Abstract

Multidim nsional upwinding t chniqu s [?, ?] hav b n d v lop d with th obj ct of th solution of th Eul r quations. How v r, th y can qually w ll b us d to solv oth r hyp rbolic syst ms of quations. R c ntly, th m thod has b n adapt d for th solution of th shallow wat r quations [?], but du to th subtly diff r nt natur of th s quations th lin arisation of th syst m us d impli d that th sch m was not quit cons rvativ .

This r port d scrib s a m thod by which th shallow wat r quations can b lin aris d in a truly cons rvativ mann r, nabling th us of wav mod ls and fluctuation distribution sch m s to giv a cons rvativ multidim nsional upwinding sch m for th shallow wat r quations.

Contents

1 Introduction

This report is intended as a supplement to reference [?] in which the multi-dimensional upwinding techniques developed b

2 A Conserv tive Line ris tion

The shallow water equations without friction, like the Euler equations, constitute a nonlinear hyperbolic system of equations and can be written in the form

$$\underline{\mathbf{u}}_t + \underline{\mathbf{F}}_x + \underline{\mathbf{G}}_y = \underline{\mathbf{0}}, \tag{2.1}$$

where

$$\underline{\mathbf{u}} = \left(\begin{array}{c} h \\ uh \\ \end{array} \right.$$

However, it is also implicit in the paper that all integrations carried out to evaluate these averages have to be done exactly, otherwise the 'telescopic' property required for conservation is not satisfied precisely. This fourth criterion can be simply expressed by requiring that

$$\sum_{\triangle} V_{\triangle}(\widehat{\underline{\mathbf{F}}_x} + \widehat{\underline{\mathbf{G}}_y}) = \oint_{\partial\Omega} (\underline{\mathbf{F}}, \underline{\mathbf{G}}) \cdot d\vec{n}, \tag{2.7}$$

where $V_{\triangle}$ is the area of the cell, $\partial\Omega$ is the outer boundary of the domain and $\vec{n}$ is the inward pointing normal to the boundary. It was never necessary to impose this condition explicitly in the treatment of the Euler equations because it is automatically satisfied by the approximation, but the condition becomes important when the procedure for creating a conservative linearisation is followed for the shallow water equations. In [**?**] an approximation was used which satisfied the three criteria specified above but used inexact integration and so was not conservative.

As with the Euler equations, assuming linearity of the conservative variables would involve the evaluation of unnecessarily complicated integrals before the linearised Jacobians could be constructed. Furthermore, the derivations of the wave models are based on the matrix $A(\underline{\mathbf{u}})\cos\theta + B(\underline{\mathbf{u}})\sin\theta$. This linearisation does not lead to an approximation to this matrix of the form $A(\overline{\underline{\mathbf{u}}})\cos\theta + B(\overline{\underline{\mathbf{u}}})\sin\theta$, where $\overline{\underline{\mathbf{u}}}$ is some average of the conserved variables, so it is not immediately obvious how the wave decomposition could be applied under these circumstances. Therefore, as in the case of the Euler equations, it is assumed that the parameter vector,

$$\underline{\mathbf{w}} = h^{\frac{1}{2}}(1, u, v)^T, \tag{2.8}$$

is the quantity that varies linearly. However, the consequences of this assumption differ slightly from those for the Euler equations because some of the components of $\underline{\mathbf{u}}$, $\underline{\mathbf{F}}$ and $\underline{\mathbf{G}}$ are polynomials in the components of $\underline{\mathbf{w}}$ of order higher than two (although they are far simpler than those resulting from assuming linearity of the conservative variables).

Multidimensional upwinding methods on triangular grids in two dimensions rely on the evaluation within each triangular cell of the so-called fluctuation

$$\Phi = \int \Big($$

Define now the Jacobian matrices,

$$Q = \frac{\partial \underline{\mathbf{u}}}{\partial \underline{\mathbf{w}}} = \begin{pmatrix} 2w_1 & 0 & 0 \\ w_2 & w_1 & 0 \\ w_3 & 0 & w_1 \end{pmatrix}, \tag{2.10}$$

and

$$R = \frac{\partial \underline{\mathbf{F}}}{\partial \underline{\mathbf{w}}} = \begin{pmatrix} w_2 & w_1 & 0 \\ 2g{w_1}^3 & 2w_2 & 0 \\ 0 & w_3 & w_2 \end{pmatrix}, \qquad S = \frac{\partial \underline{\mathbf{G}}}{\partial \underline{\mathbf{w}}} = \begin{pmatrix} w_3 & 0 & w_1 \\ 0 & w_3 & w_2 \\ 2g{w_1}^3 & 0 & 2w_3 \end{pmatrix}, \tag{2.11}$$

in terms of w_i, the components of the parameter vector (2.8). Note here that all the components of these matrices are linear functions of $\underline{\mathbf{w}}$ except for a single cubic term which appears in both R and S.

Thus the fluctuation can be written

$$\begin{aligned} \Phi &= -\int\!\!\int_{\triangle} (R\underline{\mathbf{w}}_x + S\underline{\mathbf{w}}_y)\,dx\,dy & (2.12)\\ &= -\left(\int\!\!\int_{\triangle} R\,dx\,dy\right)\underline{\mathbf{w}}_x - \left(\int\!\!\int_{\triangle} S\,dx\,dy\right)\underline{\mathbf{w}}_y, & (2.13) \end{aligned}$$

since the gradients of the parameter vector variables are constant within each cell.

At this point the theory deviates slightly from that associated with the Euler equations because the assumption of linear variation of $\underline{\mathbf{w}}$ no longer implies that

$$\int\!\!\int_{\triangle} R\,dx\,dy = V_{\triangle} R(\overline{\underline{\mathbf{w}}}), \qquad \int\!\!\int_{\triangle} S\,dx\,dy = V_{\triangle} S(\overline{\underline{\mathbf{w}}}), \tag{2.14}$$

where $V_{\triangle}$ is the area of the cell and $\overline{\underline{\mathbf{w}}}$ is the value of the parameter vector at the centroid of the triangle.

However, it is still possible to do the integration exactly, simply by using a higher order quadrature for the two cubic terms. Note that this only involves the evaluation of one integral and so is not prohibitively expensive.

Now, defining

$$\tilde{R} = \frac{1}{V_{\triangle}} \int\!\!\int_{\triangle} R\,dx\,dy, \qquad \tilde{S} = \frac{1}{V_{\triangle}} \int\!\!\int_{\triangle} S\,dx\,dy, \tag{2.15}$$

leads to

$$\begin{aligned} \Phi &= -V_{\triangle}(\tilde{R}\underline{\mathbf{w}}_x + \tilde{S}\underline{\mathbf{w}}_y) & \\ &= -V_{\triangle}(\widehat{\underline{\mathbf{F}}_x} + \widehat{\underline{\mathbf{G}}_y}). & (2.16) \end{aligned}$$

In fact, $\tilde{R}$ and $\tilde{S}$ only differ from $R(\overline{\underline{\mathbf{w}}})$ and $S(\overline{\underline{\mathbf{w}}})$ in one component each.

All the elements of the matrix Q are linear in $\underline{\mathbf{w}}$ so a similar argument to the one used with the Euler equations gives

$$\widehat{\underline{\mathbf{u}}_x} = Q(\overline{\underline{\mathbf{w}}})\underline{\mathbf{w}}_x, \qquad \widehat{\underline{\mathbf{u}}_y} = Q(\overline{\underline{\mathbf{w}}})\underline{\mathbf{w}}_y. \tag{2.17}$$

If the source term were completely ignored, the approximation would still satisfy the three criteria of the two-dimensional version of Property U, but because the integration is inexact the scheme is no longer conservative, even though the error is likely to be very small. This is effectively the simplification which was made when the concepts of multidimensional upwinding were originally transferred from the Euler equations to the shallow water equations in [?]. In fact, in [?] it was assumed that the primitive variables varied linearly. This assumption is equally valid and the preceding analysis can still be carried out, resulting in a different and slightly more complicated source term.

All that remains is the decomposition and distribution of the fluctuation which, with the aid of a wave model, can be written

$$\Phi = -\int\int_{\triangle} (\underline{\mathbf{F}}_x + \underline{\mathbf{G}}_y)\, dx\, dy = \sum_{k=1}^{N_e} \phi^k \overline{\underline{\mathbf{r}}}^k + \underline{\xi}, \tag{2.26}$$

where N_e is the number of effective waves, ϕ^k is the fluctuation of the k^{th} wave, $\overline{\underline{\mathbf{r}}}^k$ is the vector corresponding to the projection of ϕ^k on to the conservative variables and ξ is the 'source' term in (2.23). It must be remembered that average values of the primitive variables $\underline{\mathbf{q}}$ and their gradients, which are used to calculate all other cell-averaged quantities necessary for the decomposition [?], such as the conservative variables and the flux balance, must be computed exactly from the parameter vector variables, giving

$$\overline{\underline{\mathbf{q}}} = \begin{pmatrix} \overline{w_1}^2 \\ \overline{w_2}/\overline{w_1} \\ \overline{w_3}/\overline{w_1} \end{pmatrix}, \qquad \widehat{\vec{\nabla}\underline{\mathbf{q}}} = \begin{pmatrix} 2\overline{w_1}\vec{\nabla} w_1 \\ (\overline{w_1}\vec{\nabla} w_2 - \overline{w_2}\vec{\nabla} w_1)/\overline{w_1}^2 \\ (\overline{w_1}\vec{\nabla} w_3 - \overline{w_3}\vec{\nabla} w_1)/\overline{w_1}^2 \end{pmatrix}. \tag{2.27}$$

If linearity of the primitive variables is assumed, as it was in [?], then these quantities must be calculated directly.

The results in this report were obtained using the 'Froude angle splitting' wave model [?] and the PSI scalar distribution scheme [?], while the source term was distributed very simply, sending one third to each of the vertices of the triangular cell.

3 Results

Brief results are presented here for a single test case, that of an oblique hydraulic jump, induced by means of the interaction between a supercritical flow and a wall at an angle $\theta = 8.95°$ to the flo

4 Conclusions

In this report, a method has been described by which the concepts of multidimensional upwinding can be used in the solution of the shallow water equations whilst retaining conservation - a property not satisfied by the first algorithms devised [?, ?]. The resulting scheme has been tested on one test case and gives good results in comparison with the exact analytic solution.

It is suggested that in future, the 'source' terms should be distributed in a more sophisticated manner than has been used here. This may well help to reduce the magnitude of the residuals when the numerical solutions ha