1. Introduction

The testing of numerical methods for computing the steady flow in an open channel has been hindered by the fact that, until now, there have been no readily available non-trivial test problems with known analytic solutions. Here we shall introduce a simple technique for constructing such test problems, including solutions with hydraulic jumps. To carry out the construction, we in effect invert the problem and ask the following question: Given a hypothetical depth profile throughout the channel, what must the slope of the channel be in order for this profile to be a steady solution of the Saint-Venant equations.

2. Smooth Solutions

In smooth regions of flow, steady solutions of the Saint-Venant equations satisfy a differential equation of the form

$$\frac{dy}{dx} = \frac{S_0(x) - \gamma_1(x,y)}{\gamma_2(x,y)}, \tag{2.1}$$

where $y(x)$ is the depth at distance x along the channel. If $z(x)$ is the height of the channel bottom above some horizontal datum, then $S_0(x)$ is the slope of the channel given by

$$S_0 = -\frac{dz}{dx}. \tag{2.2}$$

The particular form of γ_1 and γ_2, for a general channel, is not important; however, note that these functions are independent of z, and in general are well defined for all positive y.

The crux of our method is the following observation. Suppose we have some interval $[x_1, x_2]$ and we choose some shape of channel for this interval; this fixes γ_1 and γ_2. Now let the slope of the channel be given by

$$S_0(x) = \gamma_2(x, \hat{y}(x))\frac{d}{dx}\hat{y}(x) + \gamma_1(x, \hat{y}(x)), \tag{2.3}$$

where $\hat{y}$ is some positive function in $C^1[x_1, x_2]$. By definition an exact solution to equation (2.1) for the stretch of channel in this interval is $y(x) \equiv \hat{y}(x)$.

For simplicity we apply this method only to uniform rectangular channels here, although the method is not restricted to such simple channels. In our examples we use a channel in the interval $[0, L]$ where $L = 100m$. The channel has width $B = 10m$ and discharge $Q = 20m^3s^{-1}$ and we use Manning's friction law, with friction coefficient $n = 0.03$. Under these circumstances we have (see Chow[1])

$$\gamma_1(x,y) = S_f(x,y) = Q|Q|n^2\frac{(2y+B)^{4/3}}{(By)^{10/3}} \tag{2.4}$$

and

$$\gamma_2(x,y) = 1 - \left(\frac{y_c}{y}\right)^3, \tag{2.5}$$

where $_c$ is the critical depth, given by

$$_c = \sqrt[3]{\frac{\;^2}{\;^2}} \qquad (2\;6)$$

and is the acceleration due to gravity. For our examples below, $_c = 0\;741617$.

Here we take $\hat{}$ of the form

$$\hat{}(\;) = \;_c\;\; 1 + \;\; \exp \;\;\; ^2 - \;\; \frac{1}{2} \;\; ^2 \qquad (2\;7)$$

where $= \frac{1}{2}$ and $= 2$. Figure 1a shows $\hat{}$ as well as the corresponding channel slope, calculated from equation (2.3). Figure 1b shows the profile of the chan-

$\frac{1}{4}$

$_c \quad \overline{\quad} \quad - \quad \frac{\quad}{2} \quad - \quad ^2$

$\frac{L}{2}$

$\star$

$^1 \quad L \quad R$

$\star$

$L \quad R \quad \star$

channel slope. This assumes that we have enough smoothness from $_1$, $_2$, $\hat{}_L$ and $\hat{}_R$ to be able to differentiate equation (2M3[1)equation ,ee

1

L R

$$L \quad c \quad \sum_{r=0}^{n_L} r \quad \star\, r$$

$$R \quad c \quad \sum_{r=0}^{n_R} r \quad \star\, r$$

0 0

0 0

$$0 \quad \frac{}{0} \quad \frac{}{\frac{}{3}_0}$$

$$0 \quad \frac{}{0} \quad \frac{}{\frac{}{3}_0}$$

$\star$

$$2 \;\star\; 0\, c \;\; 1\, c \;\; 1 \;\star\; 0\, c \;\; 2 \;\star\; 0\, c \;\; 1\, c \;\; 1 \;\star\; 0\, c$$

0 0

1 1 $\frac{dS_0}{dx}$

$$\frac{2}{} \;\star\; 0\, c \;\; 1\, c \frac{2}{} \;\star\; 0\, c \;\; 1\, c \;\; 2 \;\star\; 0\, c \;\; 2\, c$$

$$\frac{1}{} \;\star\; 0\, c \;\; 1\, c \frac{1}{} \;\star\; 0\, c$$

$$\frac{2}{} \;\star\; 0\, c \;\; 1\, c \frac{2}{} \;\star\; 0\, c \;\; 1\, c \;\; 2 \;\star\; 0\, c \;\; 2\, c$$

$$\frac{1}{} \;\star\; 0\, c \;\; 1\, c \frac{1}{} \;\star\; 0\, c$$

0 0 1 1

2 2

In this example we specify $\hat{\ }_L$ completely and then choose an appropriate $\hat{\ }_R$ to satisfy our three conditions. We take $\ ^\star = \frac{2L}{3}$ and let

$$\hat{\ }_L(\ ) = {}_c\ \frac{4}{3} - \frac{9}{10\ ^2}(\qquad ^\star) \tag{3 15}$$

quation (3.11) now gives us $_0 = 1\ 4305$. Using equation (3.13) we get $_1 = 0\ 14492$ and finally equation (3.14) gives us $_2 = \ 0\ 00217112$. These coefficients define a quadratic, but if we just use this quadratic as $\hat{\ }_R$, we have no control over what the right side of the solution looks like (it may become negative). To give us control we can arbitrarily add higher order terms; here we make $\hat{\ }_R$ a

R ^{3} ^{4}

3 ^{3} 4 ^{4}

R

L * $\frac{L}{3}$

R $_c$ $-$ $\overline{\qquad}$ $\overline{\qquad}$ ^{2} *

0 1 3

2 L 3

4 4

$3 \quad -1$

$4/3$

$0 \qquad \frac{\qquad}{3} \; , \qquad \frac{\qquad}{2} \; - \; \frac{\quad}{}$

$1/3 \qquad\qquad 2$

$- \qquad - \qquad \frac{\quad}{} \; -$

$1/3 \qquad\qquad 2$

$- \quad \frac{\quad}{} \; \frac{\quad}{} \; - \qquad \frac{\quad}{} \; -$

$3 \quad 1$

$4/3$

$0 \qquad \frac{\qquad}{3} \qquad \frac{\qquad}{2} \; - \; \frac{\quad}{}$

$1/3 \qquad\qquad 2$

$- \qquad - \qquad \frac{\quad}{} \; -$

and

$$\hat{y}'(x) = \left(\frac{4}{g}\right)^{1/3} \frac{1}{50} \left(\frac{x}{100} - \frac{1}{2}\right) \exp\left[-4\left(\frac{x}{100} - \frac{1}{2}\right)^2\right].$$

Manning's friction coefficient for the channel is 0.03. The flow is supercritical at inflow, with depth $\hat{y}(0)$ and supercritical at outflow.

The exact solution for this problem is $y(x) \equiv \hat{y}(x)$ and is shown in figure 2.

Example 3 *Transcritical Flow*
A rectangular channel, $0 \leq x \leq 100m$, has width $10m$ and a discharge of $20m^3s^{-1}$. The slope of the channel is given by

$$S_0(x) = \left(1 - \frac{4}{g[\hat{y}(x)]^3}\right)\hat{y}'(x) + \frac{9}{2500[\hat{y}(x)]^2}\left(\frac{1}{5} + \frac{1}{\hat{y}(x)}\right)^{4/3},$$

where

$$\hat{y}(x) = \left(\frac{4}{g}\right)^{1/3}\left(1 - \frac{(x-50)}{200} + \frac{(x-50)^2}{30000}\right),$$

and

$$\hat{y}'(x) = \left(\frac{4}{g}\right)^{1/3}\left(-\frac{1}{200} + \frac{(x-50)}{15000}\right).$$

Manning's friction coefficient for the channel is 0.03. The flow is subcritical at inflow and supercritical at outflow.

The exact solution for this problem is $y(x) \equiv \hat{y}(x)$ and is shown in figure 3.

Example 4 *Sub-Super-Subcritical Flow with Hydraulic Jump*
A rectangular channel, $0 \leq x \leq 100m$, has width $10m$ and a discharge of $20m^3s^{-1}$. The slope of the channel is given by

$$S_0(x) = \left(1 - \frac{4}{g[\hat{y}(x)]^3}\right)\hat{y}'(x) + \frac{9}{2500[\hat{y}(x)]^2}\left(\frac{1}{5} + \frac{1}{\hat{y}(x)}\right)^{4/3},$$

where

$$\hat{y}(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3}\left(\frac{4}{3} - \frac{x}{100}\right) - \frac{9x}{1000}\left(\frac{x}{100} - \frac{2}{3}\right) & x \leq \frac{200}{3} \\ \\ \left(\frac{4}{g}\right)^{1/3}\Big(0.674202\left(\frac{x}{100} - \frac{2}{3}\right)^4 + 0.674202\left(\frac{x}{100} - \frac{2}{3}\right)^3 & x > \frac{200}{3} \\ -21.7112\left(\frac{x}{100} - \frac{2}{3}\right)^2 + 14.492\left(\frac{x}{100} - \frac{2}{3}\right) + 1.4305\Big) & \end{cases},$$

and

$$\hat{y}'(x) = \begin{cases} \frac{-1}{100}\left(\frac{4}{g}\right)^{1/3} - \frac{9}{500}\left(\frac{x}{100} - \frac{1}{3}\right) & x \leq \frac{200}{3} \\ \\ \left(\frac{4}{g}\right)^{1/3}\Big(0.02696808\left(\frac{x}{100} - \frac{2}{3}\right)^3 + 0.02022606\left(\frac{x}{100} - \frac{2}{3}\right)^2 & x > \frac{200}{3} \\ -0.434224\left(\frac{x}{100} - \frac{2}{3}\right) + 0.14492\Big) & \end{cases}.$$

Manning's friction coefficient for the channel is 0.03. The flow is subcritical at outflow, with depth $\hat{y}(100)$, and subcritical at inflow.

The exact solution for this problem is $y(x) \equiv \hat{y}(x)$ and is shown in figure 4.

Example 5 *Super-Sub-Supercritical Flow with Hydraulic Jump*
A rectangular channel, $0 \le x \le 100m$, has width $10m$ and a discharge of $20m^3s^{-1}$. The slope of the channel is given by

$$S_0(x) = \left(1 - \frac{4}{g[\hat{y}(x)]^3}\right)\hat{y}'(x) + \frac{9}{2500[\hat{y}(x)]^2}\left(\frac{1}{5} + \frac{1}{\hat{y}(x)}\right)^{4/3},$$

where

$$\hat{y}(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3}\left(-10.7872\left(\frac{x}{100} - \frac{1}{3}\right)^4 + 18.8777\left(\frac{x}{100} - \frac{1}{3}\right)^3\right. & x \le \frac{100}{3} \\ \left. +17.9329\left(\frac{x}{100} - \frac{1}{3}\right)^2 + 3.1725\left(\frac{x}{100} - \frac{1}{3}\right) + 0.850042\right) & \\ & \\ \left(\frac{4}{g}\right)^{1/3}\left(\frac{5}{6} + \frac{(100-x)}{200}\right) + \frac{4}{10}\left(\frac{x}{100} - \frac{1}{3}\right)\left(\frac{x}{100} - 1\right) & x > \frac{100}{3} \end{cases},$$

and

$$\hat{y}'(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3}\left(-0.431488\left(\frac{x}{100} - \frac{1}{3}\right)^3 + 0.566331\left(\frac{x}{100} - \frac{1}{3}\right)^2\right. & x \le \frac{100}{3} \\ \left. +0.358658\left(\frac{x}{100} - \frac{1}{3}\right) + 0.031725\right) & \\ & \\ \frac{-1}{200}\left(\frac{4}{g}\right)^{1/3} + \frac{4}{500}\left(\frac{x}{100} - \frac{2}{3}\right) & x > \frac{100}{3} \end{cases}.$$

Manning's friction coefficient for the channel is 0.03. The flow is supercritical at inflow, with depth $\hat{y}(0)$ and supercritical at outflow.

The exact solution for this problem is $y(x) \equiv \hat{y}(x)$ and is shown in figure 5.