A GALERKIN BOUNDARY ELEMENT METHOD FOR HIGH FREQUENCY SCATTERING BY CONVEX POLYGONS

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Abstract. n p pe e con de e p o e of e on c co c c e n n o d en on y con e po y on nd d o nd y o n e e e en e od fo co c c e n p o e e co p on co o e ne y e pec o e f eq ency of e nc den e e e e p e en no e e n o nd y e e en e od n pp o on p ce con n of e p od c of p ne e p ece e po yno ppo ed on ded e e e e en c o e o e co ne of e po y on e de on e o o o e o n y nd n e c e e n e of de ee of f eedo eq ed o

sues, including conditioning and evaluation of the integrals that arise. We finish the paper with some concluding remarks and open problems.

We note that the Galerkin method is, of course, not the only way to select a then [41] there is also a well-defined normal derivative operator, the unique bounded linear operator $\partial_n : H^1(G;\Delta) \to H^{-1/2}(\partial G)$ which satisfies

$$\partial_n v = \frac{\partial v}{\partial n} := n \cdot \nabla v,$$

almost everywhere on Γ, when $v \in C^\infty(\bar{G})$. $H^1_{\mathrm{loc}}(G)$ denotes the set of measurable $v : G \to \mathbb{C}$ for which $\chi v \in H^1(G)$ for every compactly supported $\chi \in C^\infty(\bar{G})$.

The polygonal domain Y is Lipschitz as is its exterior $D := \mathbb{R}^2 \setminus \bar{Y}$. Let $\gamma_+ : H^1(D) \to H^{1/2}(\Gamma)$ and $\gamma_- : H^1(Y) \to H^{1/2}(\Gamma)$ denote the exterior and interior trace operators, respectively, and let $\partial_n^+ : H^1(D;\Delta) \to H^{-1/2}(\Gamma)$ and $\partial_n^- : H^1(Y;\Delta) \to H^{-1/2}(\Gamma)$ denote the exterior and interior normal derivative operators, respectively, the unit normal vector n directed out of Y. Then the boundary value problem we seek to solve is the following: given $k > 0$ (the wave number) find $u \in C^2(D) \cap H^1_{\mathrm{loc}}(D)$ such that

$$\Delta u + k^2 u = 0 \quad \text{in } D, \tag{2.1}$$

$$\gamma_+ u = 0 \quad \text{on } \Gamma, \tag{2.2}$$

and the scattered field, $u^s := u - u^i$, satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u^s}{\partial r}(x) - iku^s(x)\right) = 0, \tag{2.3}$$

where $r = |x|$ and the limit holds uniformly in all directions $x/|x|$.

THEOREM 2.1. (see e.g. [41, theorem 9.11]). The boundary value problem (2.1)–(2.3) has exactly one solution.

Suppose that $u \in C^2(D) \cap H^1_{\mathrm{loc}}(D)$ satisfies (2.1)–(2.3). Then, by standard elliptic regularity estimates [32, §8.11], $u \in C^\infty(\bar{D} \setminus \Gamma_C)$, where $\Gamma_C := \{P_1, \ldots, P_n\}$ is the set of corners of Γ. It is, moreover, possible to derive an explicit representation for u near the corners. For $j = 1, \ldots, n$, let $R_j := \min(L_{j-1}, L_j)$ (with $L_{-1} := L_N$). Let (r, θ) be polar coordinates local to a corner P_j, chosen so that $r = 0$ corresponds to the point P_j, the side Γ_{j-1} lies on the line $\theta = 0$, the side Γ_j lies on the line $\theta = \Omega_j$, and the part of $\bar{D}$ within distance R_j of P_j is the set of points with polar coordinates $\{(r,\theta) : 0 \le r < R_j,\ 0 \le \theta \le \Omega_j\}$. Choose R so that $R \le R_j$ and $\rho := kR < \pi/2$, and let G denote the set of points with polar coordinates $\{(r,\theta) : 0 \le r < R, 0 \le \theta \le \Omega_j\}$ (see figure 2.2). The following result, in which J_ν denotes the Bessel function of the first kind of order ν, follows by standard separation of variables arguments.

THEOREM 2.2 (representation near corners). Let $g(\theta)$ denote the value of u at the point with polar coordinates (R, θ). Then, where (r,θ) denotes the polar coordinates of x, it holds that

$$u(x) = \sum_{n=1}^{\infty} a_n J_{n\pi/\Omega_j}(kr) \sin\left(\frac{n\theta\pi}{\Omega_j}\right), \quad x \in G, \tag{2.4}$$

where

$$a_n := \frac{2}{\Omega_j J_{n\pi/\Omega_j}(kR)} \int_0^{\Omega_j} g(\theta) \sin\left(\frac{n\theta\pi}{\Omega_j}\right) d\theta, \quad n \in \mathbb{N}. \tag{2.5}$$

REMARK 2.3. The condition $\rho = kR < \pi/2$ ensures that $J_{n\pi/\Omega_j}(kR) \neq 0$, $n \in \mathbb{N}$, in fact (see (3.12)) that $|a_n J_{n\pi/\Omega_j}(kr)| \le C(r/R)^{-n}$, where the constant C is

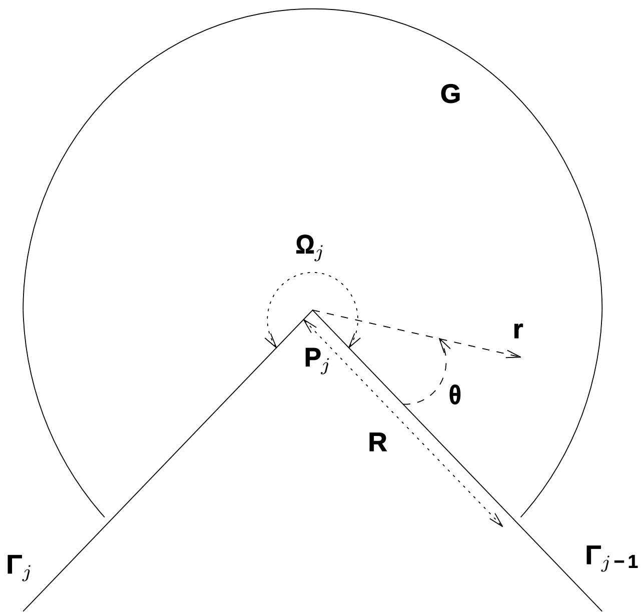
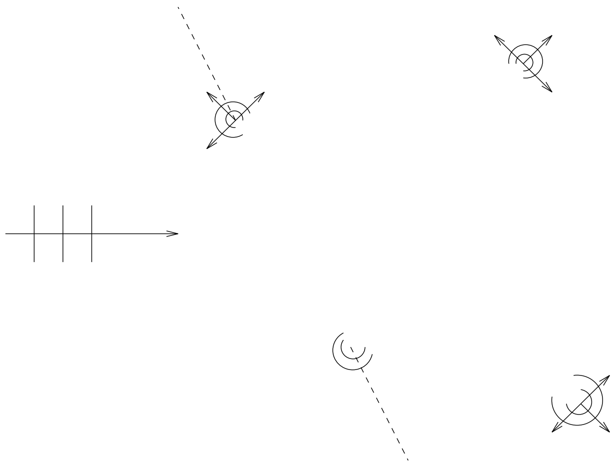


FIG. 2.2 **Neighbourhood of a corner.**

independent of n and x

to an exterior scattering problem for the Helmholtz equation dates back to Brakhage



where $\tilde{L}_0 := 0$ and, for $j = 1,\dots,n$, $\tilde{L}_j := \sum_{m=1}^{j} L_m$ is the arc-length distance from P_1 to P_{j+1}. Define

$$\varphi(s) := \frac{1}{k}\frac{\partial u}{\partial n}(x(s)), \quad \text{for } s \in [0, L], \tag{3.8}$$

where $L := \tilde{L}_n$, so that $\varphi(s)$ is the unknown function of arc-length whose behaviour we seek to determine. Let

$$\Psi(s) := \begin{cases} \frac{2}{k}\frac{\partial u^i}{\partial n}(x(s)), & \text{if } s \in (\tilde{L}_{n_s}, L) \\ 0, & \text{if } s \in (0, \tilde{L}_{n_s}), \end{cases}$$

so that $\Psi(s)$ is the physical optics approximation to $\varphi(s)$, and set $\psi_j(s) := u(\tilde{x}_j(s))$, $s \in \mathbb{R}$, where $\tilde{x}_j(s) \in \Gamma_j^+ \cup \Gamma_j \cup \Gamma_j^-$ is the point

$$\tilde{x}_j(s) := P_j + \left(s - \tilde{L}_{j-1}\right)\left(\frac{P_{j+1} - P_j}{L_j}\right), \quad -\infty < s < \infty.$$

From (3.5) and (3.6) we have the explicit representation for φ on the side Γ_j, that

$$\varphi(s) = \Psi(s) + \frac{i}{2}[e^{iks}v_j^+(s) + e^{-iks}v_j^-(s)], \quad s \in [\tilde{L}_{j-1}, \tilde{L}_j], \quad j = 1,\dots,n, \tag{3.9}$$

where

$$v_j^+(s) := k\int_{-\infty}^{\tilde{L}_{j-1}} \mu(k|s-t|)e^{-ikt}\psi_j(t)\,dt, \quad s \in [\tilde{L}_{j-1}, \tilde{L}_j], \quad j = 1,\dots,n,$$
$$v_j^-(s) := k\int_{\tilde{L}_j}^{\infty} \mu(k|s-t|)e^{ikt}\psi_j(t)\,dt, \quad s \in [\tilde{L}_{j-1}, \tilde{L}_j], \quad j = 1,\dots,n.$$

The terms $e^{iks}v_j^+(s)$ and $e^{-iks}v_j^-(s)$ in (3.9) are the integrals over Γ_j^+ and Γ_j^-, respectively, in equation (3.5), and can be thought of as the contributions to $\partial u/\partial n$ on Γ_j due to the diffracted rays travelling from P_j to P_{j+1} and from P_{j+1} to P_j, respectively, including all multiply diffracted ray components.

So the equation we wish to solve is (2.9), and we have the explicit representation (3.9) for its solution. At first glance this may not appear to help us, since the unknown solution u appears (as ψ_j) on the right hand side of (3.9). However, (3.9) is extremely helpful in understanding how φ behaves since it explicitly separates out the oscillatory part of the solution. The functions $v_j^\pm$ are not oscillatory away from the corners, as the following theorem quantifies. In this theorem and hereafter we let

$$u_M := \sup_{x \in D} |u(x)| < \infty \tag{3.10}$$

and note that $\|\psi_j\|_\infty \le u_M$, $j = 1,\dots,n$.

THEOREM 3.2 (**solution behaviour away from corners**). For $\epsilon > 0$, $j = 1,\dots,n$, and $m = 0,1,\dots$, it holds for $s \in [\tilde{L}_{j-1}, \tilde{L}_j]$ that

$$|v_j^{+(m)}(s)| \le 2C_\epsilon\, m!\, u_M k^m (k(s - \tilde{L}_{j-1}))^{-1/2-m}, \quad k(s - \tilde{L}_{j-1}) \ge \epsilon,$$
$$|v_j^{-(m)}(s)| \le 2C_\epsilon\, m!\, u_M k^m (k(\tilde{L}_j - s))^{-1/2-m}, \quad k(\tilde{L}_j - s) \ge \epsilon,$$

where C_ϵ is given by (3.7).

confirming that the series (2.4) converges for $0 \leq r < R$. Further, the bound (3.14) justifies differentiating (2.4) term by term to get that, for $x \in \Gamma_{j-1} \cap G$, $\frac{\partial u}{\partial n}(x) = kF(kr)$, where

$$F(z) := \frac{\pi}{\Omega_j z} \sum_{n=1}^{\infty} n a_n J_{n\pi/\Omega_j}(z), \quad \mathrm{Re}\, z > 0, \quad |z| < \rho. \tag{3.15}$$

Since $|\cos z| \leq e^{|\mathrm{Im}\, z|}$, $z \in \mathbb{C}$, so that $|\cos zt| \leq e^{|\mathrm{Im}\, z|}$ for $z \in \mathbb{C}$, $0 \leq t \leq 1$, we see from (3.13) that, for $\mathrm{Re}\, z > 0$,

$$\left| n a_n J_{n\pi/\Omega_j} z \right.$$

with $R = R(s,t) := \sqrt{(a}$

and, using (4.6) and since $N/(j-1) \le 2N/j$,

$$\text{(4.12)} \quad \|f^{(\ +1)}\|_{\infty,(y_{j-1},y_j)} \le c_{\ +1} k^{-} \, y_{j-1}^{- \ - \ -1} \le c_{\ +1} k^{\ +1} \left(\frac{2N}{j}\right)^{q(\ + \ +1)}.$$

Combining (4.10)-(4.12) we see that, for $j = 2, \dots, N$,

$$\text{(4.13)} \qquad e_j \le \frac{C \ c^2_{\ +1}}{kN^{2 \ +3}}.$$

For $j = N+1, \dots, N_{A,q}$, recalling (4.4) and the choice of N, and then using (4.11),

$$y_j - y_{j-1} = y_{j-1}\left(\frac{y_j - y_{j-1}}{y_{j-1}}\right) \le y_{j-1}\left(\frac{y_N - y_{N-1}}{y_{N-1}}\right) \le y_{j-1}\frac{q}{N-1} \le 2y_{j-1}\frac{q}{N}.$$

Also, from (4.6),

$$\|f^{(\ +1)}\|_{\infty,(y_{j-1},y_j)} \le c_{\ +1} k^{-1/2} y_{j-1}^{- \ -3/2}.$$

Using these bounds in (4.10), we see that the bound (4.13) holds also for $j = N + 1, \dots, N + N_{A,q}$. Combining (4.8), (4.9), and (4.13),

$$\|f - P_N f\|^2_{2,(0,A)} \le \frac{C \ \tilde{c}^2 (N + N_{A,q})}{kN^{2 \ +3}} \le \frac{C \ \tilde{c}^2 (1 + \log(kA/c))}{kN^{2 \ +3}},$$

using (4.5). Hence the result follows. □

We assume through the remainder of the paper that $c > 0$ is chosen so that

$$\text{(4.14)} \qquad kL_j \ge c, \quad j = 1, \dots, n.$$

For $j = 1, \dots, n$, recalling (3.11), sume tN

C

We then have the following error estimate, in which $\mathbf{u}_M$ is as defined in (3.10) and we abbreviate $\|\cdot\|_{2,(0,L)}$ by $\|\cdot\|_2$.

THEOREM 4.3.

for $N \geq N^{*}$, where N^{*} and C_s are as defined in theorem 5.2.

Note that we will take $c^{*} = 1$ and $\eta = k$

6. Numerical results. There has been much work on the optimal choice of the parameter η in (2.9) (see e.g. [3, 37]). Here we choose $\eta = k$ as in [28]. We also set $c^{*} = 1$ and restrict attention to the case $\nu = 0$. For higher values of ν the implementation of the scheme is similar. Note that, with $c^{*} = 1$ and $\nu = 0$, there are approximately N degrees of freedom used to represent the solution on the intervals

meshes Γ_p^+ and Γ_p^-, for some side p), it holds that $|(\rho_j, \rho_m)| = \sin(ko)\sqrt{o/(kS_1S_2)}$, where S_1 and S_2 are the lengths of the two-subintervals, o the length of the overlap.

As a numerical example, we consider the problem of scattering by a square with sides of length 2π. In this case $n = 4$ and $\Omega_j = 3\pi/2$, $j = 1, 2, 3, 4$. The corners of the square are $P_1 := (0, 0)$, $P_2 := (2\pi, 0)$, $P_3 := (2\pi, 2\pi)$, $P_4 := (0, 2\pi)$, and the incident angle is $\theta = \pi/4$, so the incident field is directed towards P_4, with P_2 in shadow (as shown in figure 6.1, where the total acoustic field is plotted for $k = 10$).

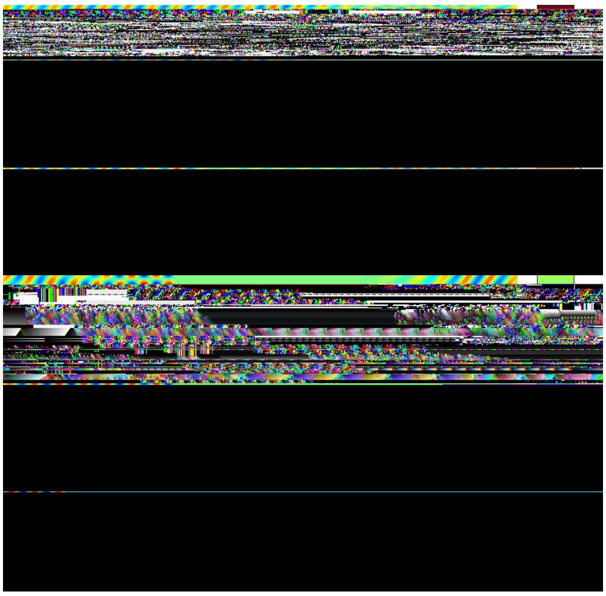


FIG. .1 Total acoustic field, scattering by a square, . Incident field is directed from the top left corner towards the bottom right corner.

In figure 6.2 we plot $|\phi_N(s)|$ against s for $k = 10$ and $N = 4, 16, 64, 256$. As we expect, $|\phi_N(s)|$ is highly peaked at the corners of the polygon, $s = 0$, 2π, 4π, 6π and 8π (which is the same corner as $s = 0$), where $\phi(s)$ is infinite. Except at these corners, $|\phi_N(s)|$ appears to be converging pointwise as N increases. (We do not plot $\phi_N(s)$ itself which is highly oscillatory.)

In order to test the convergence of our scheme, we take the "exact" solution to be that computed with a large number of degrees of freedom, namely with $N = 256$. For $k = 5$ and $k = 320$ the relative L^2 errors $\|\phi_N - \phi_{256}\|_2/\|\phi_{256}\|_2$ are shown in table 6.1 (all L^2 norms are computed by approximating by discrete L^2 norms, sampling at 100000 evenly spaced points around the boundary of the square). For this example, theorem 5.3 predicts that, for $N \geq N^*$, $\|\phi - \phi_N\|_2 \leq CN^{-1}$, where C is a constant. Thus theorem 5.3 predicts that, for $N > N^*$, the average rate of convergence,

$$\mathrm{EOC} := \frac{\log(\|\phi - \phi_N\|_2/\|\phi - \phi_{N^*}\|_2)}{\log(N/N^*)} \geq 1 - \frac{\hat{C}}{\log(N/N^*)} \approx 1$$

as $N \to \infty$, where $\hat{C} := \log(\|\phi - \phi_N\|_2/C)$. This behaviour is clearly seen in the EOC values (defined with $N^* = 8$) in table 6.1, /R298427]TJΩ/R717.97011TfΩ.89017(g)-5.89017(()3760T332(d)-59.9-1.22332

analysis will have relevance for representing certain components of the total field. For example, in the case of 2D convex curvilinear polygons, something close to the mesh grading we use may be appropriate on each side of the polygon, especially at higher frequencies when the waves diffracted by the corners become more localised near the corners. In the case of three-dimensional scattering by convex polyhedra it seems to us likely that the mesh we propose will be useful in representing the variation of edge scattered waves in the direction perpendicular to the edge.

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