx \quad \text{for all } v \in \mathcal{H}^1(\Omega).$$





Let $\mathbf{u} \in \mathbf{H}^1(\Omega; \mathbb{R}^d)$ be a displacement field satisfying the homogeneous elasticity equations (5.13). Then the function $\mathbf{u}^{\text{elast}}$ is defined by

$$\mathbf{u}^{\text{elast}} = \mathbf{u} + \mathbf{u}^{\text{ppr}}, \quad \text{where } \mathbf{u}^{\text{ppr}} \in \mathbf{H}^1(\Omega; \mathbb{R}^d) \text{ is the ppw of } \mathbf{u}.$$

Moreover, $\mathbf{u}^{\text{elast}}$ is the unique function in $\mathbf{H}^1(\Omega; \mathbb{R}^d)$ satisfying the homogeneous elasticity equations (5.13) and the boundary conditions $\mathbf{u}^{\text{elast}} = \mathbf{0}$ on Γ_D . The ppw $\mathbf{u}^{\text{ppr}}$ is defined by the boundary conditions $\mathbf{u}^{\text{ppr}} = \mathbf{u}$ on Γ_D and $\mathbf{u}^{\text{ppr}} = \mathbf{0}$ on Γ_N .

$$\mathbf{u}^{\text{elast}} = \mathbf{u} + \mathbf{u}^{\text{ppr}} \quad \text{with } \mathbf{u}^{\text{ppr}} \in \mathbf{H}^1(\Omega; \mathbb{R}^d) \text{ satisfying } \mathbf{u}^{\text{ppr}} = \mathbf{u} \text{ on } \Gamma_D \text{ and } \mathbf{u}^{\text{ppr}} = \mathbf{0} \text{ on } \Gamma_N.$$

Moreover, $\mathbf{H}_p$ is the space of ppws of $\mathbf{H}^1(\Omega; \mathbb{R}^d)$ satisfying the homogeneous elasticity equations (5.13) and the boundary conditions $\mathbf{u} = \mathbf{0}$ on Γ_D .

Theorem 1 Let $\Omega \subset \mathbb{R}^d$ be star-shaped with respect to $\mathbf{0}$. Let $\mathbf{u} \in \mathbf{H}^1(\Omega; \mathbb{R}^d)$ satisfy an exterior cone condition with angle α . Let $\mathbf{u} \in \mathbf{H}^1(\Omega; \mathbb{R}^d)$ and assume that the displacement field $\mathbf{u} \in \mathbf{H}^1(\Omega; \mathbb{R}^d)$ satisfies the homogeneous elasticity equations (5.13). Then the function $\mathbf{u}^{\text{elast}}$ is well approximated from $\mathbf{H}_p^{\text{elast}}$ such that

$$\|\mathbf{u} - \mathbf{u}^{\text{elast}}\|_{\mathbf{H}^1(\Omega; \mathbb{R}^d)} \leq C \|\mathbf{u}\|_{\mathbf{H}^1(\Omega; \mathbb{R}^d)} \quad \text{with } C = C(\alpha, d, \mu, \lambda).$$

Proof. See [10, Theorem 4.1].

Remark 5.18. For $\mathbf{u} \in \mathbf{H}^1(\Omega; \mathbb{R}^d)$ satisfying the homogeneous elasticity equations (5.13) and the boundary conditions $\mathbf{u} = \mathbf{0}$ on Γ_D , the function $\mathbf{u}^{\text{elast}}$ is the unique function in $\mathbf{H}^1(\Omega; \mathbb{R}^d)$ satisfying the homogeneous elasticity equations (5.13) and the boundary conditions $\mathbf{u}^{\text{elast}} = \mathbf{0}$ on Γ_D .

Further examples

Let $\mathbf{u} \in \mathbf{H}^1(\Omega; \mathbb{R}^d)$ be a displacement field satisfying the homogeneous elasticity equations (5.13). Then the function $\mathbf{u}^{\text{elast}}$ is defined by $\mathbf{u}^{\text{elast}} = \mathbf{u} + \mathbf{u}^{\text{ppr}}$, where $\mathbf{u}^{\text{ppr}} \in \mathbf{H}^1(\Omega; \mathbb{R}^d)$ is the ppw of $\mathbf{u}$.

Moreover, $\mathbf{u}^{\text{elast}}$ is the unique function in $\mathbf{H}^1(\Omega; \mathbb{R}^d)$ satisfying the homogeneous elasticity equations (5.13) and the boundary conditions $\mathbf{u}^{\text{elast}} = \mathbf{0}$ on Γ_D .

Moreover,

$\mathbf{u}^h \in \mathbf{U}^h$ and $\mathbf{v}^h \in \mathbf{V}^h$ are the approximations of $\mathbf{u}$ and $\mathbf{v}$ in the finite element space $\mathbf{U}^h \times \mathbf{V}^h$. The error $\mathbf{e}^h = \mathbf{u} - \mathbf{u}^h$ is the difference between the exact solution $\mathbf{u}$ and the approximation $\mathbf{u}^h$. The error $\mathbf{e}^h$ is the difference between the exact solution $\mathbf{u}$ and the approximation $\mathbf{u}^h$.

for some numbers $\alpha_{ij} \in \mathbb{R}$ and $\beta_0 \in \mathbb{R}^{1 \times k}$.

Proof. Let $\mathcal{P} = \{p_1, \dots, p_n\}$ be a partition of $\mathcal{P}$ with n parts. Then

[illegible]

Exercise 6.2. Let $\mathcal{A} = \langle A, \wedge, \vee, \neg, \rightarrow, \perp, \top \rangle$ be a Boolean algebra. Define a relation R on A by $a R b$ if and only if $a \leq b$. Prove that R is a partial order on A .

[illegible]

$$\varphi = \sum_{j=1}^{\infty} \frac{\chi_j}{i_{-j}^{1/2}} \quad \text{if } i_j \downarrow i_j \quad \varphi = 0$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{...}$$

$$N \rightarrow \begin{pmatrix} p \\ 0 \end{pmatrix} T_j \quad \text{and} \quad f_{j,i,j} = \frac{1}{j} g_{0,j}$$

$$\mathbf{V}_N \cdot \mathbf{k} \leq \mathbf{k}_H \cdot \mathbf{k} \leq \min\{\mathbf{p}; \mathbf{k}\} \cdot \mathbf{k}$$

$$T_{\alpha} = T_{\alpha}^{(1)} + T_{\alpha}^{(2)} + \dots + T_{\alpha}^{(N)} \quad (1)$$

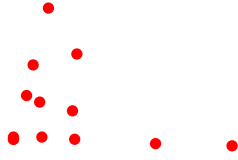
$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}$ $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Exercise 6.3. Let $\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{2 \times 2}$, $\mathbf{f} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$, $\mathbf{g} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$. Compute $\mathbf{f} \cdot \mathbf{f}$, $\mathbf{g} \cdot \mathbf{g}$, $\mathbf{f} \cdot \mathbf{g}$, $\mathbf{T} \mathbf{f}$, $\mathbf{T} \mathbf{g}$, $\mathbf{T}^T \mathbf{f}$, $\mathbf{T}^T \mathbf{g}$, $\mathbf{T} \mathbf{f} \cdot \mathbf{T} \mathbf{g}$, $\mathbf{T}^T \mathbf{f} \cdot \mathbf{T}^T \mathbf{g}$, $\mathbf{T} \mathbf{f} \cdot \mathbf{g}$, $\mathbf{T}^T \mathbf{f} \cdot \mathbf{f}$, $\mathbf{T} \mathbf{g} \cdot \mathbf{f}$, $\mathbf{T}^T \mathbf{g} \cdot \mathbf{g}$.

$$N = \frac{p_0^{p+1}}{p!} T_p \quad \text{and} \quad f_{j,m} = \frac{2}{j} \quad \text{for } j \geq 1$$

$$I^{\text{A}} = \begin{pmatrix} A & 0 \\ 0 & N \end{pmatrix}, I^{\text{B}} = \begin{pmatrix} B & 0 \\ 0 & N \end{pmatrix}, I^{\text{C}} = \begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix}, I^{\text{D}} = \begin{pmatrix} D & 0 \\ 0 & N \end{pmatrix},$$

[illegible]



[illegible]

Rena rⁱ 6.5.

2 P_p

7 Enforcement of essential boundary conditions

[illegible]

1 Conforming methods

$$\bar{u} = r(t^M, \dots, t^N, \lambda) \text{ (ppr)} \quad \text{for } \lambda \in \mathbf{N}^M \text{ (} t^M, \dots, t^N \text{)} \\ \text{and } \lambda \in \mathbf{N}^M \text{ (} \frac{1}{0}, \dots, \frac{1}{0} \text{)}.$$

A simple approach $\mathbf{B} = \mathbf{f}(\mathbf{g}(\mathbf{A}))$

$$N_0 = p \cdot f(i) \cdot p(i) \cdot g(i)$$

Exercise 7.1. Let $\mathbf{N} = \mathbf{p} \mathbf{f} \mathbf{i} \mathbf{j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{p} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{f} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{i} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Compute $\mathbf{N} \mathbf{g} \mathbf{p} \mathbf{f} \mathbf{i} \mathbf{j}$.

Example 7.5. Let $\mathbb{R} \rightarrow \mathbb{R}^n$ be a linear map. Let $\mathbf{N} = \mathbb{R}^n$.

[illegible]

$$2^{-1}; \quad k r \ll k_L \infty \quad n^{-1}$$
$$p; N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad N = \begin{pmatrix} p; 1 & T \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$v_{\alpha} v_{\beta;N} = k^{\alpha} \quad k_H = 1 \quad k^{-1} k^{\alpha} k_H = 1$$
$$k' = \varphi_N k_L \left(\frac{1}{k} \right) \quad k' = \varphi_N k_H \left(\frac{1}{k} \right) \quad k' = k_H k \left(\frac{1}{k} \right)$$
[illegible]

2 Non conforming methods: Lagrange Multiplier methods and collocation techniques

$$\begin{aligned}
 & \text{Let } \mathbf{u} \in \mathbf{H}^1(\Omega) \text{ be the exact solution of the problem } \mathcal{P}(\mathbf{u}) = \mathbf{f} \text{ and } \mathbf{u}_h \in \mathbf{H}_h^1(\Omega) \text{ be the finite element approximation.} \\
 & \text{The error } \mathbf{e}_h = \mathbf{u} - \mathbf{u}_h \text{ satisfies the following identity:} \\
 & \int_{\Omega} \nabla \mathbf{e}_h \cdot \nabla \mathbf{v} = \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} - \int_{\Omega} \nabla \mathbf{u}_h \cdot \nabla \mathbf{v} \\
 & \text{for all } \mathbf{v} \in \mathbf{H}_h^1(\Omega). \text{ Using the Cauchy-Schwarz inequality, we obtain:} \\
 & \|\mathbf{e}_h\|_{\mathbf{H}^1(\Omega)} \leq \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}^1(\Omega)} \leq C \|\mathbf{f} - \mathbf{f}_h\|_{\mathbf{L}^2(\Omega)} \\
 & \text{where } \mathbf{f}_h \text{ is the finite element approximation of } \mathbf{f}. \text{ The error estimate is:} \\
 & \|\mathbf{e}_h\|_{\mathbf{H}^1(\Omega)} \leq C h^{\frac{1}{2}} \|\mathbf{f}\|_{\mathbf{H}^1(\Omega)} \\
 & \text{where } h \text{ is the mesh size. The error estimate is:} \\
 & \|\mathbf{e}_h\|_{\mathbf{H}^1(\Omega)} \leq C h^{\frac{1}{2}} \|\mathbf{f}\|_{\mathbf{H}^1(\Omega)}
 \end{aligned}$$

[illegible]

— on j_0 — on .

$$\begin{array}{ccccccc} v_{\frac{1}{2}} v_N & k & k_L & k_r & k & k & \\ v_{\frac{1}{2}} v_N & k & k_L & k_r & k & k-1 & \end{array}$$
$$k' = \varphi_N(k_H)_{k=1}^n \quad -1 \quad -1 = k- = \quad 1 = k-1 = \quad k-1^0$$

ith the optimal α we $\rightarrow \frac{k-1}{k}$ gives

$$k = \frac{1}{N} k_H \frac{1}{\Delta t} \quad (1)$$

$$\text{Res}_{\mathcal{D}^A} \tilde{\gamma}_{10} : \mathcal{P}^A \rightarrow \mathcal{P}^A \quad \text{with} \quad \mathbf{2}^{k-1} \mid \text{rank}(\tilde{\gamma}_{10})$$

Proof of Theorem 7.9. Let $\rho \in \text{Pr}(\mathcal{A})$. Then $\rho \in \text{Pr}(\mathcal{A})$ if and only if $\rho \in \text{Pr}(\mathcal{A})$ and $\rho \in \text{Pr}(\mathcal{A})$. This is because $\rho \in \text{Pr}(\mathcal{A})$ if and only if $\rho \in \text{Pr}(\mathcal{A})$ and $\rho \in \text{Pr}(\mathcal{A})$.

$$\mathbf{k}' = \mathbf{k} - \mathbf{v}_N$$

1. $\mathbf{r} = \mathbf{r}(t)$












14. 10. 1954

$$A) \quad \frac{1}{2} \quad B) \quad \frac{1}{3} \quad C) \quad \frac{1}{4} \quad D) \quad \frac{1}{5}$$

Remark 7.11. $\mathcal{Q}_h^{\text{div}}(\mathcal{T}_h)$ is a

[illegible][illegible]
$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \nabla_{\vec{v}} \right) \left(\frac{\partial}{\partial t} + \nabla_{\vec{v}} \right) f = \left(\frac{\partial}{\partial t} + \nabla_{\vec{v}} \right) f \\ & \quad + \left(\frac{\partial}{\partial t} + \nabla_{\vec{v}} \right) f = \left(\frac{\partial}{\partial t} + \nabla_{\vec{v}} \right) f \end{aligned}$$

Lemma 1 *Set*

Let $\mathbf{v} \in \mathbb{R}^N$ and $\mathbf{v}_N \in \mathbb{R}^N$ be the vector of nodal values of $\mathbf{v}$ and $\mathbf{v}_N$ be the vector of nodal values of $\mathbf{v}_N$. Then

Exercise 7.19. Let $\mathbf{A} \in \mathbb{R}^{N \times N}$ be the stiffness matrix of the problem (7.19). Then

$$\|\mathbf{v}_N\|_{\mathbb{R}^N} \leq \|\mathbf{A}\|_{\mathbb{R}^N} \|\mathbf{v}_N\|_{\mathbb{R}^N} \quad \text{and} \quad \|\mathbf{v}_N\|_{\mathbb{R}^N} \leq \|\mathbf{A}\|_{\mathbb{R}^N} \|\mathbf{v}_N\|_{\mathbb{R}^N}.$$

Remark 7.20. Let $\mathbf{A} \in \mathbb{R}^{N \times N}$ be the stiffness matrix of the problem (7.19). Then

A Results from Analysis

Theorem A 1 (universal extension operator) Let $\mathbb{R}^d$ be a Lipschitz domain. Then there exists a linear operator $\mathbf{E} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with the following properties:

- (i) $\mathbf{E} \mathbf{j}_\Omega = \mathbf{j}_\Omega$ for $\mathbf{j}_\Omega \in \mathbb{R}^{2 \times 1}$.
- (ii) For each $\mathbf{j}_\Omega \in \mathbb{R}^{2 \times 1}$, there exists a constant c such that $\|\mathbf{E} \mathbf{j}_\Omega\|_{\mathbb{R}^{2 \times 1}} \leq c \|\mathbf{j}_\Omega\|_{\mathbb{R}^{2 \times 1}}$.

Proof. Let $\mathbf{A} \in \mathbb{R}^{N \times N}$ be the stiffness matrix of the problem (7.19). Then

Theorem A 2 (multiplicative trace theorem) Let $\mathbb{R}^d$ be a Lipschitz domain, $\mathbf{A} \in \mathbb{R}^{N \times N}$ be the stiffness matrix of the problem (7.19). Then there exists a constant c such that for

$$\|\mathbf{v}_N\|_{\mathbb{R}^N} \leq \|\mathbf{A}\|_{\mathbb{R}^N} \|\mathbf{v}_N\|_{\mathbb{R}^N} \quad \text{and} \quad \|\mathbf{v}_N\|_{\mathbb{R}^N} \leq \|\mathbf{A}\|_{\mathbb{R}^N} \|\mathbf{v}_N\|_{\mathbb{R}^N}.$$

Proof. Let $\mathbf{A} \in \mathbb{R}^{N \times N}$ be the stiffness matrix of the problem (7.19). Then

$$\|\mathbf{v}_N\|_{\mathbb{R}^N} \leq \|\mathbf{A}\|_{\mathbb{R}^N} \|\mathbf{v}_N\|_{\mathbb{R}^N} \quad \text{and} \quad \|\mathbf{v}_N\|_{\mathbb{R}^N} \leq \|\mathbf{A}\|_{\mathbb{R}^N} \|\mathbf{v}_N\|_{\mathbb{R}^N}.$$

Let $\mathbf{v}_N \in \mathbb{R}^N$ and $\mathbf{v}_N \in \mathbb{R}^N$ be the vector of nodal values of $\mathbf{v}_N$ and $\mathbf{v}_N$ be the vector of nodal values of $\mathbf{v}_N$. Then

$f \geq 2$ and $\mathbf{1}_f \leq \mathbf{1}_g$ and $\mathbf{1}_g \leq \mathbf{1}_h$, then $\mathbf{1}_f \leq \mathbf{1}_h$.

1

[illegible]

1. step: Let $\mathbf{L}_1 = \mathbf{L}$ and $\mathbf{L}_1 \mathbf{L}_1^T = \mathbf{L} \mathbf{L}^T$. *)

[illegible]
$$\begin{aligned}
 & \mathbb{H}^1(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^n, \quad \mathbb{H}^2(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^{\binom{n}{2}}, \quad \mathbb{H}^3(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^{\binom{n}{3}}, \quad \dots \\
 & \mathbb{H}^k(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^{\binom{n}{k}}, \quad \mathbb{H}^n(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}.
 \end{aligned}$$
[illegible]

1. $\psi \in \mathcal{H}^1(\mathbb{R}^n)$ so $\psi \in C.1$ for every $\mathbf{2} \mathbf{H}$.
2. For every solution ψ of (C.1) there exists a unique $\mathbf{2} \mathbf{H}$ such that $\psi = \psi_{\mathbf{2} \mathbf{H}}$.
3. $\mathbf{k} \in \mathcal{H}^1(\mathbb{R}^n)$ $\mathbf{k} \in \mathcal{H}^1(\mathbb{R}^n)$ for $\mathbf{2} \mathbf{H}$ and $\mathbf{k} \in \mathcal{H}^1(\mathbb{R}^n)$.
4. For $\mathbf{2} \mathbf{H}$, so $\psi \in C.1$, then the corresponding $\mathbf{k} \in \mathcal{H}^1(\mathbb{R}^n)$ is the one in $\mathcal{H}^1(\mathbb{R}^n)$ and $\mathbf{k} \in \mathcal{H}^1(\mathbb{R}^n)$.

[illegible][illegible]

[illegible]

$$\begin{aligned} \mu_n &= \frac{\mu_0}{n} = \frac{\mu_0}{\sum_{j=1}^n 1} \\ \mu_i &= \frac{\mu_0}{n} = \frac{\mu_0}{\sum_{j=1}^n 1} \end{aligned}$$

2. step: $\bar{r}_\ell = \frac{1}{N_\ell} \sum_{j \in \mathcal{N}_\ell} \bar{r}_j = \frac{1}{N_\ell} \sum_{j \in \mathcal{N}_\ell} \frac{1}{N_j} \sum_{i \in \mathcal{N}_j} \bar{r}_i = \frac{1}{N_\ell} \sum_{i \in \mathcal{N}_\ell} \bar{r}_i$

here depends on y on the $\mathbb{I}$ mé constants, upper bounds on γ , and lower bounds on γ .

[illegible]

$$\begin{array}{c}
\text{ap } 2 \text{ H}_p \\
\text{k}_1 \text{ ap } \text{k}_{\text{W}(\infty)} \text{L} - \text{p} \text{k}_1 \text{k}_{\text{L}(\text{Int})} \text{L} - \\
\text{k}_1 \text{ ap } \text{k}_{\text{W}(\infty)} \text{L} - \text{p} \text{k}_1 \text{k}_{\text{L}(\text{Int})} \text{L} - \\
\text{h} - \text{f} \text{ j j j} - \text{g} \text{ C n}_{-1} \text{ C n} \\
\text{r} \text{ p} \text{ 1} - \text{1} \text{ 1} \\
\text{r} \text{ p} \text{ 1} \\
\text{L} \text{ h} \frac{1}{1-n} \\
\text{r} \text{ 2 f g} \\
\text{ku} \text{ ap ap ap } \text{k}_{\text{H}(\infty)} \text{L} \\
\text{m s}
\end{array}$$

Lemma C Let $\mathbb{C}$ be a r -sheeted cover with respect to $\mathbb{R}$ and assume that $\mathbb{C}$ is a 2 -sheeted cover. Then for $\mathbb{C}^1$ is homomorphic on $\mathbb{C}$ we have

$$\mathbb{C}^1 = \frac{\mathbb{C}^1 \cdot \mathbb{C}^1}{\mathbb{C}^1} = \frac{\mathbb{C}^1 \cdot \mathbb{C}^1}{\mathbb{C}^1} = \mathbb{C}^1$$

Proof. Let $\mathbb{C}^1$ be a 2 -sheeted cover with respect to $\mathbb{R}$ and assume that $\mathbb{C}^1$ is a 2 -sheeted cover. Then for $\mathbb{C}^1$ is homomorphic on $\mathbb{C}$ we have

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