Least Squares Minimisation and Steepest Descent Methods for the Scalar Advection Equation and a Cauchy-Riemann System on an Adaptive Grid. *

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Abstract

R c ntly Ro [2] has sugg st d solving syst ms of first ord r cons rvation laws num rically and simultan ously adapting th computational grid using a l ast squar s minimisation proc dur on th fluctuations tog th r with a st p st d sc nt it ration approach to solv th r sulting minimisation probl m. In this r port, th proc dur is r p at d for th Cauchy-Ri mann syst m writt n in compl x form and sugg stions mad for oth r possibl functionals to b minimis d in th scalar cas . In ach cas , st p st d sc nt updat s ar writt n xplicitly for simpl choic s of th functional.

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1 Introduction

The problem which will be addressed in this brief report is the minimisation of the functional

$$F = \frac{1}{2} \sum_T \underline{\phi}_T^{\mathrm{T}} Q_T \, \underline{\phi}_T \, . \tag{1.1}$$

This quantity arises from the numerical solution of systems of first order conservation laws via least squares minimisation of the associated fluctuations $\underline{\phi}_T$. The sum in (1.1) is taken over the grid cells (T) of a triangulated computational domain and the Q_T are positive definite symmetric matrices. The $\underline{\phi}_T$ will be defined precisely later in this report and the Q_T will be chosen appropriately for each case considered.

F can be considered as a function of solution values stored at the grid nodes *and* of the coordinates of the nodes themselves so it can be minimised with respect to any or all of these variables. The minimisation can be achieved iteratively using

2 Split Sc l r Fluctu tions

Consider the steady state linear advection equation in two dimensions,

$$\vec{a} \cdot \vec{\nabla} u \;=\; 0 \,, \tag{2.1}$$

where the advection velocity $\vec{a} = (a,b)^{\mathrm{T}}$ is constant over the whole domain. Alternatively, in any case where $\vec{a}$ is divergence-free, equation (2.1) can be written as

$$\vec{\nabla} \cdot \vec{f} = 0 \qquad \text{where} \qquad \vec{f} = u\,\vec{a}\;. \tag{2.2}$$

The fluctuation in a triangle T associated with (2.1) is given by

$$\begin{aligned} \phi_T &= -\int\!\!\int_{\triangle} \vec{a} \cdot \vec{\nabla} u \;\mathrm{d}x\,\mathrm{d}y \\ &= \oint_{\partial\triangle} u\,\vec{a} \cdot \mathrm{d}\vec{n}\;, \end{aligned} \tag{2.3}$$

where $\vec{n}$ represents the inward pointing normal to the boundary of the cell. Under the assumption that u varies linearly over each triangle and its approximation is continuous across the cell edges the discrete fluctuation is evaluated to be

$$\begin{aligned} \phi_T &= \sum_{k=1}^{3} \frac{1}{2}\,(u_i + u_j)\,\vec{a} \cdot \vec{n}_k \\ &= \sum_{k=1}^{3} -\frac{1}{2}\,(\vec{a} \cdot \vec{n}_k)\,u_k\;, \end{aligned} \tag{2.4}$$

where k is a vertex of the triangle (i and j are the other two) and $\vec{n}_k$ is the normal to the edge opposite vertex k scaled by the length of that edge.

In [2] ϕ_T is considered as a single scalar quantity but it could be split into components before the steepest descen

The simple choice of $Q_T = \frac{1}{S_T} I$, where S_T is the area of the triangle gives

$$F \; = \; \sum_T \frac{(\phi_T^1)^2 + (\phi_T^2)^2 + (\phi_T^3)^2}{2 S_T} \; = \; \sum_T F_T \; , \tag{2.6}$$

and the individual element contributions to this sum can be written

$$\delta F_T \; = \; \sum_{k=1}^{3} \left(\frac{\phi_T^k}{S_T} \, \delta \phi_T^k - \frac{(\phi_T^k)^2}{2 {S_T}^2} \right.$$

where $(\cdot)^+$ indicates the positive part, so that only contributions to the fluctuation from inflow edges are considered in the minimisation of F (2.6).

One further option is to define the fluctuation within each triangle to be dependent only on perturbations of the variables at the upwind vertices, so if the upwind vertices of a chosen cell are 1 and 2 then

$$\delta \underline{\phi}_T \;=\; \frac{\partial \underline{\phi}_T}{\partial u_1}\delta u_1 + \frac{\partial \underline{\phi}_T}{\partial u_2}\delta u_2 \qquad (2.11)$$

and there is no dependence on δu_3. Effectively, the fluctuation is redefined to be independent of the variables at the dowstream vertices. The disadvantage of the resulting scheme, and of the process of allowing only upwind cells to contribute to the least squares iteration at a node, is that the stencil for the update to a node may change at each iteration, leading to a discontinous change in the definition of F between iterations which may even increase its value. Note that it is more likely that upwinding would be used on the solution variables rather than the grid variables since the former arises from a hyperbolic differential equation.

It may also be possible to combine the ideas behind (2.10) and (2.11) by defining an update of the form

$$\delta \underline{\phi}_T \;=\; \frac{\partial \underline{\phi}_T^1}{\partial u_1}\delta u_1 + \frac{\partial \underline{\phi}_T^2}{\partial u_2}\delta u_2 + \frac{\partial \underline{\phi}_T^3}{\partial u_3}\delta u_3\ , \qquad (2.12)$$

where the ϕ_T^k is the $k^{\rm th}$ component of a vector such as (2.10). This does not discount the possibility of discontinuities in the resulting definition of the functional being minimised but does allow more flexibility in the upwinding of the algorithm.

Also, the suggestion for splitting ϕ_T in (2.10) is not unique. Another choice, for example, might be to divide the fluctuation into components proportional to those derived from multidimensional fluctuation distribution schemes [1]. This would lead to different=exTc)tributionesTD,Tc,fi=Tj,ET,toTDmmqhTTf,TDTD,TTj,TTf,Tf,TD,Tc,=

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leads directly to the Cauchy-Riemann equations,

$$\begin{aligned} \delta' &= \tfrac{\delta}{\sqrt{1-M^2}} &= U_X + V_Y &= 0 \\ \omega' &= \omega &= V_X - U_Y &= 0\ . \end{aligned} \tag{3.3}$$

In [2] (3.3) was kept as a system of equations with real coefficients and the fluctuation was evaluated as

$$\begin{aligned} \underline{\phi}_T &= -\int\!\!\int_\triangle \underline{F}_X + \underline{G}_Y \;\mathrm{d}x\,\mathrm{d}y \\ &= \oint_{\partial\triangle} (\underline{F}, \underline{G}) \cdot \mathrm{d}\vec{n}\ , \end{aligned} \tag{3.4}$$

where $\underline{F} = (U, V)^{\mathrm{T}}$ and G

where $\vec{a} = (1, i)^{\mathrm{T}}$ and W is defined in (3.7). Note that (3.12) bears a striking resemblance to the scalar advection equation (2.1) although the coefficients are now complex.

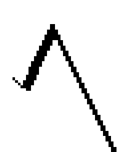
Equations (3.10) and (3.12) are integrated to give the complex fluctuation

$$\begin{aligned} \phi_T &= -\int\!\!\int_{\triangle} \vec{\nabla} \cdot \vec{f} \, \mathrm{d}x \, \mathrm{d}y \\ &= \oint_{\partial\triangle} \vec{f} \cdot \mathrm{d}\vec{n} \\ &= \oint_{\partial\triangle} W \, \vec{a} \cdot \mathrm{d}\vec{n} \, , \end{aligned} \tag{3.13}$$

and the assumption that U and V both vary linearly over each triangle leads to the discrete form of the fluctuation which is given by

$$\phi_T = \sum_{k=1}^{3} -\frac{1}{2}(\vec{a} \cdot \vec{n}_k) \, W_k \; an \; T \, \omega$$

$$\phi$$



and the fact that

$$\begin{aligned}
\phi_T &= -\frac{1}{2}\sum_{k=1}^{3}\left[(V_k\Delta_k Y + U_k\Delta_k X) + i(U_k\Delta_k Y - V_k\Delta_k X)\right] \\
&= \frac{i}{2}\sum_{k=1}^{3} W_k\Delta_k Z \\
&= -\frac{i}{2}\sum_{k=1}^{3} Z_k\Delta_k W \qquad (3.19) \\
&= \frac{1}{2}\sum_{k=1}^{3}\left[(X_k\Delta_k U + Y_k\Delta_k V) + i(Y_k\Delta_k U - X_k\right.
\end{aligned}$$

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