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Additive Schwarz Preconditioning for p-Version Triangular and Tetrahedral Finite Elements

Joachim Schöberl[1], Jens M. Melenk[2], Clemens G. A. Pechstein[3], and Sabine C. Zaglmayr[1]

[1] Radon Institute for Computational and Applied Mathematics (RICAM), Austria
{joachim.schoeberl,sabine.zaglmayr}@oeaw.ac.at
[2] The University of Reading, Department of Mathematics, UK
j.m.melenk@reading.ac.uk
[3] Institute for Computational Mathematics, Johannes Kepler University Linz, Austria clemens.pechstein@numa.uni-linz.ac.at

Summary. This paper analyzes two-level Schwarz methods for matrices arising from the *p*-version finite element method on triangular and tetrahedral meshes. The coarse level consists of the lowest order finite element space. On the fine level, we investigate several decompositions with large or small overlap leading to optimal or close to optimal condition numbers. The analysis is confirmed by numerical experiments for a model problem.

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directly. On tensor product elements, one can apply optimal preconditioners for the local sub-problems as in [KJ99, BSS04, BS04].

In the current work, we study overlapping Schwarz preconditioners with large or small overlap. The condition numbers are bounded uniformly in the mesh size h and the polynomial order p. To our knowledge, this is a new result for tetrahedral meshes. We construct explicitly the decomposition of a global function into a coarse grid part and local contributions associated with the vertices, edges, faces, and elements of the mesh.

The rest of the paper is organized as follows: In Section 2 we state the problem and formulate the main results. We prove the 2D case in Section 3 and extend the proof for 3D in Section 4. Finally, in Section 5 we give numerical results for several versions of the analyzed preconditioners.

2 Definitions and Main Result

We consider the Poisson equation on the polyhedral domain Ω with homogeneous Dirichlet boundary conditions on $\Gamma_D \subset \partial\Omega$, and Neumann boundary conditions on the remaining part Γ_N. With the sub-space $V := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, the bilinear-form $A(\cdot,\cdot) : V \times V \to \mathbb{R}$ and the linear-form $f(\cdot) : V \to \mathbb{R}$ defined as

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \qquad f(v) = \int_\Omega f v \, dx,$$

the weak formulation reads

$$\text{find } u \in V \text{ such that} \qquad A(u,v) = f(v) \qquad \forall v \in V. \tag{1}$$

We assume that the domain Ω is sub-divided into straight-sided triangular or tetrahedral elements. In general, constants in the estimates depend on the shape of the elements, but they do not depend on the local mesh-size. We define

$$\begin{aligned} \text{the set of vertices} \quad & \mathcal{V} = \{V\}, \\ \text{the set of edges} \quad & \mathcal{E} = \{E\}, \\ \text{the set of faces (3D only)} \quad & \mathcal{F} = \{F\}, \\ \text{the set of elements} \quad & \mathcal{T} = \{T\}. \end{aligned}$$

We define the sets $\mathcal{V}_f, \mathcal{E}_f, \mathcal{F}_f$ of *free* vertices, edges, and faces not completely vertices, and of edge-based, face-based, and cell-based bubble functions. The Galerkin projection onto V_p leads to a large system of linear equations, which shall be solved with the preconditioned conjugate gradient iteration.

This paper is concerned with the analysis of additive Schwarz preconditioning. The basic method is defined by the following space splitting. In Section 5 we will consider several cheaper versions resulting from our analysis. The coarse sub-space is the global lowest order space

$$V_0 := \{v \in V : v|_T \in P^1 \quad \forall T \in \mathcal{T}\}.$$

For each inner vertex we define the vertex patch

$$\omega_V = \bigcup_{T \in \mathcal{T} : V \in T} T$$

and the vertex sub-space

$$V_V = \{v \in V_p : v = 0 \text{ in } \Omega \setminus \omega_V\}.$$

For vertices V not on the Neumann boundary, this definition coincides to $V_p \cap H_0^1(\omega_V)$. The additive Schwarz preconditioning operator is $C^{-1} : V_p^* \to V_p$ defined by

$$C^{-1} d = w_0 + \sum_{V \in \mathcal{V}} w_V$$

with $w_0 \in V_0$ such that

$$A(w_0, v) = \langle d, v \rangle \quad \forall v \in V_0,$$

and $w_V \in V_V$ defined such that

$$A(w_V, v) = \langle d, v \rangle \quad \forall v \in V_V.$$

This method is very simple to implement for the p-version method using a hierarchical basis. The low-order block requires the inversion of the sub-matrix according to the vertex basis functions. The high order blocks are block-Jacobi steps, where the blocks contain all vertex, edge, face, and cell unknowns associated with mesh entities containing the vertex V.

The rate of convergence of the cg iteration can be bounded by means of the spectral bounds for the quadratic forms associated with the system matrix and the preconditioning matrix. The main result of this paper is to prove optimal results for the spectral bounds:

Theorem 1. *The constants λ_1 and λ_2 of the spectral bounds*

$$\lambda_1 \langle Cu, u \rangle \leq A(u, u) \leq \lambda_2 \langle Cu, u \rangle \quad \forall u \in V_p$$

are independent of the mesh-size h and the polynomial order p.

The proof is based on the additive Schwarz theory, which allows to express the C-form by means of the space decomposition:

$$C u, u = \inf_{\substack{u = u_0 + \sum_V u_V \\ u_0 \in V_0,\, u_V \in V_V}}$$

In the following, let V be a vertex not on the Dirichlet boundary Γ_D, and let ϕ_V be the piece-wise linear basis function associated with this vertex. Furthermore, for $s \in [0, 1]$ we define the level sets

$$\gamma_V(s) := \{y \in \omega_V : \phi_V(y) = s\},$$

and write $\gamma_V(x) := \gamma_V(\phi_V(x))$ for $x \in \omega_V$. For internal vertices V, the level set $\gamma_V(0)$ coincides with the boundary $\partial\omega_V$ (cf. Figure 1). The space of functions being constant on these sets reads

$$S_V := \{w \in L_2(\omega_V) : w|_{\gamma_V(s)} = \text{const}, \ s \in [0, 1] \ a.e.\};$$

its finite dimensional counterpart is

$$S_{V,p} := S_V \cap V_p = \text{span}\{1, \phi_V, \ldots, \phi_V^p\}.$$

We introduce the *spider averaging operator*

$$\left[\Pi^V v\right](x) := \quad 1$$

(iii) if u is continuous at V, then

$$(\Pi^V u)(V) = \Pi^V$$

$$\Pi^V u \;{}^{2}_{L_2(\omega_V)} \quad h_V \; {}_{0}^{\,1} \; {}_{\gamma_V(0)} (\Pi^V u)(F_V($$

Using $r_V(x) \simeq (1-\phi_V(x))h_V$ we derive

$$
\begin{aligned}
\int_{\omega_V} \frac{1}{r_V^2}\left(u - \Pi^V u\right)^2 dx
&\lesssim h_V \int_0^1 \int_{\gamma_V(0)} \frac{1}{r_V^2}\left(u(F_V(y,s)) - \frac{1}{|\gamma_V(0)|}\int_{\gamma_V(0)} u(F_V(x,s))\,dx\right)^2 (1-s)\,dy\,ds \\
&\lesssim h_V \int_0^1 \int_{\gamma_V(0)} \frac{h_V^2}{r_V^2}\left|\nabla_y u(F_V(y,s))\right|^2 (1-s)\,dy\,ds \\
&\lesssim h_V \int_0^1 \int_{\gamma_V(0)} \frac{1}{(1-s)^2}\left|(\nabla u)(F_V(y,s))\frac{\partial F_V}{\partial y}\right|^2 (1-s)\,dy\,ds \\
&= h_V \int_0^1 \int_{\gamma_V(0)} \left|(\nabla u)(F_V(y,s))\right|^2 (1-s)\,dy\,ds \\
&\lesssim \|\nabla u\|^2_{L_2(\omega_V)}.
\end{aligned}
$$

(iv) Since $\phi_V 1 = \Pi_0^V 1$, we can subtract the mean value $\bar{u} := \frac{1}{|\omega_V|}\int_{\omega_V} u(x)\,dx$:

$$
\begin{aligned}
\|\nabla\{\phi_V u - \Pi_0^V u\}\| &= \|\nabla\{\phi_V(u-\bar{u}) - \Pi_0^V(u-\bar{u})\}\| \\
&\le \|\nabla\phi_V(u-\bar{u})\| + \|\nabla\,\Pi^V(u-\bar{u}) - \Pi^V(u-\bar{u})|_{\gamma_V(0)}(1-\phi_V)\| \\
&\lesssim \|(\nabla\phi_V)(u-\bar{u})\| + \|\phi_V\nabla u\| + \|\nabla u\| + \left|\Pi^V(u-\bar{u})|_{\gamma_V(0)}\right| \|\nabla\phi_V\| \\
&\lesssim h^{-1}\|u-\bar{u}\| + \|\nabla u\| \\
&\lesssim \|\nabla u\|.
\end{aligned}
$$

We have used (ii) and the trace inequality for

$$
\begin{aligned}
\left|\Pi^V(u-\bar{u})|_{\gamma_V(0)}\right| &= \frac{1}{|\gamma_V(0)|}\left|\int_{\gamma_V(0)} u-\bar{u}\,dx\right| \le \\
&\le |\gamma_V(0)|^{-1/2}\|u-\bar{u}\|_{L_2(\gamma_V(0))} \lesssim \|\nabla(u-\bar{u})\| + h^{-1}\|u-\bar{u}\|.
\end{aligned}
\tag{3}
$$

(v) We finally prove $\|r_V^{-1}\{\phi_V u - \Pi_0^V u\}\|_{L_2(\omega_V)} \lesssim \|\nabla u\|_{L_2(\omega_V)}$. From the definition of r_V, we get

$$
\left\|\frac{1}{r_V}\{\phi_V u - \Pi_0^V u\}\right\| \lesssim \left\|\frac{1}{r_V}\{\phi_V u - \Pi_0^V u\}\right\| + \sum_{V' \in \omega_V\setminus\{V\}} \left\|\frac{1}{r_{V'}}\{\phi_V u - \Pi_0^V u\}\right\|.
$$

We bound the first term as follows:

$$\left\| \frac{1}{r_V}\{\phi_V u - \Pi_0^V u\} \right\|_{L_2(\omega_V)}$$
$$= \left\| \frac{1}{r_V} \left((1-(1-\phi_V))u - \Pi^V u + (1-\phi_V)(\Pi^V u)|_{\gamma_V(0)} \right) \right\|$$
$$= \left\| \frac{1}{r_V} \left((u - \Pi^V u) - (1-\phi_V)(u-\overline{u}) + (1-\phi_V) \left((\Pi^V u)|_{\gamma_V(0)} - \overline{u} \right) \right) \right\|$$
$$\left\| \frac{1}{r_V}(u - \Pi^V u) \right\| + \left\| \frac{1-\phi_V}{r_V}(u-\overline{u}) \right\| + \left\| \frac{1 - \phi_V}{r_V} \left((\Pi^V u)|_{\gamma_V(0)} - \overline{u} \right) \right\|$$
$$u \quad + h^{-1} \quad u - \overline{u} \quad + h^{-1} \left\| (\Pi^V u)|_{\gamma_V(0)} - \right.$$

Obviously, $u - \Pi_V u$ vanishes in any vertex $V \quad V_f$. These well-defined zero vertex values are reflected by the following norm definition:

$$\|\!|\cdot|\!\|^2 := \quad \cdot \;{}^2_{L_2(\Omega)} + \frac{1}{r_V} \cdot \;{}^2_{L_2(\Omega)} \tag{5}$$

Theorem 2. *Let u_1 be as in Lemma 1. Then, the decomposition*

$$u_1 = \sum_{V \quad V_f} \Pi_0^V u_1 + u_2 \tag{6}$$

is stable in the sense of

$$\sum_{V \quad V_f} \Pi_0^V u_1 \;{}^2_A + \|\!|u_2|\!\|^2 \quad u \;{}^2_A. \tag{7}$$

Proof. The vertex terms in equation (7) are bounded by

$$\Pi_0^V u_1 \;{}^2_A = \quad \Pi^V u_1$$

decomposition further, such that the remaining function, u_3, contributes only to the inner basis functions of each element.

Therefore we need a lifting operator which extends edge functions to the whole triangle preserving the polynomial order. Such operators were introduced in Babuška et al. [BCM91], and later simplified and extended for 3D by Muñoz-Sola [Mun97]. The lifting on the reference element T^R with vertices $(-1,0)$, $(1,0)$, $(0,1)$ and edges $E_1^R := (-1,1)\times\{0\}$, E_2^R, E_3^R reads:

$$(R_1 w)(x_1,x_2) := \frac{1}{2x_2}\int_{x_1-x_2}^{x_1+x_2} w(s)ds,$$

for $w \in L_1([-1,1])$. The modification by Muñoz-Sola preserving zero boundary values on the edges E_2^R and E_3^R is

$$(Rw)(x_1,x_2) := (1-x_1-x_2)(1+x_1-x_2)\left(R_1\frac{w}{1-x_1^2}\right)(x_1,x_2).$$

For an arbitrary triangle $T = F_T(T^R)$ containing the edge $E = F_T(E_1^R)$, its transformed version reads

$$R_T w := R\left(w\circ F_T\right)\circ F_T^{-1}.$$

The Sobolev space $H_{00}^{1/2}(E)$ on an edge $E = [V_{E,1}, V_{E,2}]$ is defined by its corresponding norm

$$\|w\|^2_{H_{00}^{1/2}(E)} := \|w\|^2_{H^{1/2}(E)}$$

Lemma 5. *The edge-based interpolation operator Π_0^E defined in (8) is bounded in the $|||\cdot|||$-norm:*

$$\|\Pi_0^E u\|_{L_2(\omega_E)} \preceq |||u|||_{\omega_E}$$

Proof. Foremost, we apply Lemma 4 on a single triangle $T \subset \omega_E$:

$$\begin{aligned}\|\Pi_0^E u\|^2_{L_2(T)} &= \|R_T \operatorname{tr}_E u\|^2_{L_2(T)} \\ &\preceq \|\operatorname{tr}_E u\|^2_{H^{1/2}(E)} + \int_E \frac{1}{r_{V_E}} (\operatorname{tr}_E u)^2 \, ds.\end{aligned}$$

For the first term, the trace theorem can be immediately applied.

The second term, the weighted L_2-norm on the edge, can be bounded by a weighted norm on the triangle. We transform onto the reference triangle,

$$\int_E \frac{1}{r_{V_E}} u^2 \, ds \simeq \int_{E_1^R} \frac{1}{r_{V_{E_1^R}}} (u \circ F_T)^2 \, ds,$$

and write $u^R := u \circ F_T$. Due to symmetry, we consider only the right half of the edge E_1^R, where $r_{E_1^R} = \frac{1}{1-x_1}$, and finally apply a trace inequality:

$$\begin{aligned}&\int_0^1 \frac{1}{1-x_1} u^R(x_1, 0)^2 \, dx_1 \\ &\preceq \int_0^1 \frac{1}{1-x_1} \int_0^{1-x_1} (1-x_1) \left(\frac{\partial u^R}{\partial x_2}\right)^2 + \frac{1}{1-x_1} [u^R]^2 \, dx_2 \, dx_1 \\ &\preceq |||u^R|||^2_{T^R} \simeq |||u|||^2_T.\end{aligned}$$

This leads us immediately to

Theorem 3. *Let u_2 be as in Theorem 2. Then, the decomposition*

$$u_2 = \sum_{E \in \mathcal{E}_f} \Pi_0^E u_2 + u_3 \tag{9}$$

satisfies $u_3 = 0$ on $\bigcup_{E \in \mathcal{E}_f} E$ and is bounded in the sense of

$$\sum_{E \in \mathcal{E}_f} \|\Pi_0^E u_2\|^2_{L_2} + \|u_3\|^2_{L_2} \preceq |||u_2|||^2. \tag{10}$$

3.4 Main result

Proof (Proof of Theorem 1 for the case of triangles). Summarizing the last subsections, we have

$$u_1 = u - \Pi_h u, \qquad u_2 = u_1 - \sum_{V \in \mathcal{V}_f} \Pi_0^V u_1, \qquad u_3 = u_2 - \sum_{E \in \mathcal{E}_f} \Pi_0^E u_2,$$

and the decomposition

$$u = \Pi_h u + \sum_{V \in \mathcal{V}_f} \Pi_0^V u_1 + \sum_{E \in \mathcal{E}_f} \Pi_0^E u_2 + \sum_{T \in \mathcal{T}} u_3|_T. \tag{11}$$

is stable in the $\|\cdot\|_A$-norm.

For any edge E or triangle T, we can find a vertex V, such that the corresponding summand is in V_V. Since for each vertex only finitely many terms appear, we can use the triangle inequality and finally arrive at the missing spectral bound

$$\langle Cu, u\rangle = \inf_{\substack{u = u_0 + \sum_V u_V \\ u_0 \in V_0, u_V \in V_V}} \|u_0\|_A^2 + \sum_V \|u_V\|_A^2 \preceq \langle Au, u\rangle.$$

4 Sub-space splitting for tetrahedra

Most of the proof for the 3D case follows the strategy introduced in Section 3, so we use the definitions thereof. The only principal difference is the edge interpolation operator, which shall be treated in more detail.

4.1 Coarse and vertex contributions

We define the level surfaces of the vertex hat basis functions

$$\Gamma_V(x) := \Gamma_V(\phi_V(x)) := \{y : \phi_V(y) = \phi_V(x)\}.$$

As in 2D, we first subtract the coarse grid function

$$u_1 = u - \Pi_h u,$$

and secondly the multi-dimensional vertex interpolant to obtain

$$u_2 = u_1 - \Pi_V u_1,$$

where the definitions of Π^V, Π_0^V, Π_V are the same as in Section 3, only the level set lines γ_V are replaced by the level surfaces Γ_V. With the same arguments, one easily shows that

$$\sum_{v \in \mathcal{V}_f} \|\Pi_0^V u_1\|_A^2 + \|u_2\|_{L_2}^2 + \|r_V^{-1} u_2\|_{L_2}^2 \preceq \|u\|_A^2. \tag{12}$$

4.2 Edge contributions

Let $F := \{(s,t) : s \geq 0, t \geq 0, s + t \leq 1\}$ be the reference triangle in Figure 3. For $(s,t) \in F$, we define the level lines

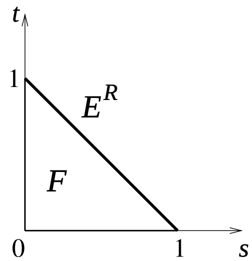

Fig. 3. Reference triangle

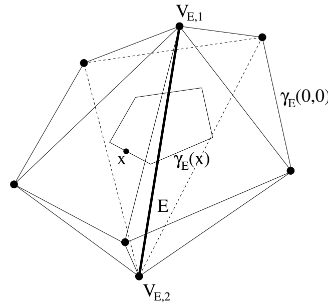

Fig. 4. Edge patch

$$\gamma_E(s,t) := \{x : \phi_{V_{E,1}}(x) = s \text{ and } \phi_{V_{E,2}}(x) = t\},$$

and write

$$\gamma_E(x) := \gamma_E(\phi_{V_{E,1}}(x), \phi_{V_{E,2}}(x))$$

for the level line corresponding to a point x in the edge-patch ω_E, see Figure 4.

Define the space of constant functions on these level lines,

$$S_E := \{v : v|_{\gamma_E(x)} = \text{const}\}$$

and its polynomial subspace $S_{E,p} :=$ S_γ γ

The proof is analogous to the proofs of Lemma 2 and Lemma 3.

Next, the edge-interpolation operator is modified to satisfy zero boundary conditions on $\partial\omega_E$. By the isomorphism

$$v_F(s,t) := v|_{\gamma_E(s,t)}, \qquad \text{for } v \in S_E, \tag{13}$$

the function space S_E can be identified with a space on the triangle F.

Lemma 7. *The isomorphism (13) fulfills the following equivalences for functions $v \in S_E$:*

(i)

$$\|v\|_{L_2(\omega_E)} \simeq h^{3/2}\,\|r_{E^R}^{1/2} v_F\|_{L_2(F)},$$

(ii)

$$\|\nabla v\|_{L_2(\omega_E)} \simeq h^{1/2}\,\|r_{E^R}^{1/2}\,\nabla v_F\|_{L_2(F)},$$

(iii)

$$\|r_{V_E}^{-1} v\|_{L_2(\omega_E)} \simeq h^{1/2}\left\|\frac{r_{E^R}^{1/2}}{r_{V_{E^R}}} v_F\right\|_{L_2(F)},$$

(iv)

$$\|r_E^{-1} v\|_{L_2(\omega_E)} \simeq h^{1/2}\,\|r_{E^R}^{-1/2} v_F\|_{L_2(F)},$$

where

$$E^R := \{(s,t) \in F : s+t=1\},$$

$$r_{E^R}(s,t) := 1-s-t, \quad \textit{and} \quad (r_{V_{E^R}})^{-1} := \tfrac{1}{1-s} + \tfrac{1}{1-t}.$$

Proof. We parameterize the edge-patch ω_E by

$$\begin{aligned} F_E : \gamma_E(0,0) \times F &\to \omega_E \\ (z,(s,t)) &\to z + s(V_{E,1}-z) + t(V_{E,2}-z). \end{aligned}$$

Note that functions $v \in S_E$ do not depend on the parameter $z \in \gamma_E(0,0)$ and $v_F(s,t) = (v \circ F_E)(z,s,t)$ for any $z \in \gamma_E(0,0)$. Equivalence (i) holds due to the transformation of the integrals

$$\begin{aligned} \int_{\omega_E} |v|^2\,dx &\simeq h^2 \int_{\gamma_E(0,0)} \int_0^1 \int_0^{1-s} |v \circ F_E|^2 (1-s-t)\,dt\,ds\,dz \\ &\simeq h^3 \int_F |v_F|^2\, r_{E^R}\, d(s,t). \end{aligned}$$

Derivatives evaluate to $\frac{\partial v_F}{\partial s} = (\nabla v) \cdot \frac{\partial F_E}{\partial s} = (\nabla v) \cdot (V_{E,1} - z)$, and thus

$$|(\nabla v) \circ F_E| \simeq h^{-1} |\nabla v_F|.$$

In combination with (i), we have proven (ii). Finally, equivalences (iii) and (iv) follow from $r_{V_E} \circ F_E \simeq h\, r_{V_{E^R}}$.

We now modify the function

$$u_F(s,t) := (\Pi^E u_2)|_{\gamma_E(s,t)} \qquad (14)$$

to obtain a function $u_{F,00}$ which satisfies zero boundary conditions on the edges $s = 0$ and $t = 0$, and coincides with u_F on the edge $s + t = 1$. This modification is done in such a way that it is continuous in the weighted H^1-norm.

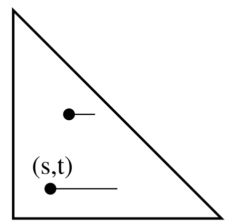

Fig. 5.

First, we prove the corresponding estimates for the smoothing operator S_s. We observe that derivatives of $S_s v$ depend on derivatives of v, only:

$$
\begin{aligned}
\frac{\partial(S_s v)}{\partial s} &= \int_0^1 (\nabla v)\cdot\left(1-\frac{\tau}{2},0\right)d\tau,\\
\frac{\partial(S_s v)}{\partial t} &= \int_0^1 (\nabla v)\cdot\left(-\frac{\tau}{2},1\right)d\tau.
\end{aligned}
$$

Since r_{E^R} is bounded from below and from above on the averaging line $[(s,t);(s+\frac{1}{2}(1-s-t),t)]$, the smoothing operator S_s is bounded in the weighted H^1-semi-norm. The approximation property corresponding to the weighted L_2-norm follows from Friedrichs' inequality applied on the same line.

Now, we prove the estimates for the correction $S_{s,0}-S_s$. The first is

$$
\begin{aligned}
&\left\| r_{E^R}^{1/2}\,\nabla\left(\frac{1-s-t}{1-t}(S_s v)(0,t)\right)\right\|_{L_2(F)}\\
&\leq \left\| r_{E^R}^{1/2}\left(\frac{-1}{1-t},\frac{-s}{(1-t)^2}\right)(S_s v)(0,t)\right\|_{L_2(F)} + \left\| r_{E^R}^{1/2}\frac{1-s-t}{1-t}\,\nabla(S_s v)(0,t)\right\|_{L_2(F)}\\
&\quad \left\|(1-t)^{-1/2}(S_s v)(0,t)\right\|_{L_2(F)} + \left\|(1-t)^{1/2}\frac{\partial(S_s v)}{\partial t}(0,t)\right\|_{L_2(F)}\\
&= \left\|(S_s v\right.
\end{aligned}
$$

$$\int_0^1 (1-t)^2 \left(\frac{\partial(S_s v)}{\partial t}(0,t)\right)^2 dt = \int_0^1 (1-t)^2 \left(\int_0^1 (\nabla v)\cdot(-\tau/2,1)^T\, d\tau\right)^2 dt$$

$$\int_0^1 (1-t)^2 \int_0^1 \left| \nabla v \left(\frac{1-t}{}\right.\right.$$

Theorem 4. *The decomposition*

$$u_2 = \sum_{E \in \mathcal{E}_f} \Pi_0^E u_2 + u_3 \tag{18}$$

fulfills the stability estimate

$$\sum_{E \in \mathcal{E}_f} \|\Pi_0^E u_2\|_A^2 + |u_3|^2 + \|r_E^{-1} u_3\|^2 \lesssim |u_2|^2 + \|r_V^{-1} u_2\|^2. \tag{19}$$

Moreover, $u_3 = 0$ *on* $\bigcup_{E \in \mathcal{E}_f} E$.

Proof. The result is an immediate consequence of Lemma 6, Lemma 8 and Lemma 9 using the argument of finite summation.

4.3 Main result

Proof (Proof of Theorem 1 for the case of tetrahedra). The interpolation on faces in 3D and its analysis follows the line of the edge interpolation in 2D, see Section 3.3.

Summarizing, we obtain

$$u_1 = u - \Pi_h u, \qquad u_2 = u_1 - \sum_{V \in \mathcal{V}_f} \Pi_0^V u_1,$$

$$u_3 = u_2 - \sum_{E \in \mathcal{E}_f} \Pi_0^E u_2, \qquad u_4 = u_3 - \sum_{F \in \mathcal{F}_f} \Pi_0^F u_3,$$

where $\mathcal{F}_f = \{F \in \mathcal{F} : F \not\subset \Gamma_D\}$. As a consequence of the last subsections, the decomposition

$$u = \Pi_h u + \sum_{V \in \mathcal{V}_f} \Pi_0^V u_1 + \sum_{E \in \mathcal{E}_f} \Pi_0^E u_2 + \sum_{F \in \mathcal{F}_f} \Pi_0^F u_3 + \sum_{T \in \mathcal{T}} u_4|_T \tag{20}$$

is stable in the $\|\cdot\|_A$-norm.

5 Numerical results

In this section, we show numerical experiments on model problems to verify the theory elaborated in the last sections and to get the absolute condition numbers hidden in the generic constants. Furthermore, we study two more preconditioners.

We consider the $H^1(\Omega)$ inner product

$$A(u, v) = (\nabla u, \nabla v)_{L_2} + (u, v)_{L_2}$$

on the unit cube $\Omega = (0,1)^3$, which is subdivided into 69 tetrahedra, see Figure 6. We vary the polynomial order p from 2 up to 10. The condition numbers of the preconditioned systems are computed by the Lanczos method.

Example 1: The preconditioner is defined by the space-decomposition with big overlap of Theorem 1:

$$V = V_0 + \sum_v V_v$$

The condition number is proven to be independent of h and p. The computed numbers are drawn in Figure 7, labeled 'overlapping V'. The inner unknowns have been eliminated by static condensation. The memory requirement of this preconditioner is considerable: For p

$$V = V_0 + \sum_{V \in \mathcal{V}} \mathrm{span}\{\phi_V^{l.e.}\} + \sum_{E \in \mathcal{E}} V_E.$$

The computed values are drawn in Figure 7, labeled 'overlapping E, low energy V', and show a moderate growth in *p*. Low energy vertex basis functions ob-

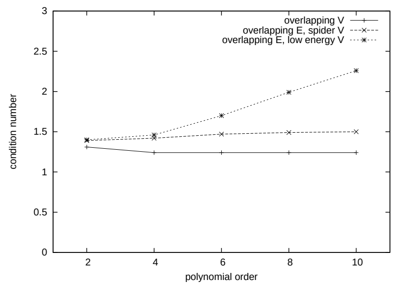

Fig. 7. Overlapping blocks

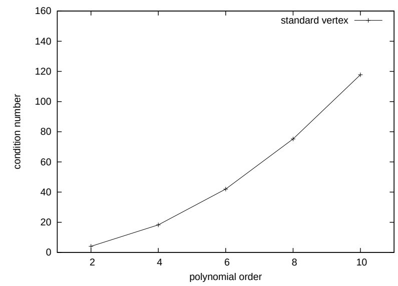

Fig. 8. Standard vertex

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